Fourier series

Fourier series of a **periodic function** $f(t)$ with period T and corresponding angular frequency $\omega = 2\pi/T$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right),$$

- Fourier series is a linear sum of cosine and sine functions with **discrete frequencies** that are **integer multiples** of the frequency of $f(t)$.
- This gives rise to a **discrete frequency spectrum** given by the Fourier coefficients (=frequency amplitudes).

Notes

Complex Fourier series

Complex representation of Fourier series of a function $f(t)$ with period T and corresponding angular frequency $\omega = 2\pi/T$:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \text{ where}$$

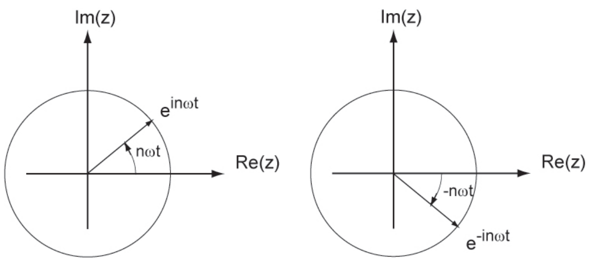
$$c_n = \begin{cases} (a_n - ib_n)/2, & n > 0, \\ a_0/2 & n = 0, \\ (a_{|n|} + ib_{|n|})/2 & n < 0 \end{cases}$$

- Note that the summation goes from $-\infty$ to ∞.
- Now have a negative frequencies as well.

Notes

Positive and negative frequencies

Positive frequencies ($n > 0$): $e^{in\omega t} = \cos(n\omega t) + i\sin(n\omega t)$
Negative frequencies ($n < 0$): $e^{-i|n|\omega t} = \cos(|n|\omega t) - i\sin(|n|\omega t)$



Notes

Fourier series demonstration

Applet for Fourier Series:

http://www.westga.edu/~jhasbun/osp/Fourier.htm

Notes

The Fourier transform

For non-periodic (or periodic) functions $f(t)$, we can define the Fourier transform

$$\text{Fourier transform} \quad \mathfrak{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$\text{inverse transform} \quad \mathfrak{F}^{-1}[g(\omega)] = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega,$$

where $g(\omega)$ corresponds to a **continuous frequency spectrum**.

Notes

Properties of the Fourier transform

Linearity	$\mathfrak{F}[f(t)+g(t)] = \mathfrak{F}[f(t)] + \mathfrak{F}[g(t)]$
Shifting	$\mathfrak{F}[f(t-t_0)] = e^{-i\omega t_0}g(\omega)$ $\mathfrak{F}^{-1}[g(\omega-\omega_0)] = e^{i\omega_0 t}f(t)$
Scaling	$\mathfrak{F}[f(\alpha t)] = \frac{1}{\lvert\alpha\rvert}g(\omega/\alpha)$ $\mathfrak{F}^{-1}[g(\beta t)] = \frac{1}{\lvert\beta\rvert}f(t/\beta)$
Power spectrum	$\mathfrak{F}[f(t)]\,(\mathfrak{F}[f(t)])^* = \lvert\mathfrak{F}[f(t-t_0)]\rvert^2 = \lvert g(\omega)\rvert^2$ cannot reconstruct f(t) from power spectrum since phase information is lost. Note: $\lvert\mathfrak{F}[f(t-t_0)]\rvert^2 = \lvert\mathfrak{F}[f(t)]\rvert^2$
Parseval's theorem	$\int_{-\infty}^{\infty} \underbrace{\lvert f(t)\rvert^2}_{\text{Power}} dt = \int_{-\infty}^{\infty} \underbrace{\lvert g(\omega)\rvert^2}_{\text{Power spectrum}} d\omega$
δ-function	$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}dt$ $\delta(-x) = \delta(x)$

Notes

Fourier transform of a derivative

Fourier transform can be used to solve differential equations. Consider

$$\mathfrak{F}[f'(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f'(t)e^{-i\omega t}dt \qquad (1)$$

where $f'(t) = df/dt$. Integrating by parts, we obtain

$$\mathfrak{F}[f'(t)] = \frac{e^{-i\omega t}}{\sqrt{2\pi}}f(t)|_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \qquad (2)$$

Since $f(t)$ vanishes as $t \to \pm\infty$, the first term vanishes and we have

$$\mathfrak{F}[f'(t)] = i\omega\mathfrak{F}[f(t)] \qquad (3)$$

Can show that for n^{th} derivative $\mathfrak{F}[f^n(t)] = (i\omega)^n\mathfrak{F}[f(t)]$

