Lecture 5: Entanglement & Quantum Resource Theories (QRTs)

5.1) Entanglement is useful

Quantum teleportation

$|\chi\rangle_S |\phi^+\rangle_{AB}$ $\qquad\qquad$ $|\chi\rangle = c_0|0\rangle + c_1|1\rangle$

Goal: Use operations in labs SA (Alice) and B (Bob) to bring state to Bob's side.

1. Alice measures in Bell-basis $\{\phi^+, \phi^-, \psi^+, \psi^-\}$ on SA.
2. Alice sends result r to Bob.
3. Bob corrects with U_r on B.

What is U_r for $r \in \{\phi^+, \phi^-, \psi^+, \psi^-\}$?

Superdense coding

$|\phi^+\rangle_{AB}$, classical message $m \in \{00, 01, 10, 11\}$

Goal: Share m with Bob by transferring only a single qubit.

Show that m can be fully recovered by Bob from $|\psi_m\rangle := U_m^{(A)} \otimes \mathbb{1}^{(B)} |\phi^+\rangle_{AB}$ by Bell-basis measurement after Alice transfers her part of the state. $U_m \in \{\mathbb{1}, X, iY, Z\}$.

Morale: Entanglement enables tasks not possible without. It gets used up in the process.

Entanglement is a resource.

5.2) Quantum Resource Theories (QRTs)

Definition

A QRT is a tuple $\mathcal{R} = (\mathcal{O}, \mathcal{F})$ where:

- $\mathcal{O}$: free operations
- $\mathcal{F}$: free states

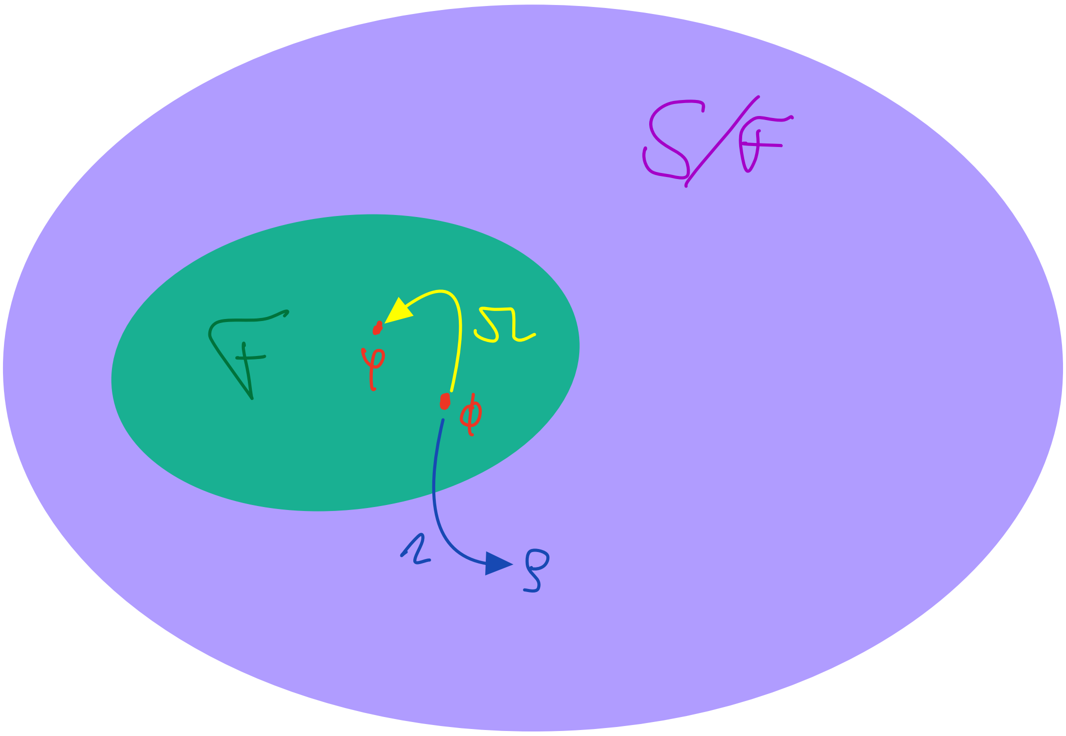
Questions, e.g.:

- Given some ρ, what states σ can it be transformed to under $\mathcal{O}$?
  ↳ assymptotic conversion?
  ↳ Rate $r = \frac{n}{m}$ $\quad \rho^{\otimes m} \xrightarrow{\mathcal{O}} \sigma^{\otimes n}$?
- How much resource is contained in $\rho \notin \mathcal{F}$?
- What tasks does $\rho \notin \mathcal{F}$ enable?

