Quantum Information Theory

$$\rho \;-\!\boxed{\Lambda}\!-\; \rho'$$

What information is contained in ρ?

↳ (What is information?)

What (other) resources does ρ contain?

↳ (For which ~~tasks~~?)

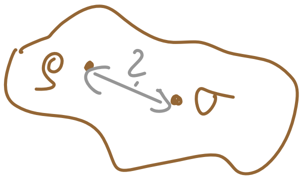
What transformations Λ are possible

↳ How to characterise them?

↳ How to optimise them?

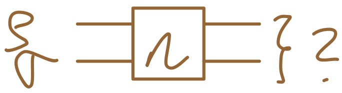
↳ (For what?)

How similar/different are ρ & σ?



Can we recover ρ, σ after Λ was applied?

What about spatial correlations?



And temporal ones?

Quantum Information

- abstracted from physics of information substrate

- most fundamental theory for information processing in nature

- includes everything classical

Course Overview

(subject to change)

31 Jan (Felix)
  Introduction & recap
  quantum states, state space, operators, composite systems

7 Feb (Felix)
  more on quantum states

14 Feb (Simon)
  Dynamics: quantum channels, measurement

21 Feb (Simon)
  channel representations

28 Feb (Felix)
  entanglement, quantum resource theories

— — — — — — — — — — —study week — — — — — — — — — — — — — —

13 Mar (Alessandro)
  Measurement disturbance, classical-quantum states,
  Holevo information, quantum discord

20 Mar (Alec)
  Recovery Maps

27 Mar (Simon)
  Semidefinite Programming

3 Apr (Simon)
  Higher-order quantum maps

10 Apr (Simon)
  Quantum stochastic processes / process tensor

References

Books (all CUP):

Mark Wilde:
Quantum Information Theory
(aka From Classical to Quantum Shannon Theory)

John Watrous:
The Theory of Quantum Information

Michael Nielsen & Isaac Chuang:
Quantum Computation and Quantum Information

Ingemar Bengtsson & Karol Zyczkowski:
Geometry of Quantum states

Lecture notes:
John Goold (Trinity):
- Introduction to Qu. Info. (MSc)
- Introduction to Qu. Info. & Qu. Optics (UG: SS)

Mark Mitchison (Trinity):
Open Qu. Systems (MS.)

John Preskill (Caltech):
Physics 219

Michael Wolf (TU Munich):
Qu. Channels & Operations

Hilbert space (finite dim)

complex inner-product space
vectors $v, w \in \mathbb{C}^d$
with addition & scalar multiplication

inner product:

1) conjugate symmetry:
$$\langle v, w\rangle = \langle w, v\rangle^* \leftarrow \text{complex conjugate}$$
2) linearity in the second argument:
$$\langle v, \lambda_1 w_1 + \lambda_2 w_2\rangle = \lambda_1 \langle v, w_1\rangle + \lambda_2 \langle v, w_2\rangle$$
3) positive definiteness:
$$\langle v, v\rangle \geq 0$$

norm $\|v\| = \|v\|_2 = \sqrt{\langle v, v\rangle}$

orthogonality: $\langle v, w\rangle = 0$

Cauchy-Schwartz: $|\langle v, w\rangle| \leq \|v\| \, \|w\|$

1, 2 $\rightarrow$ antilinearity in first argument
$$\langle \lambda_1 v_1 + \lambda_2 v_2, w\rangle = \lambda_1^* \langle v_1, w\rangle + \lambda_2^* \langle v_2, w\rangle$$

For quantum states:

$\langle \psi, \phi\rangle = \langle \psi | \phi\rangle$

$\langle \psi | : \mathcal{H} \rightarrow \mathbb{C}$

$\mathcal{H}^*$: dual space

Pure quantum states:
rays on Hilbert space

$|\psi\rangle \sim \alpha|\psi\rangle \quad \forall \alpha \in \mathbb{C}\setminus 0$

e.g. $|\psi\rangle \sim e^{i\varphi}|\psi\rangle$

$\Rightarrow$ pick $|\psi\rangle$: $\langle\psi|\psi\rangle = 1$

Basis $\{|b_j\rangle\}$: $|\psi\rangle = \sum c_j |b_j\rangle \quad \forall |\psi\rangle \in \mathcal{H}$

orthonormal basis: $\{|j\rangle\}$ basis
and $\langle j|k\rangle = \delta_{jk}$

Linear operators

e.g. $\mathbb{1}$, $|1\rangle\langle 0|$

$L(X, Y) = \{A : X \to Y\}$
(again: complex vector space)
addition: $(A+B)v = Av + Bv \quad \forall v \in X$
scalar multiplication:
$$(\alpha A)v = \alpha A v \qquad \forall v \in X$$