Notes

Fourier Transform - Symmetry properties

The Fourier transform obeys certain symmetries. Consider the Fourier transform and its complex conjugate:

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$g^*(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(t)e^{i\omega t}dt$$

If $f(t)$ is real ($f^*(t) = f(t)$), then

$$g^*(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i(-\omega)t}dt = g(-\omega)$$

Notes

Fourier Transform - Symmetry properties

$g^*(\omega) = g(-\omega)$ means the real part of the transform $Re(g(\omega))$ is even, while $Im(g(\omega))$ is odd:

$$\begin{aligned} g(\omega) &= Re(g(\omega)) + iIm(g(\omega)) \\ g(-\omega) &= Re(g(-\omega)) + iIm(g(-\omega)) \\ g^*(\omega) &= Re(g(\omega)) - iIm(g(\omega)) \end{aligned}$$

Conversely, if $f(t)$ is purely imaginary, then

$$g^*(\omega) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i(-\omega)t} dt = -g(-\omega)$$

Hence, $g(-\omega) = -g^*(\omega)$, which means that $Re(g(\omega))$ is odd, while $Im(g(\omega))$ is even.

Notes

Fourier Transform - Symmetry properties

Let $f(t)$ be even: $f(-t) = f(t)$, then can use scaling property to show:

$$\begin{aligned} \mathfrak{F}[f(-t)] &= \mathfrak{F}[f((-1)t)] = \frac{1}{|-1|} g(\omega/(-1)) \\ &= g(-\omega) \end{aligned}$$

But,

$$\mathfrak{F}[f(-t)] = \mathfrak{F}[f(t)] = g(\omega)$$

Therefore, $g(\omega) = g(-\omega)$. The transform is even too. Similarly, can show that if $f(t)$ is odd, then $g(\omega)$ is odd as well.

Notes

Fourier Transform - Symmetry properties

The symmetry properties of the Fourier transform can be summarized as follows:

$f(t)$ real	$Re(g(\omega))$ even and $Im(g(\omega))$ odd
$f(t)$ imaginary	$Re(g(\omega))$ odd and $Im(g(\omega))$ even
$f(t)$ even	$g(\omega)$ even
$f(t)$ odd	$g(\omega)$ odd
$f(t)$ real and even	$g(\omega)$ real and even
$f(t)$ real and odd	$g(\omega)$ imaginary and odd
$f(t)$ imaginary and even	$g(\omega)$ imaginary and even
$f(t)$ imaginary and odd	$g(\omega)$ real and odd

Notes

Fourier Series vs. Fourier transform

Let's compare Fourier series, complex representation of Fourier series and Fourier transform:

Consider $f(t) = \sin(\omega t)$, where $\omega = 2\pi/T$.

What are its real Fourier coefficients?

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)), \qquad (4)$$

By inspection, $a_n = 0$, $b_1 = 1$. $b_n = 0$ for $n \neq 1$.

Notes

Now let's find the Fourier coeffcients of $f(t) = \sin(\omega t)$ for the complex representation of the Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \quad (5)$$

where,

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega t} dt \quad (6)$$

Substituting $f(t)$, we obtain

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} \sin(\omega t)(\cos(n\omega t) - i\sin(n\omega t)) dt \\ &= \frac{1}{T}\left(\underbrace{\int_{-T/2}^{T/2} \sin(\omega t)\cos(n\omega t) dt}_{=0} - i\int_{-T/2}^{T/2} \sin(\omega t)\sin(n\omega t)) dt\right) \end{aligned}$$

Notes

$$\begin{aligned} c_n &= -\frac{i}{2\pi}\underbrace{\int_{-\pi}^{\pi} \sin(x)\sin(nx)) dx}_{\pi\delta_{1n}} \\ &= \begin{cases} -\frac{i}{2} & n = 1 \\ +\frac{i}{2} & n = -1 \end{cases} \end{aligned}$$

Hence, $c_1 = -i/2$ and $c_{-1} = i/2$, all other c_n are zero.
Note: A single harmonic (sin or cos) is represented by **two** Fourier coefficients in the complex Fourier series.

