E&M Lecture 11

Topics:
(1) Introduction to Magnetostatics
(2) Biot-Savart Law
(3) Field of circular current loop
(4) Magnetic dipole moment
(5) B-S Law in terms of current density
(6) Ampere’s Law & Examples
(7) Relative permeability
Introduction to Magnetostatics

As far as possible, by analogy with Electrostatics:

\[ \text{Lorentz Force} : \quad \mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

The first term defines Electric Field \( \mathbf{E} \), the second defines a cross product of velocity \( \mathbf{v} \) and Magnetic Field \( \mathbf{B} \).

More properly, \( \mathbf{B} \) is “magnetic flux density” or “magnetic induction” (see Faraday’s Law, later);
Units: weber per square metre (Wb⋅m\(^{-2}\)) or tesla (T)

Origin of \( \mathbf{B} \) : current (“relativistic transformation of \( \mathbf{E} \”)"

Plan: Magnetostatics in vacuum, then magnetic media based on “magnetic dipole moment”
Magnetostatics in vacuum

The analogue of Coulomb’s Law is the Biot-Savart Law:

Consider a current (I) loop:

For element $dl$ there is an associated element field $dB$:

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl \times \hat{r}}{r^2}$$

$dB$ perpendicular to both $dl$ and $r$; same $4\pi r^2$ dependence, but $\mu_0$ (“above the line”) is “permeability of free space” defined as $4\pi \times 10^{-7}$ Wb·A$^{-1}$·m$^{-1}$

Integrate to get B-S Law:

$$B = \frac{\mu_0 I}{4\pi} \oint dl \times \hat{r}$$
B-S Law examples

(2) Infinitely long straight conductor

both $dl$ and $r$ in the page:
$dB$ is out of the page

$B$ is “azimuthal”, forming circles centred on the conductor;
deﬁne $\rho$, radius of such circles

Apply B-S Law to get:
$B = \frac{\mu_o I}{2\pi \rho}$

(see any E&M text for detail)

(example to illustrate Ampere’s Law later)
B-S Law examples

(2) “on-axis” field of circular loop

Loop perpendicular to page, radius $a$

dl out of page and r in the page:
On-axis element $dB$ is in the page, perpendicular to r, at $\theta$ to axis.

Magnitude of element $dB$:

$$dB = \frac{\mu_o I \ dl}{4 \pi \ r^2} \Rightarrow dB_z = \frac{\mu_o I \ dl}{4 \pi \ r^2} \cos \theta$$

Integrating around loop, only z-components of $dB$ survive:
The on-axis field is “axial”, $B_{on-axis} = \int dB_z$
On-axis field of circular loop

\[ B_{\text{on-axis}} = \oint dB_z = \frac{\mu_o I}{4\pi r^2} \cos \theta \oint dl \]

\[ = \frac{\mu_o I}{4\pi r^2} \cos \theta (2\pi a) = \frac{\mu_o I a^2}{2 r^3} \]

Introduce axial distance \( z \), where \( r^2 = a^2 + z^2 \):

2 limiting cases:

\[ B_{\text{on-axis}}^{z=0} = \frac{\mu_o I}{2a} \quad \text{and} \quad B_{\text{on-axis}}^{z\gg a} \approx \frac{\mu_o I a^2}{2z^3} \]

recall \( r^3 \) dependence of field of electric dipole?
Magnetic dipole moment

The off-axis field of circular loop is much more complex. For $z >> a$ it is identical to that of the electric dipole:

$$E = \frac{p}{4\pi \varepsilon_0 r^3} \left[ 2\cos \theta \hat{r} + \sin \theta \hat{\theta} \right]$$

$$\Rightarrow B = \frac{\mu_0 m}{4\pi r^3} \left[ 2\cos \theta \hat{r} + \sin \theta \hat{\theta} \right]$$

where $m = \pi a^2 I$ or $m = \pi a^2 I \hat{z}$

“current times area” vs “charge times distance”

$m$ is basic building block of magnetic media

(note change $(r, \theta)$ coordinates; circularity defines direction)
B-S Law in terms of $J$, Ampere’s Law

$$dB = \frac{\mu_0 I \, dl \times \hat{r}}{4\pi \, r^2} = \frac{\mu_0 \, Idl \times \hat{r}}{4\pi \, r^2} = \frac{\mu_0 \, (J \cdot da) \, dl \times \hat{r}}{4\pi \, r^2}$$

$$= \frac{\mu_0 \, (J \, dv) \times \hat{r}}{4\pi \, r^2} = \frac{\mu_0 \, J \times \hat{r}}{4\pi \, r^2} \, dv$$

$$\Rightarrow B = \frac{\mu_0}{4\pi} \int_{v} \frac{J \times \hat{r}}{r^2} \, dv$$

or

$$\nabla \times \underline{B} = \mu_0 J$$

(the latter differential form requires complex derivation!)

Integrate over surface; apply Stokes’ Theorem to obtain Ampere’s Law

$$\oint_{s} (\nabla \times \underline{B}) \cdot da = \mu_0 \oint_{s} J \cdot da$$

$$\oint_{s} \underline{B} \cdot dl = \mu_0 I_{encl}$$
Ampere’s Law examples

(1) Infinitely long straight conductor:

$B$ is azimuthal, constant on circle of radius $r$

$$\oint B \cdot dl = \mu_0 I_{encl} \implies B 2\pi r = \mu_0 I \implies B = \frac{\mu_0 I}{2\pi r}$$

Exercise: find radial profile of $B$ inside and outside conductor of radius $R$

$$B_{r<R} = \frac{\mu_0 I r}{2\pi R^2}$$

$$B_{r>R} = \frac{\mu_0 I}{2\pi r}$$
**Solenoid**

Distributed-coiled conductor:

Key parameter: \( n \) loops/metre

If finite length, sum individual loops via B-S Law

If infinite length, apply Ampere’s Law:

- \( B \) constant and axial inside, zero outside;
- Rectangular path, axial length \( L \)

\[
\oint B \cdot dl = \mu_0 I_{encl} \quad \Rightarrow \quad B_{vac} L = \mu_0 (nL)I \quad \Rightarrow \quad B_{vac} = \mu_0 nI
\]

(use label \( B_{vac} \) to distinguish from core-filled solenoids)

“solenoid is to magnetics what capacitor is to electrics”
Relative permeability

Recall how field in vacuum capacitor is reduced when dielectric medium is inserted; always reduction, whether medium is polar or non-polar:

\[ E = \frac{E_{\text{vac}}}{\varepsilon_r} \quad \Rightarrow \quad B = \mu_r B_{\text{vac}} \]

is the analogous expression when magnetic medium is inserted in the vacuum solenoid.

Complication: the \( B \) field can be reduced or increased, depending on the type of magnetic medium:

\( \mu_r < 1 \), medium analogous to non-polar ("induced")
\( \mu_r > 1 \), medium analogous to polar ("orientation")