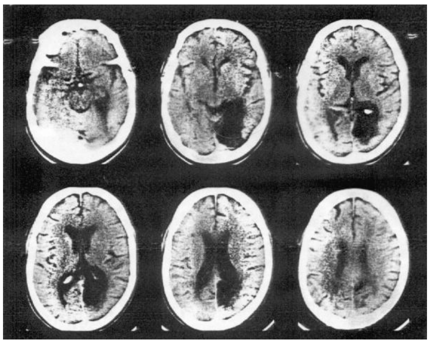
X-ray tomography

Important application of the Fast Fourier transform: X-ray tomography. Also referred to as CAT scan (Computerized Axial Tomography)

Notes

X-ray tomography

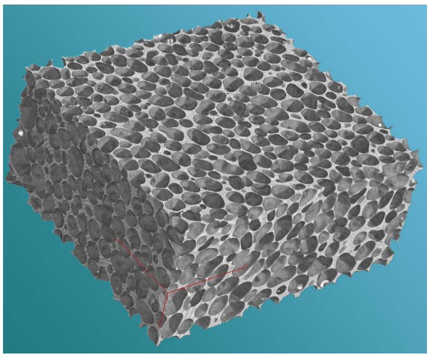
This has revolutionized medical diagnosis. Invented in the 1960's and 70's. Nobel price in medicine for Cormack and Hounsfield in 1979.



Notes

Applications in Science

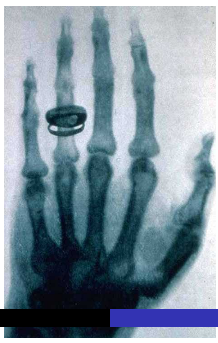
Also used in scientific research in order to infer the structure of opaque materials non-invasively (e.g. metal foams).



Notes

X-Rays

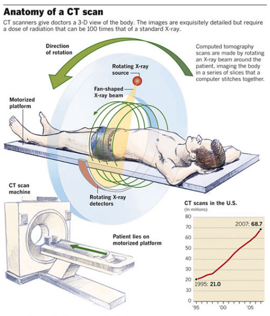
X-rays discovered in 1895 by Wilhelm Roentgen (won the first Nobel price in Physics 1901). X-rays are electromagnetic waves with wavelengths $\approx 0.01 - 10$ nm. X-Rays are absorbed/scattered from high Z-nuclei who have a higher electron density (such as bones).



Notes

How does tomography work?

Basic idea: Reconstruct 3D information by recording the sample slice by slice. 3D image created by combining the slices.
Basic problem: How do you reconstruct one slice?



Notes

Reconstructing the slice

In the following we will see how the Fourier transform is used to do image reconstruction for one particular slice.

When X-rays pass through a uniform slab of material its intensity I decreases exponentially (attenuation):

$$I = I_0 e^{-\lambda d} \tag{1}$$

where I_0 is the initial intensity and λ is the absorption coefficient (many processes such as scattering and photoelectric effect can contribute to λ).

Notes

The reconstruction problem

As X-rays pass through the sample, their intensity decreases, depending on the **local attenuation** function $f(x, y)$. The X-ray detector is constructed in such a way that X-rays that enter at an angle are rejected. The attenuation will be of the form

$$I = I_0 e^{-\int f(x,y) d\eta} \tag{2}$$

where $d\eta$ is an element along the path.

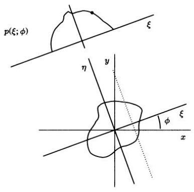


FIGURE 6.19 CAT scan geometry. The path of a typical beam is shown by the dotted line; its attenuation is represented by a single point in the projection $p(\xi;\phi)$.

Notes

The projection at angle ϕ

Each *parallel* beam offset by ξ from the center traverses a different portion of the sample and is attenuated according to eq. (2). The attenuated signal at different ξ is then recorded at the detector, which is the *projection* at a given angle ϕ:

$$p(\xi;\phi) = -\ln\left(\frac{I}{I_0}\right) = \int f(x, y) d\eta \tag{3}$$

Goal

Reconstruct $f(x, y)$ from the various projections $p(\xi;\phi)$ obtained at different angles ϕ

Notes

Relation between coordinate systems

The two coordinate systems are related by a simple rotation

$$\xi = x\cos(\phi) + y\sin(\phi) \quad (4)$$
$$\eta = -x\sin(\phi) + y\cos(\phi) \quad (5)$$

Likewise,

$$x = \xi\cos(\phi) - \eta\sin(\phi) \quad (6)$$
$$y = \xi\sin(\phi) + \eta\cos(\phi) \quad (7)$$

Notes

Fourier transforming the attenuation

Let's look at the Fourier transform of $f(x,y)$

$$\mathfrak{F}[f(x,y)] = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int\int_{-\infty}^{+\infty} f(x,y)e^{-i(k_x x + k_y y)}dxdy \quad (8)$$

Want to express k_x, k_y in rotated frame k_ξ, k_η.

