E&M Lecture 6

Topics:
(1) Local Field in a dielectric medium
(2) Clausius-Mossotti equation
(3) non-uniform polarisation
(4) Electric displacement
(5) Origins of volume bound charge density
Local Field

the field creating dipole (called the “local field” or $E_{loc}$) is not the same as the actual field $E$ because the latter contains the effect of all the dipoles created! Recall the Capacitor where $E = E_{vac} - E_{pol} = E_{vac} / \varepsilon_r$

The key word above is all! In fact $E_{loc}$ is the actual field $E$ less the effect of one dipole (essentially the one “being created”) - think about one solitary individual dipole being created within a capacitor, the creating field ($E_{vac}$) is larger than the actual field ($E = E_{vac} - E_{ indiv}$) by the contribution of this one individual dipole.

Vectorially, obtain $E_{loc}$ by subtracting $E_{ indiv}$ from $E$:

$$E_{loc} = E - E_{ indiv}$$

Estimate for $E_{ indiv}$?-some sort of average?
Local Field continued

Estimate for $E_{\text{indiv}}$ comes from L4: recall the average field of a point charge inside a spherical volume. Or, more generally, the average field of a spherical volume containing a specific dipole moment $p$:

$$E_{\text{indiv}} = \langle E \rangle = -\frac{p}{4\pi \epsilon_0 R^3} = -\frac{p}{4\sqrt[3]{\frac{3\pi R^3}}(3\epsilon_0)} \approx -\frac{P}{3\epsilon_0}$$

$$E_{\text{loc}} = E - E_{\text{indiv}} \approx E - \left(-\frac{P}{3\epsilon_0}\right) \approx E + \frac{P}{3\epsilon_0}$$

$$E_{\text{loc}} \approx E + \frac{\chi e \epsilon_0 E}{3\epsilon_0} \approx \left(1 + \frac{\chi_e}{3}\right)E$$

Approximation arises from attempt to fill space with spheres, more correct to replace 3 by $b \approx 3$
Clausius-Mossotti equation

A practical equation relating $\varepsilon_r$ to mass density $\rho_m$:

$$\chi_e = \frac{E_{loc}}{\varepsilon_r} = \left(1 + \frac{\chi_e}{3}\right) \frac{E}{E} = 1 + \frac{\chi_e}{3}$$

$$\Rightarrow N\alpha = \frac{\chi_e}{\left(1 + \frac{\chi_e}{3}\right)} = \frac{3\chi_e}{3 + \chi_e} \quad (\approx \chi_e \text{ when } \chi_e \text{ small})$$

(or when $E_{loc} \approx E$: eg gases)

Substituting $\chi_e = \varepsilon_r - 1$

$$\frac{N\alpha}{3} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \quad \text{but } N = \frac{N_A \rho_m}{M}$$

$$\Rightarrow \frac{N_A \alpha}{3} = \frac{M}{\rho_m} \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

where $N_A$ is Avogadro’s number, $M$ is Molecular Weight
Non-uniform polarisation

Uniform polarisation results in \textbf{surface} bound charge density only; non-uniform polarisation results in \textbf{surface and volume} bound charge densities. Focus on latter ($\rho_b$), origin later.

Consider box, cornered on $(x,y,z)$ volume $\Delta x$ by $\Delta y$ by $\Delta z$, small but still containing large number of molecules and exhibiting non-uniform polarisation:

Focus on $x$-component of $P$, and assume it is independent of $y$ or $z$

<table>
<thead>
<tr>
<th><strong>value at $x$ is</strong></th>
<th>$P_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>value at $x + \Delta x$ is</strong></td>
<td>$P_x + \frac{\partial P_x}{\partial x} \Delta x$</td>
</tr>
</tbody>
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From L5, charge crossing area = $P \cdot da$
Non-uniform polarisation continued

Charge entering LH $yz$ face is \( P_x \Delta y \Delta z \)

Charge exiting RH $yz$ face is

\[
\left( P_x + \frac{\partial P_x}{\partial x} \Delta x \right) \Delta y \Delta z
\]

Net charge entering box is

\[
- \frac{\partial P_x}{\partial x} \Delta x \Delta y \Delta z
\]

Total charge on box (including $zx$ and $xy$ pairs of faces) is

\[
- \left( \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) \Delta x \Delta y \Delta z = - (\nabla \cdot \mathbf{P})(\text{volume})
\]

\[\Rightarrow \rho_b = -\nabla \cdot \mathbf{P} \]

looks a bit like $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$?
Electric displacement \((D)\)

….a convenience vector?

