Problem set 1 - PY4C01

(1) The periodic rectified sine wave $f(t)$, is given by:

$$f(t) = \begin{cases} \sin(\omega t) & 0 > \omega t > \pi \\ -\sin(\omega t) & -\pi < \omega t < 0 \end{cases}$$

Compute the coefficients a_n and b_n of the Fourier series.

(2) Two signals p and q, are given by

$$p(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases} \quad \text{,and} \quad q(t) = \begin{cases} 0, & t < 0 \\ 1-t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

. Compute the correlation $p \odot q$.

(3) Consider the function $s(x) = \sin(x)$.

(a) What will the autocorrelation look like? (Note: use the average correlation function).

(b) What is the period of the autocorrelation function?

(c) Let $s(x)$ be corrupted by some white noise $n(x)$, such that $f(x) = s(x) + n(x)$. What is the autocorrelation function of $f(x)$?

(d) How would you find the period of a noisy periodic signal?

(4) Let $f(x)$ be some function that vanishes as $x \to \pm\infty$ and $n(x)$ is uncorrelated noise. What is the power spectrum of $f(x)+n(x)$ in terms of the power spectra of $f(x)$ and $n(x)$?

(5) Using the orthogonality of the sums, $\sum_{k=0}^{N-1} e^{i2\pi km/N} e^{-i2\pi kn/N} = N\delta_{m,n}$, show that the following relation holds: $\sum_{m=0}^{N-1} x_m y_m^* = \frac{1}{N}\sum_{n=0}^{N-1} X_n Y_n^*$, where X_n and Y_n are the discrete Fourier transforms of x_m, y_m.

(6) Compute the Fourier transform of $f(x) = e^{-\alpha|x|}$

(7) Apply the Crank-Nicholson difference scheme to the time dependent Schrödinger equation in one spatial dimensions:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

What is the amplification factor ξ for this scheme? What is $|\xi|^2$?

(8) Consider the following difference scheme for the diffusion equation:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^n + u_{j-1}^n - u_j^{n+1} - u_j^{n-1}}{(\Delta x)^2}$$

(a) Solve it for u_j^{n+1}. Is this an implicit or explicit difference scheme?

(b) What is the stability criterion for this scheme?

(9) Consider the explicit scheme for the wave equation as discussed in the lectures. Substitute a plane wave $u_j^n = e^{ikj\Delta x - i\omega n\Delta t}$ into the difference scheme and compare your answer to the known dispersion relation $\omega/k = \pm c$. Are there any values for $\Delta x/\Delta t$ for which your result reduces to $\omega/k = \pm c$?