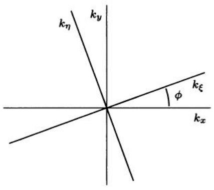


FIGURE 6.20 The relevant coordinate systems in transform space.

Notes

Relation between coordinate systems in Fourier space

k_ξ, k_η coordinate system is rotated with respect to the k_x, k_y system the same way as the spatial coordinate system:

$$k_\xi = k_x\cos(\phi) + k_y\sin(\phi) \quad (9)$$
$$k_\eta = -k_x\sin(\phi) + k_y\cos(\phi) \quad (10)$$

$$k_x = k_\xi\cos(\phi) - k_\eta\sin(\phi) \quad (11)$$
$$k_y = k_\xi\sin(\phi) + k_\eta\cos(\phi) \quad (12)$$

Notes

Using these coordinate transformations, we can transform $\mathfrak{F}[f(x,y)]$ to the (ξ,η) coordinate system.

Can easily show that the exponential appearing in the FT of $f(x,y)$ is simply

$$e^{-i(k_x x + k_y y)} = e^{-i(k_\xi \xi + k_\eta \eta)} \quad (13)$$

and $dxdy = d\xi d\eta$ (The Jacobean is 1)

Therefore,

$$\begin{aligned} F(k_x,k_y) &= F(k_\xi\cos(\phi) - k_\eta\sin(\phi), k_\xi\sin(\phi) + k_\eta\cos(\phi)) \\ &= \frac{1}{2\pi}\int\int_{-\infty}^{+\infty} f(x,y)e^{-i(k_\xi\xi + k_\eta\eta)}d\xi d\eta \end{aligned}$$

Notes

The Projection theorem

We will now restrict ourselves: We will keep the angle ϕ fixed and only evaluate the Fourier transform along the k_ξ axis, so $k_\eta = 0$. This corresponds to integrating along the radial coordinate, except that the integral extends from $-\infty$ to $+\infty$:

$$\mathfrak{F}[f(k_\xi \cos(\phi), k_\xi \sin(\phi))] = \frac{1}{2\pi} \int \int_{-\infty}^{+\infty} f(x,y) e^{-ik_\xi \xi} d\xi d\eta \quad (14)$$

Now recall, that the projection $p(\xi;\phi)$ is given by eq.(3):

$$p(\xi;\phi) = \int f(x,y) d\eta \quad (15)$$

Notes

The Projection theorem

Combining the last two equations we obtain the *projection theorem*:

Projection theorem

$$F(k_\xi \cos(\phi), k_\xi \sin(\phi)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p(\xi;\phi) e^{-ik_\xi \xi} d\xi \quad (16)$$

The Fourier Transform of the projection $p(\xi;\phi)$ at a given angle ϕ is the Fourier transform of the attenuation function along ϕ in transform space.
Measuring the projections at various angles ϕ will yield the FT of the attenuation function throughout the k-plane, one radial line at a time.

Notes

Reconstruction problem

However, there is a practical problem implementing it via the DFT/FFT as illustrated below

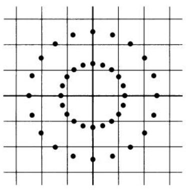


FIGURE 6.21 The Fourier transform is known on the polar grid, indicated by the black dots, while the straightforward FFT requires the data to be known on a Cartesian grid.

Notes

Reconstruction problem

Using the projection theorem, we obtain the transform of $p(\xi;\phi)$ in Fourier space, but **not on a cartesian grid** as required by the discrete Fourier transform.
In principle one could use *interpolation* to approximate the Fourier transform on the Cartesian grid. In practice this is usually not done for two reasons:

1. Interpolation introduces rounding errors that propagate through the whole image when the inverse DFT is applied to the transform of the projection.
2. Interpolation requires additional computational resources.

To circumvent these problems, so called *back projection* methods are employed. We will eventually need an interpolation, but it will be done in the coordinate space.

Notes

Back projection

In the following we will make a change to polar coordinates to the inverse FT that gives us $f(x,y)$:

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(k_x,k_y)e^{i(k_x x + k_y y)}dk_x dk_y \qquad (17)$$

We express x, y in terms of the polar coordinates r, θ and the conjugate variables k_x, k_y in terms of ρ, ϕ

Notes

Back projection

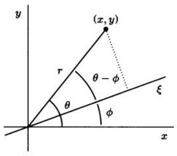


FIGURE 6.22 Geometry relating (x, y) and (r, θ) to ξ.

