Physical Optics

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- Textbook:

- RECOMMENDED: Optics by Hecht, Addison Wesley

- Also: Optics by Hecht and Zajac, Addison Wesley

- Insight into Optics by Heavens and Ditchburn, Wiley
SF Physical Optics (12 lectures)


- Interference and coherence-, complex degree of coherence. Visibility of fringes, temporal coherence and spectral bandwidth, spatial coherence. Interference- wavefront and amplitude splitting interferometers, types and location of fringes, and applications of interferometry. Multiple beam interference. Fabry-Perot interferometer.

- Diffraction - near-field and far-field diffraction, Fourier treatment of far-field diffraction, spatial filtering and image formation.
Light as a Wave

- Isaac Newton thought light consisted of a stream of \textit{particles} traveling in straight lines from a source.

- 1801 Thomas Young showed that light could undergo \textit{interference}. This definitively showed that light was somehow a \textit{wave}.

- \textbf{James Clerk Maxwell} in the 1860s. He saw a link between \textit{Electricity} and \textit{Magnetism} and thought it might have something to do with light.
Electricity and Magnetism can be summarised using 4 Laws:

- Gauss’s Law
- The fact that magnetic charges (monopoles) don’t exist
- Amperes Law
- Faraday’s Law
The four laws....

Gauss’s Law

No magnetic monopoles

Ampere’s Law

Faraday’s Law

\[ \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon} \]

\[ \oint \vec{B} \cdot d\vec{A} = 0 \]

\[ \oint \vec{B} \cdot d\vec{l} = \mu \left[ I + \epsilon \frac{d(E \cdot A)}{dt} \right] \]

\[ \oint E \cdot d\vec{l} = -\frac{d(B \cdot A)}{dt} \]

\[ B \cdot A = \Phi \]

The Magnetic Flux
Electromagnetic wave: electromagnetic disturbance time varying electric and magnetic field that can propagate through space

- Amperes law says that a time varying Electric field, \( E(t) \) generates a Magnetic field, \( B \).

- But if \( \frac{dE}{dt} \) varies in time then \( B \) will vary in time, ie \( B(t) \).

- BUT Faradays law says that a time varying Magnetic field, \( B(t) \) generates a time varying Electric field, \( E(t) \).

- This in turn generates a time varying Magnetic field, \( B(t) \) and so on.

- Maxwell used his theory to calculate the speed of such waves and found it to be \( v = 3 \times 10^8 \text{ m/s} \).
Looking at electromagnetic waves...

Consider a plane parallel to yz moving in the +x direction (red) with velocity $c$.

At any time, all points left of the plane experience E and B fields as shown.
This plane...the wavefront

- In time $dt$, the wavefront moves a distance $cdt$ and sweeps out part of the blue rectangle. (*as dt is small, E, B are constant over dt*)

- In this time the Magnetic flux, $\Phi=BA$, through the blue rectangle rises by $d\Phi=BdA=Bacdt$
This means...

- \[ \frac{d\Phi}{dt} = Bac \]
- But Faraday's law says \[ \oint E \cdot dl = -\frac{d\Phi}{dt} \]
- Taking \[ \oint E \cdot dl \] anti-clockwise (Right hand rule) around the perimeter of the blue square gives \[ \oint E \cdot dl = -Ea \]

(E is zero to right of wavefront)
Therefore: $\mathbf{Bac} = \mathbf{E}a$ giving: $\mathbf{E} = c\mathbf{B}$

Now we consider the same system but the blue rectangle is now in the $xz$ plane

Ampere’s Law says:

$$\oint B \cdot dl = \mu \left[ I + \varepsilon \frac{d(E \cdot A)}{dt} \right]$$

But if this is in free space, there are no currents and so $I=0$
Therefore \( \oint B \cdot dl = \mu \varepsilon \frac{d(E \cdot A)}{dt} \)

The electric flux through the blue rectangle, \( EA \) has increased by
\[ d(EA) = E \cdot dA = Ebc \, dt \]
during this time. Therefore
\[ \frac{d(E \cdot A)}{dt} = Ebc \]

Taking \( \oint B \cdot dl \) anticlockwise around the perimeter of the blue square gives:
\[ \oint B \cdot dl = Bb \]
(B is zero to the right of the wavefront.)

