Lecture 2: Principles of Magnetic Sensing

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1. Basic Concepts in Magnetism
2. Sensor Principles
2.1 Basic concepts in Magnetism

Magnetic field sources are

- distributions of electric current (including moving charged particles)
- time-varying electric fields
- permanently magnetized material
Magnetic fields

Two sources of $H$ - currents

Biot-Savart Law

$$\delta H = -\frac{1}{4\pi} \frac{r \times j}{|r^3|} \delta V$$

$$\delta H = -\frac{1}{4\pi} I \frac{r \times \delta \ell}{|r^3|}$$

Unit of $H$ - Am$^{-1}$

Right-hand corkscrew
Scalar product

\[ m.r \]

\[ m_x x + m_y y + m_z z \]

\[ \nabla \cdot B \]

\[ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \]

The ‘divergence’ of \( B \)

\[ \nabla = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \]
Vector product

\[ r \times j \rightarrow r \times j \]

Right-hand rule

\[ \nabla \times B \]

The 'curl' of \( B \)

\[
\begin{vmatrix}
 e_x & e_y & e_z \\
x & y & z \\
j_x & j_y & j_z
\end{vmatrix}
\]

\[
\begin{vmatrix}
 e_x & e_y & e_z \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
B_x & B_y & B_z
\end{vmatrix}
\]

\[
(y_j - z_j)e_x - (x_j - z_j)e_y + (x_j - y_j)e_z \\
(\partial B_z/\partial y - \partial B_y/\partial z)e_x - (\partial B_z/\partial x - \partial B_x/\partial z)e_y + \\
+ (\partial B_y/\partial x - \partial B_x/\partial y)e_z
\]
In a steady state (no time-dependent electric field)

\[ \nabla \times \mathbf{H} = j \]

\[ \int (\nabla \times \mathbf{H}) \, d\mathcal{A} = \int j \, d\mathcal{A} \quad \text{Stokes theorem} \rightarrow \]

\[ \oint \mathbf{H} \cdot d\mathbf{l} = I \]

\[ H = I / 2\pi r \]

The field at a distance 5cm from a wire carrying a current of 1 A is \( \sim 3 \text{ A m}^{-1} \)
In free space $B = \mu_0 H$

Unit of B - Tesla
Unit of $\mu_0 \ T/\text{Am}^{-1}$

$\mu_0 = 4\pi \times 10^{-7} \ T/\text{Am}^{-1}$

$1 \ T = 10^7/4\pi \approx 800,000 \ \text{Am}^{-1}$

$B = \mu_0 I/2\pi r$

The field at a distance 5cm from a wire carrying a current of 1 A is $4 \ \mu T$
Two sources of $H$ - magnets

The magnetic moment $m$ is the elementary quantity in solid state magnetism.

Define a local moment density - magnetization - $M(r,t)$ which fluctuates wildly on a sub-nanometer and a sub-nanosecond scale.

Define a mesoscopic average magnetization $M$, averaging over a few nm and $\sim 1\mu s$

$$\delta m = M \delta V$$

The continuous medium approximation

$M$ can be the spontaneous magnetization $M_s$ within a ferromagnetic domain

A macroscopic average magnetization is the domain average

$$M = \Sigma_i M_i V_i / \Sigma_i V_i$$

The mesoscopic average magnetization
The magnetic moment

Ampère: A magnetic moment \( m \) is equivalent to a current loop.

Provided the current flows in a plane

\[
m = I \mathbf{A}
\]

Units: Am^2

In general:

\[
m = (1/2) \int r \times j(r) \, d^3r
\]

where \( j \) is the current density; \( I = j \cdot \mathbf{A} \)

so \( m = 1/2 \int r \times I \, dl = I \int d\mathbf{A} = m \)

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<td>Axial vector</td>
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Magnetization