We find

$$k_x x + k_y y = \rho\cos(\phi) r\cos(\theta) + \rho\sin(\phi) r\sin(\theta) = \rho r\cos(\theta - \phi)$$

Notes

Back projection

Therefore, $f(x,y) = f(r\cos(\theta), r\sin(\theta))$ becomes

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} F(\rho\cos(\phi),\rho\sin(\phi))e^{i\rho r\cos(\theta-\phi)}\rho d\phi d\rho \\
&= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\pi} F(\rho\cos(\phi),\rho\sin(\phi))e^{i\rho r\cos(\theta-\phi)}\rho d\phi d\rho \\
&\quad + \frac{1}{2\pi}\int_0^{\infty}\int_{\pi}^{2\pi} F(\rho\cos(\phi),\rho\sin(\phi))e^{i\rho r\cos(\theta-\phi)}\rho d\phi d\rho \\
&= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\pi} F(\rho\cos(\phi),\rho\sin(\phi))e^{i\rho r\cos(\theta-\phi)}\rho d\phi d\rho \\
&\quad + \frac{1}{2\pi}\int_0^{\infty}\int_0^{\pi} F(-\rho\cos(\phi),-\rho\sin(\phi))e^{-i\rho r\cos(\theta-\phi)}\rho d\phi d\rho
\end{aligned}$$

Notes

Back projection

Therefore, $f(r\cos(\theta), r\sin(\theta))$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{\pi}\left(\int_0^{\infty} F(\rho\cos(\phi),\rho\sin(\phi))e^{i\rho r\cos(\theta-\phi)}\rho d\phi d\rho\right. \\
&\quad \left. + \int_0^{\infty} F(-\rho\cos(\phi),-\rho\sin(\phi))e^{-i\rho r\cos(\theta-\phi)}\rho d\rho\right) d\phi
\end{aligned}$$

If we do the ϕ integration last, then ρ integration is done at constant ϕ. At fixed ϕ, however, k_ξ lies along ρ.

Therefore, make the following substitutions: $\rho \to k_\xi$ in the first integral and $\rho \to -k_\xi$ in the second (and reverse limits of integration).

Also, through the choice of our coordinate system we have

$$r\cos(\theta - \phi) = \xi \qquad (18)$$

Notes

Back projection

Notes

Finally we obtain $f(r\cos(\theta), r\sin(\theta))$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{\pi}\left(\int_0^{\infty} F(k_\xi\cos(\phi), k_\xi\sin(\phi))e^{ik_\xi\xi}k_\xi dk_\xi\right. \\
&\quad \left. + \int_{-\infty}^{0} F(k_\xi\cos(\phi), k_\xi\sin(\phi))e^{ik_\xi\xi}(-k_\xi)dk_\xi\right) d\phi \\
&= \frac{1}{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} F(k_\xi\cos(\phi), k_\xi\sin(\phi))e^{ik_\xi\xi}|k_\xi|dk_\xi d\phi
\end{aligned}
$$

Back projection

Notes

So

$$f(r\cos(\theta), r\sin(\theta)) = \int_0^{\pi} \tilde{p}(\xi;\phi)d\phi \tag{19}$$

where

$$\tilde{p}(\xi;\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k_\xi\cos(\phi), k_\xi\sin(\phi))e^{ik_\xi\xi}|k_\xi|dk_\xi \tag{20}$$

is the *modified projection* at ϕ, which is just the inverse Fourier transform of $F(k_\xi\cos(\phi), k_\xi\sin(\phi))|k_\xi|$.
This can be computed numerically with the inverse DFT.

Back projection - summary

Notes

Let's summarize the steps one has to take to reconstruct the slice:

1. Initialize everything. The outermost loop will integrate over the various angles ϕ (eq. 19)
2. Measure and obtain the projection $p(\xi;\phi)$ at the current angle ϕ. It will be measured at discrete points ξ_i.
3. Using the FFT, obtain $F(k_\xi\cos(\phi), k_\xi\sin(\phi)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} p(\xi;\phi)e^{-ik_\xi\xi}d\xi$ (Projection theorem)
4. Using the inverse FFT compute the modified projection $\tilde{p}(\xi;\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k_\xi\cos(\phi), k_\xi\sin(\phi))e^{ik_\xi\xi}|k_\xi|dk_\xi$
5. Evaluate the contribution to the ϕ integral: $f(r\cos(\theta), r\sin(\theta)) = \int_0^{\pi}\tilde{p}(\xi;\phi)d\phi$
6. For this we need to determine ξ for a particular x, y coordinate through the coordinate transformation and then find the value of the modified projection at ξ by the interpolation of $\tilde{p}(\xi_i;\phi)$

Tomography - summary

Notes

- Powerful tool to probe matter non-invasively in three dimensions.
- Mathematically well defined by use of the projection theorem.
- Numerical implementation is more complicated: Data is taken at discrete points in a polar coordinate system, but the discrete Fourier transform assumes cartesian coordinates.
- Interpolation in Fourier space would lead to image artifacts in the reconstruction.
- Back projection relies on a coordinate transformation which enables us to use the DFT.
- Even though interpolation is still required, it is done in coordinate space which is less prone in introducing artifacts.