Direct correspondence
lin. operators $\Leftrightarrow$ matrices

$$A|x_j\rangle = \sum_i a_{ij} |y_i\rangle$$
$$\Rightarrow A = \sum_{ij} a_{ij} |y_i\rangle\langle x_j|$$

adjoint $A^\dagger$:
$$\langle\psi|A\varphi\rangle = \langle A^\dagger\psi|\varphi\rangle$$

$\leftarrow$ obtain by transpose & entry-wise complex conjugation conjugate linearity:

$$(\alpha A + \beta B)^\dagger = \alpha^* A^\dagger + \beta^* B^\dagger$$

anti-distributivity:

$$(AB)^\dagger = B^\dagger A^\dagger$$

$$\text{rank}(A) = \dim(\underbrace{\text{im}(A)}_{\{Av : v\,\mathcal{X}\}})$$

composite spaces

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \rightarrow \quad \langle \psi_1 \otimes \psi_2 | \phi_1 \otimes \phi_2 \rangle$$

$$= \langle \psi_1 | \phi_1 \rangle \langle \psi_2 | \phi_2 \rangle$$

ON basis:

$$|e_{12}^{(ij)}\rangle = |e_1^{(i)}\rangle \otimes |e_2^{(j)}\rangle$$

operators:

$$(A_1 \otimes A_2)(|\psi_1\rangle \otimes |\psi_2\rangle)$$

$$= A_1 |\psi_1\rangle \otimes A_2 |\psi_2\rangle$$

$$A_1, A_2 \in L(X, Y) \qquad B_1, B_2 \in L(Y, Z)$$

$$\Rightarrow (A_1 \otimes A_2)(B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2$$

$$(A_1 \otimes A_2)^\dagger = A_1^\dagger \otimes A_2^\dagger$$

Singular value decomposition

Any $m \times n$ matrix M can be written as

$M = U \Lambda V$

U $m \times m$

V $n \times n$

Λ diagonal with λ_j

U unitary

$\Leftrightarrow$

$U^\dagger U = U U^\dagger = \mathbb{1}$

Schmidt decomposition

$\forall |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$\exists\ \lambda_j > 0$ $\{|j\rangle_A\}$: ONB, $\{|j\rangle_B\}$: ONB s.t.:

schmidt rank

$$|\psi\rangle = \sum_{k=1}^{r} \lambda_k |k\rangle_A \otimes |k\rangle_B$$

$r \leq \min(d_A, d_B)$

In generic ON bases $\{|\alpha\rangle\}$ & $\{|\beta\rangle\}$:

$$|\psi\rangle = \sum_{\alpha=1}^{d_A} \sum_{\beta=1}^{d_B} \gamma_{\alpha,\beta} |\alpha\rangle \otimes |\beta\rangle$$

$\gamma_{\alpha,\beta} = U \Lambda V$ (SVD)

$$\gamma_{\alpha,\beta} = \sum_{\alpha,\beta,k} U_{\alpha,k} \lambda_k V_{k,\beta}$$

$$|\psi\rangle = \sum_{\alpha,\beta,k} U_{\alpha,k} \lambda_k V_{k,\beta} |\alpha\rangle \otimes |\beta\rangle$$

$$= \sum_k \lambda_k \underbrace{\sum_\alpha U_{\alpha,k} |\alpha\rangle}_{\equiv |k\rangle_A} \underbrace{\sum_\beta V_{k,\beta} |\beta\rangle}_{\equiv |k\rangle_B}$$

$$\langle k|j\rangle_A = \sum_{\alpha,\alpha'} U^*_{k,\alpha'} U_{\alpha,j} \underbrace{\langle\alpha'|\alpha\rangle}_{\delta_{\alpha,\alpha'}}$$

$$\boxed{U^\dagger U = \mathbb{1}} = \sum_\alpha U^*_{k,\alpha} U_{\alpha,j} = \delta_{k,j}$$

$\langle k|j\rangle_B$: analogous $\square$

Square operators

$S \in L(X,X) \equiv L(X)$

Normal operators

$N \in L(X)$ normal if $[N, N^\dagger] = 0$ => spectral theorem!

