Fuchs – van de Graaf relations

recap:

trace norm: $\|A\|_1 = \mathrm{tr}\sqrt{A^\dagger A}$

$\hookrightarrow$ trace distance: $D_{tr}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1$

$\hookrightarrow$ (Uhlmann) fidelity: $F(\rho,\sigma) = \|\sqrt{\sigma}\sqrt{\rho}\|_1 = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$

$\hookrightarrow D_{BD}^2(\rho,\sigma) = 2(1-F(\rho,\sigma))$

$$\underbrace{1-F(\rho,\sigma)}_{=\frac{1}{2}D_{BD}^2(\rho,\sigma)} \le D_{tr}(\rho,\sigma) \le \sqrt{1-F^2(\rho,\sigma)}$$

upper bound

Equality for pure states:

write $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$

$|\phi\rangle = \sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$ $\qquad 0 \le \theta \le \frac{\pi}{2}$

$F(\psi,\phi) = |\langle\psi|\phi\rangle| = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta$

$\sqrt{1-F^2(\psi,\phi)} = \cos\theta$

$$\underset{|\psi\rangle\langle\psi| \ \ldots}{\psi - \phi} = \begin{pmatrix} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}$$

$\Rightarrow D_{tr}(\psi,\phi) = \cos\theta$ $\square$

optimal purifications

$\sqrt{1-F^2(\rho,\sigma)} = \sqrt{1-|\langle\psi_\rho|\psi_\sigma\rangle|^2}$

$= D_{tr}(\psi_\rho,\psi_\sigma)$

$\ge D_{tr}(\rho,\sigma)$

lower bound

some other time...

$F_{cl}(q,p) = \sum \sqrt{q_i}\sqrt{p_i}$