Partial Differential Equations

Partial differential equations (PDE's) describe fundamental laws of physics.

Examples:

- Hydrodynamics - Navier Stokes equations.
- Electrodynamics - Maxwell's equations.
- Thermodynamics - Diffusion equation.
- Quantum Mechanics - Schrödinger equation.

Typically, these equations determine the evolution of a scalar or vector field that depends on space and time.

Steady state problems in two or three spatial dimensions are also described by PDE's.

Notes

Basic properties of partial differential equations

The most general *first order* partial differential equation of a *dependent* variable u which is a function of two *independent* variables x, y, is

$$F(x, y, u, u_x, u_y) = 0$$

, where $u_x \equiv \partial u/\partial x$ etc.
The most general second order PDE with two independent variables is

$$F(x, y, u, u_{xx}, u_{yy}, u_{xy}, u_x, u_y) = 0$$

The order denotes the highest derivative appearing in the equation.

Many PDE's in physics are **linear**.

Notes

Basic properties of partial differential equations

Linearity: Express the PDE in terms of a differential operator $\mathfrak{L}$:

$$\mathfrak{L}u = f$$

, where f is some function of the independent variables.
e.g. $\mathfrak{L} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \ldots$, the Laplacian.

Linearity means that $\mathfrak{L}$ is a linear operator:
For any function u and v

$$\mathfrak{L}(u + v) = \mathfrak{L}(u) + \mathfrak{L}(v)$$

and

$$\mathfrak{L}(cu) = c\mathfrak{L}(u)$$

, where c is constant.

Notes

Homogeneity

The PDE is homogeneous if it can be expressed as

$$\mathfrak{L}u = 0$$

The PDE is inhomogeneous if it has a so called source term:

$$\mathfrak{L}u = f$$

, where f is some function of the independent variables (e.g. Poisson's equation in electrostatics).
Note that one can add solutions of the homogeneous PDE to the inhomogeneous solution.
IF a PDE is both linear and homogeneous then it obeys the principle of *superposition:* If f and g are two solutions of a linear, homogeneous PDE, then $c_1 f + c_2 g$ is also a solution, where c_1, c_2 are some constants.

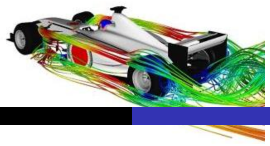
Notes

Example: Turbulence

At undergraduate level, typically deal with "easy" geometries that have symmetries.In many problems, however, numerical solutions are the only way to proceed, because the geometry may be complicated and/or the PDE is non-linear.

Example: Navier-Stokes equations - Fundamental equation of hydrodynamics.

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} &= -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \vec{u} + \vec{g} \\ \nabla.\vec{u} &= 0 \end{aligned}$$



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Classes of differential equations

Many physical systems are governed by a *linear, second order* partial differential equations (PDE's). For a variable u that depends on x and t the general form is

$$a_{11}\frac{\partial^2 u}{\partial u^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial t} + a_{22}\frac{\partial^2 u}{\partial t^2} + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial t} + a_0 u = 0 \quad (1)$$

Examples: Maxwell's equations, Schrödinger equation etc.

Notes

Classes of differential equations

By linear transformation of the independent variables linear, second order PDE's reduce to 3 different categories:

1. **Elliptic case**. $a_{12}^2 < a_{11}a_{22}$
$$u_{xx} + u_{tt} + \ldots = 0$$
2. **Hyperbolic case** $a_{12}^2 > a_{11}a_{22}$
$$u_{xx} - u_{tt} + \ldots = 0$$
3. **Parabolic case** $a_{12}^2 = a_{11}a_{22}$
$$u_{xx} + \ldots = 0$$
(unless $a_{11} = a_{22} = a_{12} = 0$)

This classification is analogous to conic sections in analytic geometry. Can also have mixed type PDE's if the coefficients depend on the linear variables.