Free operations

Free operations postulate (FOP) "golden rule of QRTs"

$$\forall \rho = \Omega(\phi),\ \Omega \in \mathcal{O},\ \phi \in \mathcal{F}:\ \rho \in \mathcal{F}$$



Def.: (one-way LOCC $[\mathcal{O}_{1\text{-LOCC}}]$)

local ops. and classical communication

Instrument $\mathcal{I} = \{\mathcal{F}_j\}_j \in \mathcal{O}_{1\text{-LOCC}}$

$\Leftrightarrow \exists\, \mathcal{I}^{(k)} = \{\mathcal{E}_j^{(k)}\}_j$ s.t.

$$\mathcal{F}_j = \bigotimes_{m<k} \Lambda_j^{(m)} \otimes \mathcal{E}_j^{(k)} \bigotimes_{m>k} \Lambda_j^{(m)} \quad \text{with } \Lambda_j^{(m)} \text{ CPTP}$$

Def. (LOCC $[\mathcal{O}_{LOCC}]$)

$\Lambda \in \mathcal{O}_{LOCC} \Leftrightarrow \Lambda$ can be implemented by successive rounds of 1-LOCC, where coarse-graining an conditioning on previous measurements is allowed.

Def. (separable states $[\mathcal{F}_S]$)

For $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \ldots \otimes \mathcal{H}^{(N)}$:

$$\sigma = \sum p_i\, \sigma_i^{(1)} \otimes \sigma_i^{(2)} \otimes \ldots \otimes \sigma_i^{(N)} \qquad p_i \geq 0,\ \sum p_i = 1$$

create σ by LOCC:

1) sample $\{p_i\}_i$ locally
2) broadcast
3) prepare $\sigma_i^{(k)}$ at each site k.

Def. (separable operations [$\mathcal{O}_{SEP}$])

For $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \ldots \otimes \mathcal{H}^{(n)}$:

CPTP map $\Lambda \in \mathcal{O}_{SEP} \Leftrightarrow \exists$ kraus representation with $K_i = \bigotimes_k A_i^{(k)}$

E.g. bipartite $\Lambda(\rho) = \sum_i A_i \otimes B_i \, \rho \, A_i^\dagger \otimes B_i^\dagger$

Note: 1) $\mathcal{O}_{LOCC} \subset \mathcal{O}_{SEP}$

2) $\mathcal{F}_S$ closed under $\mathcal{O}_{SEP}$

$$\forall \Lambda \in \mathcal{O}_{SEP}, \ \phi \in \mathcal{F}_S : \Lambda(\phi) \in \mathcal{F}_S$$

Maximal set of free operations $\mathcal{O}_{max}$:

Def. (non-entangling operations [$\mathcal{O}_{ne}$])

$$\mathcal{O}_{ne} := \{\Lambda : \Lambda(\phi) \in \mathcal{F}_S \ \forall \phi \in \mathcal{F}_S\}$$

$\mathcal{O}_{SEP} \subsetneq \mathcal{O}_{ne}$ -> problem set

$\mathcal{O}_{ne} \stackrel{!}{=} \mathcal{O}_{max}$ for general QRTs

Def. (completely resource non-generating ops [$\mathcal{O}_{c\text{-}max}$])

$$\mathcal{O}_{c\text{-}max} = \{\Lambda : A \to A' : \forall \phi \in A \otimes B, \mathcal{F} \ \ \Lambda \otimes id_B(\phi) \in \mathcal{F}\}$$

Resource monotones & measures

idea: Quantify resource by non-negative fct. $f: \mathcal{S} \to \mathbb{R}_0^+$

Monotonicity (M)

$$f(\Omega(\rho)) \leq f(\rho) \quad \forall \Omega \in \mathcal{O}, \rho \in \mathcal{S}$$

Discriminance (D)

$$f(\varphi) = 0 \iff \varphi \in \mathcal{F} \quad \to f \text{ is faithful}$$

weak discriminance (WD)

$$f(\varphi) = 0 \quad \forall \varphi \in \mathcal{F}$$

Def. (resource monotone)

A fct. $f: \mathcal{S} \to \mathbb{R}_0^+$ is a resource montone for QRT $\mathcal{R} = (\mathcal{O}, \mathcal{F})$ iff it obeys monotonicity w.r.t. $\mathcal{O}$.