The local moment density $M$ is the magnetization

Units: $\text{A m}^{-1}$

e.g. for iron $M = 1710 \text{ kA m}^{-1}$; for BaFe$_{12}$O$_{19}$ $M = 380 \text{ kA m}^{-1}$

e.g. for a 2.5 cc BaFe$_{12}$O$_{19}$ fridge magnet ($M = 380 \text{ kA m}^{-1}$, $V = 2.5 \times 10^{-6} \text{ m}^3$),

$m \approx 1 \text{ A m}^2$

Magnetization $M$ can be induced by an applied field or it can arise spontaneously within a ferromagnetic domain, $M_s$. A macroscopic average magnetization is the domain average

The equivalent Amperian current density is

\[ j_M = \nabla \times M \]
Magnetic properties:

- **Paramagnet**: The magnetic moment increases linearly with the applied magnetic field.
- **Diamagnet**: The magnetic moment is proportional to the applied magnetic field but in the opposite direction.
- **Ferromagnet**: Exhibits hysteresis, with a magnetic moment that does not completely align with the applied field until a certain coercivity field is reached. Parameters include $M_S$ (saturation magnetization) and $M_r$ (remanent magnetization).
The Magnetic Periodic Table

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Susceptibilities of the elements
**B, H and M**

The equation used to define $H$ is $B = \mu_0(H + M)$ i.e. $H = B/\mu_0 - M$

We call the $H$-field due to a magnet; — *stray field* outside the magnet

— *demagnetizing field*, $H_d$, inside the magnet

Units: Am$^{-1}$

The total $H$-field at any point is $H = H' + H_m$ where $H'$ is the applied field
The $B$ field - magnetic induction/magnetic flux density

\[ \nabla \cdot B = 0 \]

Significance; It is the *fundamental* magnetic field.
There are no sources or sinks of $B$ i.e. *no monopoles*.

*Magnetic vector potential* \[ B = \nabla \times A \]

The gradient of any scalar $\phi$, $\nabla \phi$ may be added to $A$ without altering $B$.

*Gauss’s theorem:* The net flux of $B$ across any closed surface is zero.

\[ \int_S B \cdot dA = 0 \]

*Magnetic flux* \[ d\Phi = B \cdot dA \]

Units: Weber (Wb)

Flux quantum $\Phi_0 = 2.07 \times 10^{15}$ Wb
The equation \( \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \) valid in static conditions gives:

**Ampere’s law** \( \int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \) for any closed path

Good for calculating the field for very symmetric current paths.

Example: the field at a distance \( r \) from a current-carrying wire

\[ B = \frac{\mu_0 I}{2\pi r} \]

\( \mathbf{B} \) interacts with any *moving* charge:

**Lorentz force** \( f = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \)
The $H$ field

Significance; The magnetization of a solid reflects the local value of $H$.

In free space $B = \mu_0 H$; $\nabla \times B = \mu_0 j_c$

In a medium $B = \mu_0 \mu_r H$ (linear isotropic media only!)

$B = \mu_0 (H + M)$ (general case)

$\nabla \times B = \mu_0 (j_c + j_M)$,

but $j_M = \nabla \times M$

Define $H = (1/\mu_0) B - M$.

hence $\nabla \times H = j_c$ This is useful. We cannot measure $j_M$

$\oint H.\text{d}l = I_c$

Ampere’s law for $H$
The \( \mathbf{H} \) field

Significance; The magnetization of a solid reflects the local value of \( \mathbf{H} \).

In free space \( \mathbf{B} = \mu_0 \mathbf{H} \).

\[
\nabla \times \mathbf{B} = \mu_0 (j_c + j_M), \quad j_M = \nabla \times \mathbf{M} \text{ hence } \nabla \times \mathbf{H} = j_c
\]

\[\int \mathbf{H} \cdot d\mathbf{l} = I_c\]

Coulomb approach to calculate \( \mathbf{H} \)

\( \mathbf{H} \) has sources and sinks associated with nonuniform magnetization

\[
\nabla \cdot \mathbf{H} = - \nabla \cdot \mathbf{M}
\]

Imagine \( \mathbf{H} \) due to a distribution of magnetic charges \( q_m \) (Am)