2 distinct types of charge, “free” and “bound”,
are combined as “total”

\[ \rho_t = \rho_f + \rho_b \]

Gauss’s Law (differential form) gives

\[ \nabla \cdot E = \frac{\rho_t}{\varepsilon_o} = \frac{1}{\varepsilon_o} (\rho_f + \rho_b) = \frac{1}{\varepsilon_o} (\rho_f - \nabla \cdot P) \]

\[ \Rightarrow \rho_f = \varepsilon_o \nabla \cdot E + \nabla \cdot P = \nabla \cdot (\varepsilon_o E + P) \]

write \( D = \varepsilon_o E + P \)

Displacement: a vector whose div equals free charge density
Units: C·m\(^{-2}\) (same as \(P\)). Express \(D\) in terms of \(E\)?

\[ D = \varepsilon_o E + P = \varepsilon_o E + \chi_e \varepsilon_o E = \varepsilon_o \left(1 + \chi_e\right) E = \varepsilon_o \varepsilon_r E \]
Gauss’s Law for $E$, $P$ and $D$

$\nabla \cdot E = \frac{\rho_t}{\varepsilon_o} \quad \Rightarrow \quad \oint E \cdot da = \frac{1}{\varepsilon_o} (\text{encl. total ch.})$

$\nabla \cdot P = -\rho_b \quad \Rightarrow \quad \oint P \cdot da = -(\text{encl. bound ch.})$

$\nabla \cdot D = \rho_f \quad \Rightarrow \quad \oint D \cdot da = (\text{encl. free ch.})$

use $D$ as a route to $E$? Field in capacitor?

$E$ – Gauss: $\quad E \cdot A = \frac{1}{\varepsilon_o} (\sigma_f A) \quad \Rightarrow \quad E = \frac{\sigma_f}{\varepsilon_o}$

$D$ – Gauss: $\quad D \cdot A = (\sigma_f A) \quad \Rightarrow \quad D = \sigma_f$

$E = \frac{D}{\varepsilon_o \varepsilon_r} = \frac{\sigma_f}{\varepsilon_o} \quad (\text{as } \varepsilon_r = 1 \text{ in vacuum})$
Gauss’s Law for $E$, $P$ and $D$ continued

Trivial example but field in non-vacuum capacitor?

$E - Gauss: \quad E \cdot A = \frac{1}{\varepsilon_o} \left( \sigma_f - \sigma_b \right) A$

$\Rightarrow E = ?$

as $\sigma_b$ is not easily known

$D - Gauss: \quad D \cdot A = \left( \sigma_f A \right) \quad \Rightarrow D = \sigma_f$

$E = \frac{D}{\varepsilon_o \varepsilon_r} = \frac{\sigma_f}{\varepsilon_o \varepsilon_r}$

Still a relatively trivial example perhaps but one that “shows” how bound charge gets “hidden” in the relative permittivity!
Origins of volume bound charge density

Recall uniform dielectric has no *volume* bound charge density (although it has dipoles), only *surface* bound charge density

\[ D = \varepsilon_o E + P \quad \Rightarrow \quad P = D - \varepsilon_o E \]

\[ \rho_b = -\nabla \cdot P = -\nabla \cdot (D - \varepsilon_o E) = -\nabla \cdot \left( D - \frac{D}{\varepsilon_r} \right) \]

If no longer treat \( \varepsilon_r \) as a spatial constant:

\[ \rho_b = -\left( \nabla \cdot D - D \cdot \nabla \left( \frac{1}{\varepsilon_r} \right) - \frac{1}{\varepsilon_r} \nabla \cdot D \right) \]

\[ = -\nabla \cdot D \left( 1 - \frac{1}{\varepsilon_r} \right) + D \cdot \nabla \left( \frac{1}{\varepsilon_r} \right) = - \left( 1 - \frac{1}{\varepsilon_r} \right) \rho_f + D \cdot \nabla \left( \frac{1}{\varepsilon_r} \right) \]

2 origins:
(1) embedded \( \rho_f \) induces \( \rho_b \) of opposite sign
(2) non-uniform medium, i.e. varying \( \varepsilon_r \): either via dipole number density or magnitude of dipole moment