E&M Lecture 8

Topics:
(1) Validity of expressions
(2) Electrostatic energy (in a capacitor?)
(3) Collection of point charges
(4) Continuous charge distribution
(5) Energy in terms of electric field
(6) Energy stored in capacitor
(7) Energy density
Validity of expressions

Always valid: Gauss’ Law for $\vec{E}$, $\vec{P}$ and $\vec{D}$
relation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

Limited validity: expressions involving $\varepsilon_r$ and $\chi_e$

Have assumed that $\chi_e$ is a simple number: $\vec{P} = \chi_e \varepsilon_0 \vec{E}$
Which is only true in LIH media:

Linear: $\chi_e$ independent of magnitude of $\vec{E}$;
interesting media “non-linear”: $\vec{P} = \chi_1 \varepsilon_0 \vec{E} + \chi_2 \varepsilon_0 \vec{E}^2 + \ldots$

Isotropic: $\chi_e$ independent of direction of $\vec{E}$;
interesting media “anisotropic”: $\chi_e$ is a tensor (gen. vector)

Homogeneous: uniform medium (recall varying $\varepsilon_r$ and $\rho_b$)
Electrostatic Energy

recall energy stored in a capacitor: $U = QV/2$

Raises 2 questions:

(1) why factor of $1/2$?
(2) where exactly is the energy stored?

Re (1) “energy is charge times potential difference”

But in this case, the charge stored is creating the potential difference: the factor of $1/2$ is to avoid double-counting!

Or graph the charging process…..

can further this discussion by “collecting” point charges…..

Also, begin discussion, as always, in a vacuum…..
Collection of Point Charges in vacuum

“potential is energy per unit +ve charge”

Consider isolated point charge $q_1$

Bringing a second point charge $q_2$
from infinity to reside at $r_{12}$ from $q_1$

Potential energy is

$$q_2 \left( \frac{q_1}{4\pi\varepsilon_0 r_{12}} \right)$$

Now bring a third point charge $q_3$
from infinity to reside at $r_{13}$ from $q_1$ and at $r_{23}$ from $q_2$

Additional potential energy is

$$q_3 \left( \frac{q_1}{4\pi\varepsilon_0 r_{13}} + \frac{q_2}{4\pi\varepsilon_0 r_{23}} \right)$$

“arriving charge times existing potential”
Energy of Collection of Point Charges

Extending the process, the total potential energy is

\[ U = q_2 \left( \frac{q_1}{4\pi\varepsilon_o r_{12}} \right) + q_3 \left( \frac{q_1}{4\pi\varepsilon_o r_{13}} + \frac{q_2}{4\pi\varepsilon_o r_{23}} \right) + q_4 \left( \ldots \right) \ldots \]

Each bracket contains all point charges which pre-existed the one outside!

Re-write as:

\[ U = \frac{1}{4\pi\varepsilon_o} \sum_i q_i \sum_{j<i} \frac{q_j}{r_{ji}} \]

Note how the condition \( j < i \) prevents double counting!

Exploit this by deliberately double-counting and dividing by two - or introducing a factor of \( 1/2 \)!
Point charges vs continuous distribution

\[ U = \frac{1}{2} \frac{1}{4\pi \varepsilon_0} \sum_i q_i \sum_{j \neq i} \frac{q_j}{r_{ji}} = \frac{1}{2} \sum_i q_i \phi_i \]

Where \( \phi_i \) is the potential due to all point charges except \( q_i \)

Note
(a) for single point charge, \( U = 0 \)
(b) \( U \) can be +ve or -ve
(key assumption: “already-assembled” point charges)

Extend to continuous distribution:
\[ U = \frac{1}{2} \int_{V} \phi \rho dv \]
replace \( q \) by \( dq = \rho dv \)

Where \( V \) is any volume enclosing all the charge
This expression includes “energy of assembly” and is always +ve! Show this and obtain useful expression…..
Energy in vacuum in terms of \( E \)

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_o}
\]

and

\[
E = -\nabla \phi
\]

\[
\Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_o} \quad \Rightarrow \rho = -\varepsilon_o \nabla^2 \phi
\]

\[
\therefore U = \frac{1}{2} \int \phi \rho dv = -\frac{\varepsilon_o}{2} \int \phi \nabla^2 \phi dv
\]

Where \( \phi \) is sole parameter; expand integrand using identity:

\[
\nabla \psi F = \psi \nabla F + F \cdot \nabla \psi
\]

Exercise: write \( \psi = \phi \) and \( E = \nabla \phi \) to show:

\[
\nabla \phi \nabla \phi = \phi \nabla^2 \phi + (\nabla \phi)^2
\]

\[
\Rightarrow \phi \nabla^2 \phi = \nabla \phi \nabla \phi - (\nabla \phi)^2
\]

substitute.....
Energy in vacuum in terms of $E$

$$U = -\frac{\varepsilon_o}{2} \left[ \int_v \nabla \cdot \phi \nabla \phi \, dv - \int_v (\nabla \phi)^2 \, dv \right]$$

$$= -\frac{\varepsilon_o}{2} \left[ \oint_s (\phi \nabla \phi) \cdot \, da - \int_v (\nabla \phi)^2 \, dv \right] \quad (Green's \ Theorem)$$

Recall “where $v$ is any volume enclosing all the charge”
Free choice of $s$, the enclosing surface - so let it expand!

Think of sphere, radius $r$ : $da$ goes as $r^2$
but $\phi$ goes as $r^{-1}$ (at least) and $\nabla \phi$ goes as $r^{-2}$ (at least)
Hence surface integral goes as $r^{-1}$ (at least) For large $r$, this integral goes to zero, the volume integral remains unchanged!
Energy in vacuum in terms of $E$

$$U = -\frac{\varepsilon_o}{2} \left[ \oint (\phi \nabla \phi) \cdot da - \int (\nabla \phi)^2 \, dv \right]$$

$$= +\frac{\varepsilon_o}{2} \int (\nabla \phi)^2 \, dv = \frac{\varepsilon_o}{2} \int E^2 \, dv$$

where $v$ encloses all regions where $E$ is non-zero.

Confirms mathematically that $U$ is always +ve: $E$-squared
Only applies to vacuum!!!
Use this expression to calculate electrostatic energy……

……example of (vacuum) capacitor
Energy stored in (vacuum) capacitor

\[ U = \frac{\varepsilon_o}{2} \int E^2 \, dv = \frac{\varepsilon_o}{2} E^2 \int dv \]

\[ = \frac{\varepsilon_o}{2} \left( \frac{V}{d} \right)^2 \int dv = \frac{\varepsilon_o}{2} \left( \frac{V}{d} \right)^2 A \times d \]

\[ = \frac{1}{2} \frac{\varepsilon_o A}{d} V^2 = \frac{1}{2} CV^2 = \frac{1}{2} QV \]

As \( E \) is constant and equal to \( V/d \), and the volume is \( A \times d \)

this example is trivial, but illustrates the answer to the second question raised in the introduction: where in the capacitor does the energy reside? - it resides in the field!
Energy density in a vacuum

\[ U = \frac{\varepsilon_0}{2} \int E^2 dv \]

\[ \Rightarrow \frac{dU}{dv} = \frac{\varepsilon_0}{2} E^2 \]

This is the energy density and as field \( E \) is a point concept, the energy density is also a point concept.

In next lecture, we determine this expression for the energy density via a less mathematical and more physical argument!