But since \( \oint B \cdot dl = \mu \varepsilon \frac{d(E \cdot A)}{dt} \)
Therefore \( Bb = \mu \varepsilon Ebc \)

or \( B = \mu \varepsilon Ec \)
Already \( E = cB \)

**giving**

\[
C = \frac{1}{\sqrt{\varepsilon \mu}}
\]

In free space this becomes

\[
C = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}
\]

\[
c = 1/ \left( 8.85 \times 10^{-12} \times 4\pi \times 10^{-7} \right)^{1/2} = 3 \times 10^8 \text{ m/s}
\]
Wave motion

- What is a wave?
- A disturbance moving through space carrying energy and momentum
Transverse wave: the disturbance is perpendicular to the direction of motion

A wave on a string is a displacement of the string perpendicular to the direction of motion.

Here the “disturbance” is just a local vertical displacement of the string.

Let's call this displacement, $\psi$.

$\psi$ is a function of position along the string but is also a function of time.
That is: \( \psi = f(x, t) \)

We can describe the previous wave using the *fixed* set of axes \( S \).
Then \( \psi = f(x, t) \)

Or we can describe it using a different set of axes, \( S' \), which move along with the wave.
Then \( \psi = f(x') \)

These 2 sets of axes are clearly related.

At any time \( t \), the axes, \( S' \), have moved by a distance \( vt \) with respect to \( S \).

*Any point on the wave using axes \( S' \) is at position \( x' \).*
The same point using axes $S$ is at position $x$. These must be related by $x=x'+vt$ or $x'=x-vt$.

Therefore:

$$\psi = f(x,t) = f(x') = f(x-vt)$$

We can imagine the waveform as a moving bus.
• Then the fixed axes, S, represent the viewpoint of an observer on the footpath: They see someone in the bus moving relative to them.

• The moving axes, S’, represent the viewpoint of the observer on the bus. They see the bus as static.

• The difference between these viewpoints is controlled by the position of the bus, which is governed by its speed.
This is a completely **general** equation that describes any one dimensional wave.

At $t=0$, $\psi$ can be thought of as the shape of the wave.

$\psi(x,0) = \psi(x) = \psi(x')$.

The wave on the left has the general shape given by

$$\psi = e^{ax^2}$$

If the wave moves to the right with velocity $v$ we can find its equation just by replacing $x'$ with $x-vt$:

$$\psi = e^{a(x-vt)^2}$$
This is a general method for finding a wave equation if we know the shape of the wave. Remember:

\[ \psi = \psi(x,t) = \psi(x') = \psi(x \pm vt) \rightarrow x' = x \pm vt \]

By the chain rule for differentiation:
This gives:

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 \psi}{\partial t^2}
\]

This is known as the **differential wave equation** and is completely general.

This equation applies to any wave. Any function, \( \Psi \), that is a solution to this equation represents a wave.
HARMONIC WAVES

The simplest form is a sine or cosine:

Here $kx'$ is in radians, $x'$ is in m, so $k$ has units of rad/m

To turn this into a traveling wave just replace $x'$ with $x-\mathbf{v}t$
So a simple traveling wave can be described by
Check this is a solution to the wave equation.

\[ \psi(x, t) = A \sin[k(x - vt)] \]

where \( \varphi \) is known as the phase (angle)

What is \( k \)?

All waves repeat themselves in space after one wavelength. This means that the “disturbance”, \( \psi \) is the same at any position \( x \) and the position \( x + \lambda \).

\[ \psi(x, t) = \psi(x + \lambda, t) \]
In addition waves repeat themselves in **time** after one **period**. This means that the “disturbance”, $\psi$ is the same at any time $t$ and another time $t + \tau$. 

\[
A\sin[k(x - vt)] = A\sin[k((x + \lambda) - vt)]
\]

but waves also repeat after the **phase** moves on by $2\pi$.

\[
A\sin[k(x - vt)] = A\sin[k(x - vt) + 2\pi]
\]