Field of a point ‘charge’ \( H = q_m e / 4\pi r^2 \)

Magnetization distribution is replaced by

- surface charge distribution \( \sigma_m = \mathbf{M} \cdot \mathbf{e}_n \)
- volume charge distribution \( \rho_m = - \nabla \cdot \mathbf{M} \)
Magnetic scalar potential

When $H$ is due only to magnets i.e. $\nabla \times H = 0$
we can define a scalar potential $\varphi_m$ (Units are Amps)
Such that

\[ H = -\nabla \varphi_m \]

The potential of charge $q_m$ is

\[ \varphi_m = q_m / 4\pi r \]

If currents are present, this cannot be done.

Poisson’s equation $\nabla^2 \varphi_m = \nabla \cdot \mathbf{M}$
2.4 Boundary conditions

Gauss’s law $\int_S \mathbf{B} \cdot d\mathbf{A} = 0$ gives that the perpendicular component of $\mathbf{B}$ is continuous. 

$$(\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{e}_n = 0$$

It follows from Ampère’s law

$$\int_{\text{loop}} \mathbf{H} \cdot d\mathbf{l} = I_c = 0$$

(there are no conduction currents on the surface) that the parallel component of $\mathbf{H}$ is continuous.

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{e}_n = 0$$

Conditions on the potentials

Since $\int_S \mathbf{B} \cdot d\mathbf{A} = \int_{\text{loop}} \mathbf{A} \cdot d\mathbf{l}$ (Stoke’s theorem)

$$(\mathbf{A}_1 - \mathbf{A}_2) \times \mathbf{e}_n = 0$$

The scalar potential is continuous

$\varphi_{m1} = \varphi_{m2}$
Hysteresis

The hysteresis loop shows the irreversible, nonlinear response of a ferromagnet to a magnetic field $M = M(H)$. It reflects the arrangement of the magnetization in ferromagnetic domains. The magnet cannot be in thermodynamic equilibrium anywhere around the open part of the curve! It reflects the arrangement of the magnetization in ferromagnetic domains. The $B = B(H)$ loop is deduced from the relation $B = \mu_0(H + M)$. 
Maxwell’s equations in a material medium

\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \cdot \mathbf{D} = \rho \]
\[ \nabla \times \mathbf{H} = \mathbf{j}_c + \frac{\partial \mathbf{D}}{\partial t} \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

Written in terms of the four fields, these are valid in any medium. In vacuum \( \mathbf{D} = \varepsilon_0 \mathbf{E}, \mathbf{H} = \mathbf{B}/\mu_0 \), 
\( \rho \) is charge density (C m\(^{-3}\)), \( \mathbf{j}_c \) is conduction current density (A m\(^{-2}\))

In magnetostatics there is no time-dependence of \( \mathbf{B}, \mathbf{D} \) or \( \rho \)

Conservation of charge \( \nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t} \). In a steady state \( \frac{\partial \rho}{\partial t} = 0 \)

Magnetostatics: \( \nabla \cdot \mathbf{j} = 0; \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{H} = \mathbf{j}_c \)

Constituent relations: \( \mathbf{j}_c = \mathbf{j}(\mathbf{E}); \quad P = P(\mathbf{E}); \quad M = M(\mathbf{H}) \)
Magnetic materials

Ferromagnets and ferrimagnets have spontaneous magnetization within a domain.

The magnetization falls with increasing temperature, first gradually, then abruptly at the Curie point $T_C$.

