Limb Darkening

Stars are both redder and dimmer at the edges...

Limb Darkening

- Can also be understood in terms of temperature within the solar photosphere. Deeper $\rightarrow$ hotter.

- Since we ‘see’ ~ 1 optical depth into atmosphere along the line-of-sight, but this means different depths in the atmosphere when observing a spatially-resolved disk
  
  - At centre see hotter gas than at edges
  
  - Similar effect to line formation earlier
  
  - Centre appears hotter, brighter
  
  - Limb darkening!

Empirical Limb Darkening

Variation of intensity across solar disk

- Note the stronger limb darkening with increasing frequency
Semi-Empirical Equation

Plane-Parallel Models

Geometry of stellar atmosphere

Plane-Parallel Optical Depth

• The stellar atmosphere is thin relative to the stellar radius, and the bulk of the variation of the atmosphere is in the radial direction, so we can simplify:

→ Note that we are measuring the optical depth backwards along the z-direction, i.e. from the stellar surface down into the interior.

Plane-Parallel RTE

• Note that we can apply this idea to cases where behaviour varies with height by considering a series of plane-parallel slabs, each with well-behaved (uniform) characteristics. We can therefore solve simple problems in hydrostatic stellar atmospheres.

Plane-Parallel RTE

• Under the plane-parallel assumption we get the radiative transfer equation:

\[
\cos \theta \frac{dI_\nu}{d\tau} = I_\nu - S_\nu = \mu \frac{dI_\nu}{d\tau}
\]

• Where we must remember that the intensity is a function of both optical depth and angle.
Solution to RTE

- We can solve this equation in a similar manner to before:

\[
\mu \frac{dI_\nu(\theta)}{d\tau} \exp[-\tau / \mu] = I_\nu(\theta) \exp[-\tau / \mu] - S_\nu \exp[-\tau / \mu]
\]

- Which can be written:

\[
\frac{d(I_\nu(\theta) \exp[-\tau / \mu])}{d(\tau / \mu)} = -S_\nu \exp[-\tau / \mu]
\]

\( \rightarrow \) remember that both \( I \) and \( S \) have a dependence on optical depth.

Solution to RTE

- We can solve this equation by integrating over the full range of optical depth:

\[
[I_\nu(\theta) \exp[-\tau / \mu]]_a^b = -\int_a^b S_\nu \exp[-\tau / \mu] d(\tau / \mu)
\]

- We have found that no photons come from infinite optical depths, so we get:

\[
I_\nu(0, \theta) = \int_0^\infty S_\nu(\tau) \exp[-\tau / \mu] d(\tau / \mu)
\]

\( \rightarrow \) which represents the transfer of radiation through an optically thick medium.

Source function and optical depth

- Recall our previous observation of the trend in the value of the source function with (radial) optical depth:

- Around this region we can model the source function with a linear approximation.

Linear source function solution

- Assume a simple form for the source function of the form:

\[
S_\nu(\tau_\nu) = a_\nu + b_\nu \tau_\nu
\]

- Substituting gives a solution of the form:

\[
I_\nu(0, \theta) = a_\nu + b_\nu \mu = S_\nu(\tau_\nu = \mu)
\]

- So the intensity at any angle relative to the surface element is given by the source function for the (radial) optical depth equal to the cosine of the angle.

- This is the Eddington-Barbier relationship.

Determination of the function

- By observing over a range of angles we can determine the two parameters \( a \) and \( b \) as they have different dependencies on angle (longitude).

Eddington-Barbier Approximation
**The Linear Source Function**

- As before, assume a simple form for the source function of the form:
  \[ S_\nu(r_\nu) = a_\nu + b_\nu r_\nu \]
- We can obtain the outwardly-directed flux from the stellar surface by integration:
  \[ \pi F_\nu(0) = 2\pi \int_0^1 I_\nu(0, \theta) \mu \, d\mu \]
- Substituting in the linear form, we find:
  \[ \pi F_\nu(0) = 2\pi \int_0^1 (a_\nu + b_\nu \mu) \mu \, d\mu \]

**The Eddington-Barbier Relationship**

- Which has the solution:
  \[ \pi F_\nu(0) = \pi (a_\nu + 2/3 b_\nu) \]
- By comparing this with the original source function approximation, we see that:
  \[ F_\nu(0) = S_\nu(r_\nu = 2/3) \]
- In other words, the flux weighted across the surface of such a star is given by the source function contribution from an optical depth of 2/3.

**Gray Atmospheres**

- A special case of radiative transfer occurs when the opacity is grey, i.e. uniform with wavelength.
- For the plane-parallel case, we then have:
  \[
  \cos \theta \frac{d}{dc} \int_0^\infty I_\nu \, d\nu = \rho \int_0^\infty \kappa_\nu I_\nu \, d\nu - \rho \int_0^\infty \kappa_\nu S_\nu \, d\nu
  \]

**The Effective Temperature**

- If we can apply LTE, then the emission is that of a blackbody from the same optical depth of 2/3.
- For the case of a gray atmosphere (i.e., one for which the opacity is similar at all frequencies), we can also determine that the emission is controlled by the temperature at that point, i.e. the effective temperature is given by the temperature of the plasma at this point.

**Opacity**

- Since the temperature must have a unique value at each physical region, we can use this to identify similar regions in the atmosphere.
- For each of these regions, we can then derive the optical depth, and so map out the shape of the opacity curve in frequency/wavelength.
Limb darkening measurements of intensity vs. position (top) can be converted to source function estimates vs. optical depth for that frequency (bottom).

Under the assumption of LTE, we can use the Planck function to obtain brightness temperatures from the $S_\nu$ values, and plot these vs. $\tau_\nu$. Values of the same temperature must refer to the same physical region in the atmosphere.

For any temperature, we can replot the results in terms of relative opacity vs. frequency to determine the behaviour of the opacity source.

For the Sun, Chalonge & Kourganoff (1946) found that the result matched the $H$-opacity suggested by Wildt in 1938:

**Convection**
**Radiative Equilibrium**

- If energy is carried by radiation alone (which is nearly true in the case of the Sun), and there are no additional sources or sinks of radiation in the atmosphere, then the flux leaving the surface is simply that produced in the core, i.e. \( \frac{d}{dz} F(z) = 0 \) or: \( F(z) = F_0 \)

Where the script \( F \) represents the total energy flux in erg cm\(^{-2}\) s\(^{-1}\) (or W m\(^{-2}\)), and \( F_0 \) represents its constant value.

**Non-Radiative Equilibrium**

- Note: if convection or other energy modes (acoustic waves, Alfvén waves etc.) play an important role in energy transport, then we need a term for it too in order to obtain the total flux:

\[
\int_0^\infty F_i(z) dz + \Phi(z) = F_0
\]

- For the solar case, we can neglect the second, convective term - at least for the region near the photosphere.