Unitary operators: $U^\dagger U = \mathbb{1} = UU^\dagger$

Hermitian operators

$H \in L(X)$: $H^\dagger = H$

$$\left.\begin{matrix} H_1 + H_2 \\ \lambda H \end{matrix}\right\} \text{remains Hermitian}$$

H: also normal

Identity operator $\mathbb{1}|\psi\rangle = |\psi\rangle \ \forall |\psi\rangle \in X$

trace

$\mathrm{tr}[S] = \sum_i \langle i|S|i\rangle$ ON basis

$$\left.\begin{matrix} \bullet\ \mathrm{tr}[S] \text{ is basis-independent} \\ \bullet\ \text{cyclicity: } \mathrm{tr}[AB] = \mathrm{tr}[BA] \end{matrix}\right\} \text{Show these!}$$

Projectors

$\Pi = \Pi^2$ Hermitian

$\Rightarrow$ only eigenvalues: 1 and 0

Write as $\Pi = \sum_j |j\rangle\langle j|$

Resolution of the identity:

$$\mathbb{1}_d = \sum_{j=1}^{d} |j\rangle\langle j|$$

Spectral theorem

scalars projectors

N normal $\Rightarrow \exists \{\lambda_j\}, \{\Pi_j\}$ s.t.:

$$N = \sum_j \lambda_j \Pi_j$$

eigenvalue

Hermitian operators: real eigenvalues

$\Rightarrow$ Observables

$A = \sum_n a_n \Pi_n \qquad A^\dagger = \sum_n a_n^* \Pi_j$

$a_j^* = a_j \Leftrightarrow A^\dagger = A$

$\langle A \rangle_\psi = \mathrm{tr}(\psi A) = \langle \psi | A | \psi \rangle$

Functions of normal operators

$f: \mathbb{C} \to \mathbb{C}$

extends s.t. $f(N) = \sum_j f(\lambda_j) \Pi_j$

$Z = \mathrm{tr}[e^{-\beta H}] = \dots$

e.g. thermal state

$H = \sum_j E_j |E_j\rangle\langle E_j| \qquad \rho = \frac{1}{Z} e^{-\beta H} = \frac{1}{Z} \sum_j e^{-\beta E_j} |E_j\rangle\langle E_j|$

Positive semidefinite operators

$P \in \mathcal{L}(X)$: $\langle \psi | P | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in X$

$\Leftrightarrow$ P Hermitian with
$\lambda_k \geq 0 \quad \forall k$

Quantum states

$\rho \in S(\mathcal{H})$: positive semidefinite operator
on Hilbert space $\mathcal{H}$
of unit trace

$\mathrm{tr}[\rho] = 1$

ρ can be seen as an ensemble:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

e.g. biased coin $\quad p_H |H\rangle\langle H| + \underset{1-p_H}{p_T} |T\rangle\langle T|$

thermal state: $Z_\beta \propto e^{-\beta H}$

pure state: $|\psi\rangle\langle\psi|$

Convexity:

$(p\rho + (1-p)\sigma) \in S$ $\leftarrow$ show this

mixture vs. superposition

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad$ vs. $\quad \rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

$\mathrm{tr}(|0\rangle\langle 0| + \rangle\langle +|) = \mathrm{tr}(|0\rangle\langle 0|\rho) = \frac{1}{2}$

$\mathrm{tr}(|+\rangle\langle +|+\rangle\langle +|) = 1 \neq \mathrm{tr}(|+\rangle\langle +|\rho) = \frac{1}{2}$

Purity:

$$P(\varrho) = \mathrm{tr}[\varrho^2]$$

$$p(|\psi\rangle\langle\psi|) = 1$$

$$p\left(\tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right) = \tfrac{1}{2}$$

In general $p(\varrho) = \sum_j \lambda_j^2$

minimal for $\mu_d = \mathbb{1}/d$

$$p(\mu_d) = \frac{1}{d}$$

μ_d: maximally mixed