Def. resource measure

monotone + WD

Desiderata & Properties

- computability
- convexity

$$f(p\rho + (1-p)\sigma) \leq p f(\rho) + (1-p) f(\sigma)$$

$$\forall \rho, \sigma \in \mathcal{S}, \quad 0 \leq p \leq 1$$

- additivity

  $f(\varrho \otimes \sigma) = f(\varrho) + f(\sigma) \quad \forall \varrho, \sigma \in S$

  ↳ weaker variants:

  sub(super)additivity

  $f(\varrho \otimes \sigma) \leq (\geq) f(\varrho) + f(\sigma)$

  extensivity:

  $f(\varrho^{\otimes n}) = n f(\varrho)$

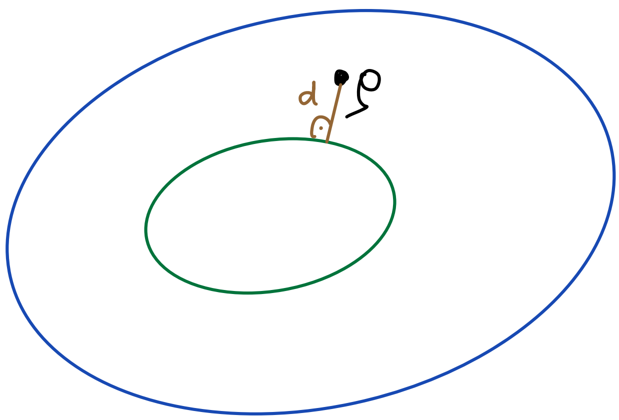
Examples for QRT of entanglement

- Entropy of entanglement (pure states only!)

  $S(\mathrm{tr}_B \psi_{AB}) = S(\mathrm{tr}_A \psi_{AB}) \qquad S := -\mathrm{tr}[\varrho \log \varrho]$

Schmidt decomposition $|\psi_{AB}\rangle = \sum_{j=1}^{r} \sqrt{\lambda_j}\, |\alpha_j\rangle |\beta_j\rangle$

- Schmidt rank r



$E_d := \inf_{\varphi \in F} d(\varrho, \varphi)$

Examples of further ORTs

	$\mathcal{O}$	$\mathcal{F}$	monotone f
coherence w.r.t. $\{\lvert j\rangle\}$	e.g. strictly incoherent ops.	incoherent states $\phi = \sum p_i \lvert i\rangle\langle i\rvert$	relative entropy of coherence
Athermality	Gibbs-preserving ops. (GPOs) $\Gamma(\tau_\beta) = \tau_\beta$ thermal ops. (TOs) $\rho' = \mathrm{tr}_B[U \rho \otimes \tau_\beta U^\dagger]$ with $[U, H_S + H_B] = 0$	$\tau_\beta \propto e^{-\beta H}$	Renyi-α free energies
Entanglement	LOCC separable ops.	separable states e.g. $\phi = \sum p_j \alpha_j \otimes \beta_j$	EoE Rel. Entropy of E. ...

For almost all of these: multiple valid $\mathcal{O}$

Recommended references for further study – week 27

Entanglement Theory

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Quantum Resource Theories

Eric Chitambar and Gilad Gour. "Quantum resource theories". Reviews of Modern Physics **91**, 25001 (2019). arXiv:1806.06107

Gilad Gour. "Resources of the Quantum World" (2024). arXiv:2402.05474

QRTs of Coherence

Alexander Streltsov, Gerardo Adesso, and Martin B. Plenio. "Colloquium: Quantum coherence as a resource". Reviews of Modern Physics **89**, 041003 (2017). arXiv:1609.02439

QRTs of Athermality

Matteo Lostaglio. "An introductory review of the resource theory approach to thermodynamics". Reports on Progress in Physics **82**, 1–31 (2019). arXiv:1807.11549