E&M Lecture 2

Exercise solution: \[ \nabla \cdot E = \frac{\rho_t}{\varepsilon_o} \quad (Maxwell) \]

\[ \int (\nabla \cdot E) \, dv = \int \frac{\rho_t}{\varepsilon_o} \, dv = \frac{1}{\varepsilon_o} \int \rho_t \, dv \]

Use Green’s Theorem
\[ \oint F \cdot da = \int (\nabla \cdot E) \, dv \]

\[ \oint E \cdot da = \frac{Q_{encl}}{\varepsilon_o} \quad (Gauss) \]

Topics:
1. CURL and Stokes’ Theorem
2. vector identities, analogue of Green’s Theorem
3. coordinate system of “del” - a subtlety
4. Equation of Continuity
**CURL (\(\nabla \times \mathbf{F}\))**

Defined as “line integral per unit area”

Put Ampere’s Law for \(\mathbf{B}\) into words:
“closed line integral of \(\mathbf{B}\) is \(\mu_o\) times the enclosed current”

closed line integral of a general vector …..?

….but has 3D complexity, so first consider xy plane only:

\[
\mathbf{F} \cdot dl = F_x \, dx + F_y \, dy
\]

\[
\oint \mathbf{F} \cdot dl = \oint F_x \, dx + \oint F_y \, dy
\]

Evaluate each part, then each pair in other two planes…..
Choose rectangular elemental path, $dx \times dy$, centred on $(x,y)$

$x$-component of $F$ only has value along horizontal sides!

Value on upper horizontal side: $$F_x + \frac{\partial F_x}{\partial y} \frac{dy}{2}$$

And value on lower horizontal side: $$F_x - \frac{\partial F_x}{\partial y} \frac{dy}{2}$$

“line integral” is value times distance, minding direction, .....choose anticlockwise for convenience...
CURL continued

\[ \oint F_x \, dx = \left( F_x - \frac{\partial F_x}{\partial y} \frac{dy}{2} \right) dx - \left( F_x + \frac{\partial F_x}{\partial y} \frac{dy}{2} \right) dx \]

\[ \oint F_y \, dy = -\frac{\partial F_x}{\partial y} \, dy \, dx \]

Note zero contributions on vertical sides and negative sign for upper side contribution. Equivalently, for \( y \) component:

\[ \oint F_y \, dy = + \frac{\partial F_y}{\partial x} \, dx \, dy \]

Hence “line integral of \( F \)” in \( xy \) plane is:

\[ \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \, dx \, dy = g_3 \, dx \, dy \]
CURL continued

Equivalently, in $yz$ and $zx$ planes, obtain:

$$g_1 = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \quad \text{and} \quad g_2 = \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right)$$

Where the total line integral is:

$$\oint F \cdot dl = g_1 dydz + g_2 dzdx + g_3 dxdy$$

But area of element is:

$$da = dydz \hat{x} + dzdx \hat{y} + dxdy \hat{z}$$

And $g$’s are components of $\nabla \times F$, hence

$$\oint F \cdot dl = (\nabla \times F) \cdot da$$

Definition of “line integral per unit area” (elemental path!)
CURL - Stoke’s Theorem

Integrate over finite path, obtain: \( \oint F \cdot dl = \iint_{S} (\nabla \times F) \cdot d\mathbf{a} \)

S is any surface (not enclosed!) bounded by path:
Theorem allowing swap between line and surface integrals!

Exercise:
Use theorem to show Maxwell’s third equation is same as Faraday’s Law of Electromagnetic Induction

But Maxwell’s fourth equation is only same as Ampere’s Law when electric field is static!

Extra term in Maxwell 4 is “magnetoelectric induction”
2 further operations with “del”

“div-grad”: \[ \nabla \cdot \nabla \psi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 \psi \]
2nd derivative of scalar (”del-squared” or “Laplacian”)

Also, Laplacian of a vector:

\[ \nabla^2 F = \nabla^2 F_x \hat{x} + \nabla^2 F_y \hat{y} + \nabla^2 F_z \hat{z} \]

….components are Laplacians of scalars (cartesian co-ords!)
Identities, products, analogues

2 useful vector identities: \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \)

(basically, cross product of identical vector is always zero)

\( \nabla \times \nabla \psi = 0 \)

Products:
\( \nabla \cdot (\psi \mathbf{F}) = \psi (\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla \psi \)

\( \nabla \times (\psi \mathbf{F}) = \psi (\nabla \times \mathbf{F}) + \nabla \psi \times \mathbf{F} \)

\( \nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \)

Analogue of Green’s theo: \[ \oint \mathbf{F} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{F}) d\mathbf{v} \]

\[ \oint \mathbf{F} \times d\mathbf{a} = -\int (\nabla \times \mathbf{F}) d\mathbf{v} \]
Source versus field

Normal coordinates of “del” are (x,y,z) and (x′,y′,z′).

Consider a vector \( \mathbf{r} \), linking points (x,y,z) and (x′,y′,z′).

Take grad of scalar \( 1/r \)

\[
\nabla \left( \frac{1}{r} \right) = -\frac{\hat{r}}{r^2}
\]

Meaning: gradient of \( 1/r \) evaluated at (x,y,z).

Can also define “del-prime” with coordinates (x′,y′,z′).

Where

\[
\nabla' \left( \frac{1}{r} \right) = +\frac{\hat{r}}{r^2}
\]

Distinction between source pt (x′,y′,z′) and field pt (x,y,z).
Source versus field: an implication

An example of source pt \((x',y',z')\) and field pt \((x,y,z)\) is current density \(\mathbf{J}\) (source) and magnetic field \(\mathbf{B}\) (field).

An important implication of this distinction is that since “del” has coordinates of \((x,y,z)\) and \(\mathbf{J}\) has coordinates \((x',y',z')\), then

\[
\nabla \times \mathbf{J} = 0
\]

When del is specified in “field” coordinates the curl of a source is always zero!

(….needed in a particular derivation)
Equation of continuity

Also known as the conservation of charge, linking current density (charge flow) with change in charge density!

Enclosed surface, charge density $\rho$

Net outward flow of charge depletes charge inside: equate rates

Apply Green’s theorem

Used in L14 to establish extra term in Maxwell’s 4th equation