Notes

Classes of differential equations

Prototypical PDE's in physics:

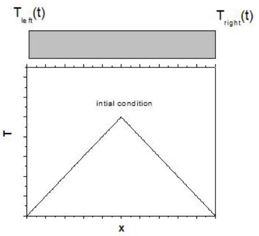
- Hyperbolic - Wave equation
$$u_{tt} = c^2\nabla^2 u$$
- Parabolic - Diffusion equation
$$u_t = \kappa\nabla^2 u$$
- Elliptic - Laplace equation
$$\nabla^2 u = 0$$
Poisson equation
$$\nabla^2 u = -\frac{\rho}{\epsilon}$$
Typically elliptic PDE's occur equilibrium (steady state) problems.

Notes

Boundary and initial conditions

In order to have a well defined problem we not only need the partial differential equation that governs the physics, but also a set of *boundary conditions* (BC) and *initial conditions* (IC) to specify the problem.

e.g. Heat equation for an insulated bar with its two ends immersed in heat baths



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Typical boundary conditions:

- **Dirichlet boundary conditions:** The value of the function is specified at the boundary. e.g. temperature of heat bath.
- **von Neumann boundary conditions:** The normal gradient of the function is specified at the boundary. e.g. the electric field from a given potential at the boundary.
- **Cauchy boundary conditions:** Both Dirichlet and von von Neumann boundary conditions are specified.

There are usually two different problems that arise in physics:

- **Initial value problem:** This deals with time evolution of u. At a certain time t_0, the value of u (and possibly its derivative) is identified.
- **Boundary value problem:** Steady state, time independent problems. The value of u and/or its derivative is determined along a boundary that encloses the region of interest. e.g. electrostatic problems where the potential at the boundaries is specified. In some cases can also have open boundaries.

Notes

Numerical techniques

There is a vast body of literature on the numerical solution of PDE's. Many different techniques available:

- *Finite difference methods:* This is the most common approach. Based on approximating derivatives by finite differences. e.g. $\frac{\partial u(x,t)}{\partial t} \approx \frac{u^n_{j+1}-u^n_j}{\Delta x}$ where $u^n_j = u(j\Delta x, n\Delta t)$
- *Finite Element method:* Basic idea is to tessellate volume of interest into discrete pieces (polygons) and approximate the solution by simple functions on these pieces. Complicated geometries with curved boundaries for example are easily implemented. Therefore, popular with engineers to model solids and structures.
- *Spectral methods:* Approximate the solution and the boundary condition with a Fourier series and substitute into the PDE. Use the inverse Fast Fourier transform to compute the Fourier coefficients of the solution.
- *Monte Carlo*: Use stochastic methods to find solutions of a PDE.

Notes

General remarks on computational methods for PDE's

The mathematical classification of second order, linear PDE's into the three canonical forms is not too useful from computational point of view. What is relevant whether we are dealing with

- **Time evolution - Initial value problem**: Wave equation, diffusion equations. Compute the time evolution for a given initial condition subject to boundary conditions.
- **Static solution - Boundary value problem**: Laplace equation, Poisson equation: Find solutions that satisfies the boundary conditions around the region of interest.

Because of the way the boundary conditions need to be enforced, the numerical methods will be different for these two types of problems.

In boundary value problems one cannot "integrate in from the boundary" in the same sense one can "integrate forward in time" for initial value problems. For static problems the goal of the numerical method is to converge on the correct solution everywhere at once

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Initial versus boundary value problems

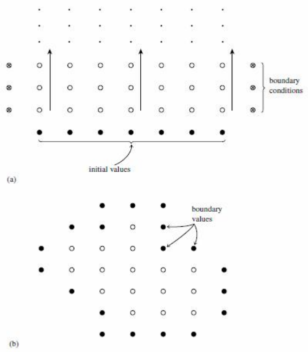

Figure 20.0.1. Initial value problem (a) and boundary value problem (b) are contrasted. In (a), initial values are given on one "time slice," and it is desired to advance the solution in time, computing successive rows of open dots in the direction shown by the arrows. Boundary conditions at the left and right edges of each row (⊗) must also be supplied, but only one row at a time. Only one, or a few, previous rows need be maintained in memory. In (b), boundary values are specified around the edge of a grid, and an iterative process is employed to find the values of all the internal points (open circles). All grid points must be maintained in memory.

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