Notes

Now consider the Fourier transform of $f(t) = \sin(\Omega t)$. Rewrite as $f(t) = \frac{e^{i\Omega t} - e^{-i\Omega t}}{2i}$:

$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{i\Omega t} - e^{-i\Omega t}}{2i} e^{-i\omega t} dt \\ &= \frac{2\pi}{2i\sqrt{2\pi}}\left(\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\Omega-\omega)t} dt}_{\delta(\Omega-\omega)} - \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\Omega+\omega)t} dt}_{\delta(-(\Omega+\omega))=\delta(\Omega+\omega)}\right) \\ &= \frac{i\sqrt{2\pi}}{2}\left(\delta(\Omega+\omega) - \delta(\Omega-\omega)\right) \end{aligned}$$

Note that $\delta(x) = \delta(-x)$.

Notes

Therefore,

$$\mathfrak{F}[\sin(\Omega t)] = \frac{i\sqrt{2\pi}}{2}\left(\delta(\Omega+\omega) - \delta(\Omega-\omega)\right)$$

yields two δ-function peaks at $\omega = \pm\Omega$ with imaginary amplitudes. This is analogous to the two complex Fourier coefficients we obtained earlier.

Similarly, one can obtain

$$\mathfrak{F}[\cos(\Omega t)] = \frac{\sqrt{2\pi}}{2}\left(\delta(\Omega+\omega) + \delta(\Omega-\omega)\right)$$

Notes

What is the Fourier transform of a constant function $f(t) = C$?

$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C e^{-i\omega t} dt \\ &= \frac{2\pi C}{\sqrt{2\pi}} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} dt}_{=\delta(-\omega)=\delta(\omega)} \\ &= C\sqrt{2\pi}\delta(\omega) \end{aligned}$$

δ-function peak at $\omega = 0$. Zero frequency mode.

Notes

Fourier transform of periodic functions

Fourier transform can operate on non-periodic, but also periodic functions, which we can express in terms of Fourier series. Let $f(t)$ be a periodic function with angular frequency Ω:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\Omega t) + b_n \sin(n\Omega t)\right),$$

Therefore,

$$\mathfrak{F}[f(t)] = \mathfrak{F}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\Omega t) + b_n \sin(n\Omega t)\right)\right],$$

Now use linearity of transform

$$\mathfrak{F}[f(t)] = \mathfrak{F}[a_0/2] + \sum_{n=1}^{\infty} \left(a_n \mathfrak{F}[\cos(n\Omega t)] + b_n \mathfrak{F}[\sin(n\Omega t)]\right)$$

Notes

Fourier transform of periodic functions

$$\begin{aligned} \mathfrak{F}[f(t)] &= \frac{a_0\sqrt{2\pi}}{2}\delta(\omega) \\ &+ \frac{\sqrt{2\pi}}{2} \sum_{n=1}^{\infty} a_n \left(\delta(n\Omega + \omega) + \delta(n\Omega - \omega)\right) \\ &+ \frac{i\sqrt{2\pi}}{2} \sum_{n=1}^{\infty} b_n \left(\delta(n\Omega + \omega) - \delta(n\Omega - \omega)\right) \end{aligned}$$

Therefore, Fourier transforms of periodic functions yield a sum of δ-functions located at integer multiples of the frequency Ω.

Notes

Gaussian peak:

$f(t) = e^{-\alpha t^2}$

$$\begin{aligned} \mathfrak{F}[f(t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(t^2 + \frac{i\omega}{\alpha} t + \left(\frac{i\omega}{2\alpha}\right)^2 - \left(\frac{i\omega}{2\alpha}\right)^2\right)} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha(t + \frac{i\omega}{2\alpha})^2} e^{\alpha\left(\frac{i\omega}{2\alpha}\right)^2} dt \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha}} \underbrace{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx}_{=\sqrt{\pi/\alpha}} \\ &= \frac{1}{\sqrt{2\alpha}} e^{-\frac{\omega^2}{4\alpha}} \end{aligned}$$

Notes

Table of common Fourier transforms

$f(t)$	$g(\omega)$
$\delta(t)$	$\frac{1}{\sqrt{2\pi}}$
constant C	$C\sqrt{2\pi}\delta(\omega)$
$\sin(\Omega t)$	$i\sqrt{2\pi}\frac{\delta(\omega+\Omega)-\delta(\omega-\Omega)}{2}$
$\cos(\Omega t)$	$\sqrt{2\pi}\frac{\delta(\omega+\Omega)+\delta(\omega-\Omega)}{2}$
Gaussian ($\alpha > 0$): $e^{-\alpha t^2}$	$\frac{1}{\sqrt{2\alpha}}e^{-\omega^2/(4\alpha)}$
Exponential ($\alpha > 0$): $e^{-\alpha\|t\|}$	$\sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+\omega^2}$

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