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<td>Fe$_3$O$_4$</td>
<td>860</td>
<td>480</td>
</tr>
<tr>
<td>BaFe$<em>{12}$O$</em>{19}$</td>
<td>740</td>
<td>380</td>
</tr>
<tr>
<td>SmCo$_5$</td>
<td>1020</td>
<td>856</td>
</tr>
<tr>
<td>Nd$<em>2$Fe$</em>{14}$B</td>
<td>588</td>
<td>1290</td>
</tr>
</tbody>
</table>
Iron Fe

The most important ferromagnetic material. Main constituent of the whole Earth, 5 wt % of crust. Usually alloyed with 6 at% Si and fabricated in 300 µm rolled laminations (isotropic or grain oriented), castings or reduced powder, Mainly used in electrical machines (motors, transformers) and magnetic circuits.
Production 5 Mt/yr for magnetic purposes (8 B€)

\[ J_s = 2.0 \, \text{T (Si-Fe)} \]
\[ M_s = 1.71 \, \text{MA m}^{-1} \, \text{(Fe)} \]
\[ T_C = 1044 \, \text{K} \, \text{(Fe)} \]
\[ K_1 = 48 \, \text{kJ m}^{-3} \, \text{(Fe)} \]
\[ \lambda_s = -8 \times 10^{-6} \]
Permalloy Fe$_{20}$Ni$_{80}$

324 pm

Multipurpose soft magnetic material, with near-zero anisotropy and magnetostriction
Sometimes alloyed with Mo, Cu …
Sputtered or electrodeposited films, sheet, powder.
Uses: magnetic recording; write heads, read heads (AMR), magnetic shields, transformer cores

$J_s = 1.0$ T $M_s = 0.8$ MA m$^{-1}$

$K_1 \approx 2$ kJ m$^{-3}$ $\lambda_s = 2 \times 10^{-6}$

$T_C = 843$ K

Compositions near Fe$_{50}$Ni$_{50}$ have larger $J_s$ but greater anisotropy
Cobalt Co

hcp; \( a = 251 \text{ pm, c} = 407 \text{ pm} \)

Highest-\( T_C \) ferromagnet, anisotropic, expensive (\( \approx 50 \text{ /kg} \)), strategic.
Useful alloying addition
Sputtered nanocrystalline thin films (with Cr, Pt, B additions) are used as media for hard discs

\[ J_s = 1.8 \text{ T} \quad M_s = 1.44 \text{ MA m}^{-1} \]
\[ K_1 = 530 \text{ kJ m}^{-3} \]
\[ T_C = 1360 \text{ K} \]
Magnetite, Fe₃O₄

Most common magnetic mineral, source of rock magnetism, main constituent of lodestones.

A ferrimagnet. with Fe²⁺ and Fe³⁺ disordered on B-site above the Verwey transition at $T_v = 120$ K, ordered below; A-B superexchange is the main magnetic interaction

$$[\text{Fe}^{3+}]_{\text{tet}} \{\text{Fe}^{2+} \text{Fe}^{3+}\}_{\text{oct}} \text{O}_4$$

$$\downarrow \uparrow \uparrow$$

$$-5 \mu_B + 4 \mu_B + 5 \mu_B = 4 \mu_B$$

A half-metal. Fe(B); $\downarrow$ electrons hop in a $t_{2g}$ band

Used as toner, and in ferrofluids. Potential for spin electronics.

$$J_s = 0.60 \, \text{T} \quad m_0 = 4.0 \, \mu_B / \text{fu}$$

$$K_1 = -13 \, \text{kJ m}^{-3} \quad \lambda_s = 40 \times 10^{-6}$$

$$T_C = 843 \, \text{K}$$
BaFe$_{12}$O$_{19}$; Hexaferrite

An hcp lattice of oxygen and Ba, with iron in octahedral (12k, 4f$_2$, 2a) tetrahedral (4f$_1$) and trigonal bipyramidal (2b) sites.

Brown ferrimagnetic insulator. All magnetic ions are Fe$^{3+}$. Also SrFe$_{12}$O$_{19}$ and La/Co substitution.

Structure is $12k \uparrow 2a \uparrow 2b \uparrow 4f_1 \downarrow 4f_2 \downarrow$

$T_C = 740$ K.

$m_0 = 20 \mu_B / fu$

Low-cost permanent magnet, the first magnet to break the ‘shape barrier’. 98% of all permanent magnets by mass are Ba or Sr ferrite. Found on every fridge door and in innumerable catches, dc motors, microwave magnetrons, ...