Therefore: $k\lambda = 2\pi$, so 

\[
k = \frac{2\pi}{\lambda}
\]
\[ \Psi(x,t) = \Psi(x,t + \tau) \]

By applying the same analysis as above, we get

\[ kv\tau = 2\pi \rightarrow \text{which gives} \rightarrow \tau = \lambda / \nu \rightarrow \text{or} \rightarrow \nu = \lambda / \tau \]

Remember that a period is the inverse of frequency:

\[ f = 1 / \tau \rightarrow \text{giving} \rightarrow \nu = f \lambda \]

Another common quantity is the \textbf{angular frequency}, \( \omega \), defined by

\[ \omega = 2\pi f = 2\pi / \tau \]

With this in mind we note that as

\[ kv\tau = 2\pi, \rightarrow k\nu = 2\pi / \tau = \omega \]

This is the most common expression
Phase:
We have been talking about waves of the form

\[ \psi(x,t) = A \sin(kx - \omega t) \]

However at \( t=0, \ x=0 \) we get \( \psi=0 \).
This is a special case, the “disturbance” is not necessarily zero at the origin.
In some cases there may be a phase shift. This means the whole wave is shifted in the x direction at \( t=0, \ x=0 \). Then \( \psi(0,0) \neq 0 \).
Thus the most general description of a travelling wave is

\[ \psi(x,t) = A\sin[kx - \omega t + \varepsilon] \]

Where \( \varepsilon \) is the initial phase.

**LIGHT AS A WAVE**

Light consists of linked oscillating \( B \) and \( E \) fields. Thus we can describe light by describing these fields in space and time by

\[
E(x,t) = E_0\sin(kx - \omega t + \varepsilon)
\]
\[
B(x,t) = B_0\sin(kx - \omega t + \varepsilon)
\]

Note that these \( E \) and \( B \) waves have the same frequency \( (f=\omega/2\pi) \) and wavelength \( (\lambda=2\pi/k) \). They also have the same initial phase, \( \varepsilon \). This means they are always in phase.
It can also be shown using Maxwell's equations that they are always mutually perpendicular and both are perpendicular to the direction of motion.

Both Electric and Magnetic fields carry energy. This energy is distributed over some region of space so we can think in terms of an energy density, \( U \) (energy per unit volume, J/m\(^3\)). The energy density associated with an electric field (ie as in a capacitor) is given by:

\[
U_E = \frac{\varepsilon}{2} E^2
\]
The energy density associated with a magnetic field (i.e., as in a solenoid) is given by

\[ U_B = \frac{1}{2\mu} B^2 \]

Thus the total energy density associated with a light wave is

\[ U = U_E + U_B = \frac{\varepsilon}{2} E^2 + \frac{1}{2\mu} B^2 \]

But \( E = cB \)

and

\( c^2 = \frac{1}{\varepsilon\mu} \)

Therefore

\[ U = \varepsilon E^2 = \frac{1}{\mu} B^2 \]
Because light travels, the associated energy must flow through space. This can be described by the amount of energy flowing per unit time across a unit area, $S$ (W/m$^2$).

Imagine a wave moving at speed, $c$, through an area $A$. 
In time $dt$, the wave travels $c dt$, therefore only the energy in the cylindrical volume will flow through $A$.

\[
S = \frac{U c dt A}{dt A} = U c
\]

\[
U = \frac{1}{\mu} B^2
\]

and \( E = c B \)

Therefore

\[
S = \frac{1}{\mu} E B = \varepsilon c^2 E B
\]

$S$ is known as the Poynting Vector (see below).

(E and $B$ mutually perpendicular and perpendicular to the direction of motion)

If we assume that the energy flows in the direction of propagation of the wave, then

\[
\vec{S} = \varepsilon c^2 \vec{E} \times \vec{B}
\]

(E and $B$ mutually perpendicular and perpendicular to the direction of motion)

For a typical light wave traveling in the $x$ direction
This flow of energy fluctuates very rapidly in time. For visible light the frequency is in the region of $10^{14}$ Hz. This is too rapid to observe, what we see is an average over a long time.

