Quantum Physics Lecture 4

The Uncertainty Principle - continued

“Thought-experiments”
- *microscope, single slit and 2-slit diffraction*

Some applications
- *propagation of wave group, minimum confinement energy*

Alternative $E-t$ form

Thermal Properties

Specific Heat of solids
- *Classical model, failure at low temperature, Einstein model*

Black Body radiation (introduction)
Uncertainty and Error?

Uncertainty looks like some sort of “experimental error” - It is not!

- Measurement of one of $x$ or $p$ alters the value of the other.

Experimental error can be arbitrarily reduced by better experiment. But uncertainty is a fundamental limit, and property of the wave nature of matter!

Note central role of Planck’s constant $h$

Commonly use “$h$-bar” $\hat{h}$ or $h/2\pi = 1.054 \times 10^{-34}$ Js

(later) will see that $\hat{h}$ is basic unit of angular momentum!

Uncertainty in ‘wave’ experiments?

- Microscope and Diffraction
Thought-Experiment (microscope)

Use optical microscope to find particle (e.g. electron) position. “see” the electron by scattering a photon into the lens.....

.....anywhere within the lens angle $2\alpha$.
Photon momentum $(p=h/\lambda)$ change causes recoil of electron!

Along horizontal, change ranges from $-p\sin\alpha$ to $+p\sin\alpha$
i.e. range in photon momentum $\Delta p = 2p\sin\alpha = 2(h/\lambda)\sin\alpha$
.......which becomes the uncertainty in particle momentum
Thought-Experiment (microscope)

Uncertainty in particle position associated with “diffraction limit”: minimum separation of points is

\[ \Delta x = \frac{\lambda}{\sin \alpha} \]

\[ \Delta x \cdot \Delta p = \left( \frac{\lambda}{\sin \alpha} \right) \left( 2 \frac{h}{\lambda} \sin \alpha \right) = 2h \]

\[ \Delta x \cdot \Delta p \geq \frac{\hbar}{2} \]

How could we “improve” microscope?

- by decreasing \( \lambda \): decreasing \( \Delta x \) but increasing \( \Delta p \)?
- by decreasing \( \alpha \), decreasing \( \Delta p \) but increasing \( \Delta x \)?

Quantum concept of photon is intrinsic:

Classically, could decrease \( \Delta x \) without increasing \( \Delta p \)

(lower intensity and wait?)
Thought-Experiment (single-slit)

Remove “complication” of photon and electron by single-slit diffraction (of either)

Slit width ($s$) is the uncertainty in position: $\Delta x = s$

At 1st diffraction minimum: $s \sin \theta = \lambda$

Therefore, $\Delta x = s = \lambda / \sin \theta = h / p \sin \theta$
Thought-Experiment (single-slit)

Electron (or photon) arriving within central maximum must be deflected through angle range 0 to $\theta$: this means uncertainty in transverse momentum:

If momentum is $p$, then $\Delta p = p\sin\theta$

$\Delta x.\Delta p = (h/p\sin\theta)(p\sin\theta) = h$

“showing” that

$\Delta x.\Delta p \geq \frac{\hbar}{2}$

Equivalent analysis of Young’s (Two) Slits using 1st maximum, Where slit separation is the uncertainty in position (exercise)

Q: “which slit does the particle (or photon) go through?” !!
Two-slit experiment

Observe:
- Close one slit (i.e. the particle must go through the other) ⇒ lose the 2-slit diffraction pattern!
- Single particle causes single point of scintillation ⇒ pattern results from addition of many particles!
- Pattern gives **probability** of any single particle location

![Diagram of two-slit experiment](image)

39.9 (a) Formation of an interference pattern for electrons incident on two slits. (b) after 28, 1000, and 10,000 electrons.

Graph shows the degree of exposure of the film, which in any region is proportional to the number of electrons striking that region.
Two slit experiment - summary

(1) Both slits required to give pattern, even for single particle

(2) Single particle arrives at single point. i.e. “explores” all regions available (see 1), but occupies only one point when actually “measured”

(3) Arrival of individual particle conforms to statistical pattern of diffraction (complementarity).

(4) Average over many particles gives standard diffraction pattern. (complementarity)

Key features of quantum mechanics!
"Practical" applications of UP: propagation of a wave group?

Establish particle position to an uncertainty $\Delta x_o$ at time zero: what is uncertainty $\Delta x_t$ at later time $t$?