80g manufactured per year for everyone on earth

$J_s = 0.48$ T. $K_1 = 450$ kJ m$^{-3}$. $B_a = 1.7$ T
Samarium-cobalt SmCo$_5$

Versatile, high-temperature permanent magnet.
Cellular intergrowth with Sm$_2$Co$_{17}$ in
Sm(Co, Fe, Zr, Cu)$_{7.6}$ alloys provides
domain-wall pinning
Dense sintered oriented material.
Uses: specialised electrical drives
Expensive (≈150 €/kg)

$J_r = 1.0$ T \hspace{1cm} (BH)_{max} = 200$ kJ/m$^3$

$K_1 = 17$ MJ m$^{-3}$ \hspace{1cm} $B_a = 30$ T

$T_C = 1020$ K

R-T exchange is direct, between the 5d and 3d shells
This is antiferromagnetic; on-site coupling of 5d and 4f spins is
ferromagnetic, hence moments couple parallel for light rare
earths ($J = L - S$) and antiparallel for heavy rare earths ($J = L + S$).
Useful alloys are of Pr, Nd, Sm with Fe, Co, Ni
Neomax, $\text{Nd}_2\text{Fe}_{14}\text{B}$

tetragonal; $a = 879$ pm, $c = 1218$ pm

The highest-performance permanent magnet. Discovered in 1983 by Sagawa (sintered) and by Croat and Herbst (melt spun) Dy, Co .. substitutions
Dense sintered oriented material, melt-spun isotropic flakes.
Voice-coil actuators, spindle motors, nmr imaging, flux sources ....
Cost $\approx 30$ €/kg, Production 50 kT/yr (1.5 B€)

$J_r = 1.4$ T \hspace{1cm} (BH)_{max} = 200-400$ kJ/m$^3$
$K_I = 4.9$ MJ m$^{-3}$ \hspace{1cm} $B_a = 7.7$ T
$T_C = 878$ K
Exchange interactions.

The interaction responsible for magnetic order is exchange. Basically it is a Coulomb interaction between the charges of electrons on adjacent ions 1, 2, subject to the symmetry constraints of quantum mechanics. It written in terms of their spins.

*Heisenberg-Dirac Hamiltonian* \( \mathcal{H} = -2J \mathbf{S}_1 \cdot \mathbf{S}_2 \)

\[ J > 0, \text{ ferromagnetic} \]

\[ J < 0, \text{ antiferromagnetic.} \]

Curie or Néel temperature \( T_c \approx 2Z J S(S+1)/3k_B \)
Exchange interactions.

The magnetic coupling in a ferromagnet can be represented by a ‘magnetic stiffness’ \( A \)

\[
E_{ex} = \int A(\nabla e_M)^2 d^3r
\]

\[
e_M = \frac{M(r)}{M_s} \quad (\theta, \phi)
\]

\[
E_{ex} = \int A[(\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2] d^3r
\]

\[
(\nabla e_{Mx})^2 + (\nabla e_{My})^2 + (\nabla e_{Mz})^2
\]

\[
l_{ex} = \sqrt{\frac{A}{\mu_0 M_s^2}} \quad \text{Exchange length}
\]

\[
A \approx 10 \text{ pJ m}^{-1}
\]

\[
l_{ex} \approx 2 - 3 \text{ nm}
\]
**Demagnetizing field**

The H-field in a magnet depends $\mathbf{M}(\mathbf{r})$ and on the shape of the magnet. $\mathbf{H}_d$ is uniform only in the case of a *uniformly-magnetized ellipsoid*.

\[
(H_d)_i = -\mathcal{N}_{ij} M_j \quad i,j = x,y,z
\]

\[
\mathcal{N}_x + \mathcal{N}_y + \mathcal{N}_z = 1
\]

**Demagnetizing factors for some simple shapes**

- Long needle, $\mathbf{M}$ parallel to the long axis: 0
- Long needle, $\mathbf{M}$ perpendicular to the long axis: 1/2
- Sphere, $\mathbf{M}$ in any direction: 1/3
- Thin film, $\mathbf{M}$ parallel to plane: 0
- Thin film, $\mathbf{M}$ perpendicular to plane: 1
- General ellipsoid of revolution $(a,a,c)$: $\mathcal{N}_c = (1 - 2\mathcal{N}_a)$
The shape barrier.