The average value of $S$ is given by:

$$\langle S \rangle = \varepsilon c^2 E_0 \times B_0 \langle \sin^2 (kx - \omega t) \rangle$$
The average of $\sin^2 \varphi$ over many cycles is 0.5.

\[
\langle \vec{S} \rangle = \frac{\varepsilon c^2}{2} E_0 \times B_0 = \frac{\varepsilon c}{2} E_0^2 \hat{x}
\]

: unit vector in the propagation direction

The average value of $S$ denotes the average energy per unit area per unit time falling on a surface. This is known as the intensity or irradiance, $I$, and is a measure of the brightness of the light.
Plane Waves
We have up to now considered waves travelling in only one dimension (the x direction)

\[ E(x, t) = E_0 \sin(kx - \omega t) \]

We won’t bother with the B part anymore, just assume where there is an E, there is a B field

However in the real world, rays travel in 3 dimensions.
We can generalise the above equation in 3 dimensions by

\[ E(x, y, z, t) = E_0 \sin(k_x x + k_y y + k_z z - \omega t) \]

This describes a wave moving in one direction in space.
The direction is given by the vector $k$, where

$$k = k_x i + k_y j + k_z k$$

The magnitude of this vector is related to the wavelength

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda}$$

Any position on the wave is described by the position vector, $r$.

$$r = xi + yj + zk$$

Thus

$$\vec{E}(r, t) = \vec{E}_0 \sin(k \cdot \vec{r} - \omega t)$$

This is the equation for a plane wave.
We can think of a plane wave as made up of an infinity set of planes moving in a direction in space. Each plane represents a surface where the E field is constant. This means the phase of the wave is constant over the plane. The set of planes above represent where the E fields are a maximum.
Each plane can be thought of as a wavefront. Lines perpendicular to the wave front give the direction of propagation. These are called rays.

**Spherical Waves**
Waves don’t have to be plane waves. A localized light source or point source emits rays in all directions. This is known as a spherical wave.

A point source in 2 dimensions. In 3 dimensions the wavefronts would be spherical.
Spherical waves must get weaker as they get further from the source, the Electric field must diminish to satisfy conservation of energy.

This means the amplitude of the wave, in this case $E_s$, must be a (diminishing) function of $r$

$$E(r) = E_s(r) \sin(k \cdot r - \omega t)$$

Wave Equation for a spherical wave

The total energy per second crossing any sphere a distance $r$ from the source must be constant.

The intensity ($W/m^2$) is proportional to $E^2$.

$$I(r) = \frac{\varepsilon c}{2} E_s(r)^2$$

The total energy per second crossing a sphere of radius $r$ is

$$4\pi r^2 I = P_0$$

where $P_0$ is the power output of the source, which is constant.
Therefore

\[ 4\pi r^2 \frac{\varepsilon c}{2} E_s(r)^2 = P_0 \]

but \( \frac{2P_0}{4\pi \varepsilon c} \) is a constant and can be written as \( E_0^2 \).

Therefore:

\[ E_s(r)^2 = \frac{2P_0}{4\pi \varepsilon c} \frac{1}{r^2} \]

\[ E_s(r) = \frac{E_0}{r} \]

\[ E(r) = \frac{E_0}{r} \sin(k \cdot r - \omega t) \]

for a spherical wave.

In the same way we could analyze a cylindrical wave from a line source.

Then we would get:
For a cylindrical wave

\[ E(r) = \frac{E_0}{\sqrt{r}} \sin(k \cdot r - \omega t) \]
The Complex Representation
A plane wave can be described as either a sine or a cosine

\[ E = E_0 \sin(kx - \omega t) \]

Or

\[ E = E_0 \cos(kx - \omega t) \]

There is however a third way

Remember **de Moivres theorem**:  
Then

\[ e^{i\varphi} = \cos \varphi + i \sin \varphi \rightarrow \rightarrow \text{Then} \rightarrow \cos \varphi = \Re e^{i\varphi} \]

Then

\[ E = E_0 \cos(kx - \omega t) = E_0 \Re e^{i(kx - \omega t)} \]

Usually, for convenience we just omit the Re and write a wave as
This can significantly simplify calculations.

When doing calculations with the complex representation just do all the maths and the answer is just the real part of the result.

\[ E = E_0 e^{i(kx - \omega t)} \]