UP implies $\Delta p \geq \frac{\hbar}{2\Delta x_0}$ and $p = mv$

So $\Delta v = \Delta p/m \geq \frac{\hbar}{2m\Delta x_0}$

uncertainty in velocity implies uncertainty in position at time $t$

$$\Delta x_t = \Delta v \cdot t \geq \frac{\hbar t}{2m\Delta x_0}$$

$\Delta x_t \propto t$: uncertainty in position increases with time (dispersion)

$\Delta x_t \propto 1/\Delta x_o$: "more you know now, less you know later"
Application of UP: minimum energy of confinement

**Rough estimate** $KE$ of electron in hydrogen atom

\[(in \ full, \ later\ lecture)\]

\[
\Delta x \sim \text{radius of H atom} = 5.3 \times 10^{-11} \text{ m}
\]

\[
\Delta p \geq \frac{h}{4\pi \Delta x} = 1 \times 10^{-24} \text{ kg m s}^{-1}
\]

Treat electron as non-relativistic, $KE = p^2/2m_o$

where $p \sim \Delta p$ at least:

\[
KE \geq \left( \frac{\Delta p^2}{2m_o} \right) = \frac{(1x10^{-24})^2}{(2)(9.1x10^{-31})} = 5.4 \times 10^{-19} \text{ J} = 3.4 \text{ eV}
\]

(see later lecture: $KE=13.6 \text{ eV}$ so correct order of magnitude)
Alternative form of UP:

\[ \Delta x \cdot \Delta p \geq \frac{h}{4\pi} \] related to **spatial** extent needed to measure \( \lambda \)

What about the **temporal** extent needed to measure \( \omega \) (or \( f \))?
- at least one period?

Estimate: \( \Delta f \cdot \Delta t \geq 1 \quad E=hf \)

\[ \Delta E = h \Delta f \quad \text{So} \quad \Delta E \Delta t \geq h \]

(correct maths gives) \[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

Eg. \( \Delta E \) is the spectral “width”

of optical emission lines,
where \( \Delta t \) is “lifetime” of transition

*(see atomic transitions, later)*
Specific heat $C_v$ of solids

Solid is $N$ atoms coupled, each having 3 degrees of freedom

Classically    $\text{Energy} = k_b T$ per oscillator

($\text{equipartition principle} - k_b T/2 \text{ per deg of freedom, oscillator has 2, PE & KE}$)

Total energy $U = 3Nk_b T$  ($N$ is number of atoms, $k_b$ Boltzmann const)

Now  $C_v = dU/dT$    So    $C_v = 3Nk_b$  (or $3R$/mole, Dulong & Petit)

$i.e.$  $a$  constant, independent of temperature $T$…

Experimental observation:
OK at high $T$

BUT…
$C_v$ falls ($\text{towards zero}$) at low temperatures…

Why is classical result wrong?
Planck - Assumed energy of oscillators is quantised!

\[ E = n\hbar\omega \] where \( n \) is a positive integer

Probability of an energy \( E \) is

\[ P(E) = e^{-\frac{E}{k_bT}} = e^{-\frac{n\hbar\omega}{k_bT}} \]

Mean (expectation) energy is

\[ \langle E \rangle = \frac{\sum_n E P(E)}{\sum_n P(E)} \]

So total Energy \( U \) is

\[ 3N \langle E \rangle = \frac{3N \sum_n n\hbar\omega e^{-\frac{n\hbar\omega}{k_bT}}}{\sum_n e^{-\frac{n\hbar\omega}{k_bT}}} = 3Nk_bT \left( \frac{\frac{\hbar\omega}{k_bT}}{e^{\frac{\hbar\omega}{k_bT}} - 1} \right) \]

i.e. Quantum term on R.H.S. freezes out energy exchange at low temperature.

Happens because the finite gap between states, \( \hbar\omega \) becomes greater than \( k_bT \)

Similar ‘quenching’ effect for molecule modes

\[ C_v = \left( \frac{\partial U}{\partial T} \right)_V = Nk_b \left( \frac{\hbar\omega}{k_bT} \right)^2 \frac{e^{\frac{\hbar\omega}{k_bT}}}{\left( e^{\frac{\hbar\omega}{k_bT}} - 1 \right)^2} \]

**Einstein** formula for specific heat
Blackbody Radiation

At finite temperature matter “glows”
i.e. emits radiation with a continuous spectrum.
\[ e.g \text{ Infrared imaging of people, planet etc.} \]

Surface dependent (emissivity, silvery, black, etc.)

Blackbody = ideal 100% emitter/absorber
in thermal equilibrium with its surroundings.
\textit{Practical realisation is a thermal cavity.}

**Measure:** spectrum energy density \( u(\omega) \)
Observe that increasing temperature
(1) increases \( u \) overall
(2) shifts peak emission to higher frequencies

\[ i.e. \text{ colour and intensity of hot objects vary with} \ T \]
\textit{Examples – Bar fire, molten iron, stars, universe \( \mu \)-wave background.....}