Daniel Bernouilli 1743

\[ N < 0.1 \]

Shen Kwa 1060

Gowind Knight 1760

New icon for permanent magnets! ⇒
Working point.
Measuring magnetization with no need for demagnetization correction

Apply a field in a direction where $\mathcal{N} = 0$

$$H = H' + H_m$$

$$(H_d)_i = -\mathcal{N}_{ij}M_j$$

$$H \approx H' - \mathcal{N}M$$
It is not possible to have a uniformly magnetized cube

When measuring the magnetization of a sample, \( H \) is the independent variable,
\[ M = M(H). \]
Response to an applied field $H'$

Susceptibility of linear, isotropic and homogeneous (LIH) materials

\[ M = \chi' H' \quad \chi' \text{ is external susceptibility} \]

\[ M = \chi H \quad \chi \text{ is internal susceptibility} \]

It follows that from \( H = H' + H_d \), dividing by \( M \), that

\[ 1/\chi = 1/\chi' - N \]

Typical paramagnets and diamagnets:

\[ \chi \approx \chi' \quad (10^{-5} \text{ to } 10^{-3}) \]

Paramagnets close to the Curie point and ferromagnets:

\[ \chi >> \chi' \quad \chi \text{ diverges as } T \rightarrow T_C \quad \text{but } \chi' \text{ never exceeds } 1/N. \]

\( \chi' = 3 \) for ferromagnetic spheres.
**Permeability**

In LIH media 

\[ B = \mu H \quad \mu = B/H \quad \text{Units: } \text{TA}^{-1}\text{m} \]

**Relative permeability**

\[ \mu_r = \frac{\mu}{\mu_0} \]

\[ B = \mu_0(H + M) \quad \text{gives} \quad \mu_r = 1 + \chi \]

\(\mu_0\) is the *permeability of free space*.

• In practice it is often easier to measure the mass of a sample than its volume. Measured magnetization is usually \(\sigma = M/\rho\), the magnetic moment per unit mass (\(\rho\) is the density).

• Likewise the mass susceptibility is defined as \(\chi_m = \chi/\rho\)

• Molar susceptibility is \(\chi_{mol} = M \chi_m / 1000\) \(M\) is the molecular weight (g/mole)
Examples.

<table>
<thead>
<tr>
<th>Susceptibility</th>
<th>Units</th>
<th>H$_2$O</th>
<th>Al</th>
<th>CuSO$_4$$\cdot$5H$_2$O</th>
<th>Gd$_2$(SO$_4$)$_3$$\cdot$8H$_2$O</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td></td>
<td>$-9.0 \times 10^{-6}$</td>
<td>$2.1 \times 10^{-5}$</td>
<td>$1.41 \times 10^{-4}$</td>
<td>$2.6 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\chi_m$</td>
<td>m$^3$ kg$^{-1}$</td>
<td>$-9.0 \times 10^{-9}$</td>
<td>$7.9 \times 10^{-9}$</td>
<td>$6.2 \times 10^{-8}$</td>
<td>$8.7 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\chi_{mol}$</td>
<td>m$^3$ mol$^{-1}$</td>
<td>$-1.62 \times 10^{-10}$</td>
<td>$2.1 \times 10^{-10}$</td>
<td>$1.57 \times 10^{-8}$</td>
<td>$6.5 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Susceptibilities of the elements
Soft and hard magnets.

The area of the hysteresis loop represents the energy loss per cycle. For efficient soft magnetic materials, this needs to be as small as possible.

For a useful hard magnet, $H_c > M_r/2$.
Any macroscopic magnet exhibiting remanence is in a thermodynamically-metastable state.
Magnetostatic energy and forces

Energy of ferromagnetic bodies

- Magnetostatic (dipole-dipole) forces are long-ranged, but weak. They determine the magnetic microstructure.

\[ M \approx 1 \text{ MA m}^{-1}, \quad \mu_0 H_d \approx 1 \text{ T}, \quad \text{hence} \quad \mu_0 H_d M \approx 10^6 \text{ J m}^{-3} \]

Products \( B.H, B.M, \mu_0 H^2, \mu_0 M^2 \) are all energies per unit volume.

- Magnetic forces do no work on moving charges \( f = q(v \times B) \)
- No potential energy associated with the magnetic force.

**Torque and potential energy of a dipole in a uniform field**

\[ \Gamma = m \times B \]

\[ \varepsilon_m = -m.B \]

**Force**

In a non-uniform field, \( f = -\nabla \varepsilon_m \) \quad \( f = m.\nabla B \)
Reciprocity theorem

The interaction of a pair of dipoles, $\varepsilon_p$, can be considered as the energy of $m_1$ in the field $B_{21}$ created by $m_2$ at $r_1$ or vice versa.

$$\varepsilon_p = -m_1 \cdot B_{21} = -m_2 \cdot B_{12}$$

So

$$\varepsilon_p = -(1/2)(m_1 \cdot B_{21} + m_2 \cdot B_{12})$$

Extending to magnetization distributions:

$$\varepsilon = -\mu_0 \int M_1 \cdot H_2 \, d^3r = -\mu_0 \int M_2 \cdot H_1 \, d^3r$$
Magnetic energy terms

Self energy of a magnet in its demagnetizing field

\[ E_m = -(1/2) \int \mu_0 H_d M d^3r \]
\[ E_m = (1/2) \int \mu_0 H_d^2 d^3r \]

Self energy of a uniformly magnetized sample

\[ E_m = (1/2) \mu_0 M^2 V \]

Energy associated with a magnetic field

\[ E_m = (1/2) \int \mu_0 H^2 d^3r \]
Energy product of a permanent magnet

Aim to maximize energy associated with the field created around the magnet, from previous slide:

\[
\frac{1}{2} \int \mu_0 H_d^2 d^3r = -\frac{1}{2} \int_V \mu_0 H_d \cdot M d^3r.
\]

Can rewrite as:

\[
\frac{1}{2} \int_0^{\infty} \mu_0 H_d^2 d^3r = -\frac{1}{2} \int_i \mu_0 H_d^2 d^3r - \frac{1}{2} \int_i \mu_0 M \cdot H_d d^3r.
\]

where we want to maximize the integral on the left. Since \( B = \mu_0 (H + M) \),

Energy product: twice the energy stored in the stray field of the magnet is

\[
-\mu_0 \int_i B \cdot H_d d^3r
\]

Optimum shape, \( \mathcal{N} = 1/2 \)
**Thermodynamics**

**First law:** \( dU = H_x dX + dQ \)

\( dQ = T dS \)

Four thermodynamic potentials;

- \( U(X,S) \) \textit{internal energy}
- \( E(H_x, S) \) \textit{enthalpy}
- \( F(X,T) = U - TS \) \textit{Helmholz free energy}
- \( G(H_x, T) = F - H_x X \) \textit{Gibbs free energy}

Magnetic work is \( H \delta B \) or \( \mu_0 H' \delta M \)

\[
\begin{align*}
\frac{dF}{dT} &= \mu_0 H' \delta M - S dT \\
\frac{dG}{dT} &= -\mu_0 M \delta H' - S dT
\end{align*}
\]

\[
\begin{align*}
S &= -(\frac{\partial G}{\partial T})_{H'} \\
\mu_0 M &= -(\frac{\partial G}{\partial H'})_T
\end{align*}
\]

Maxwell relations

\[
(\frac{\partial S}{\partial H'})_T = - \mu_0 (\frac{\partial M}{\partial T})_H \text{ etc.}
\]

\( (U, Q, F, G \text{ are in units of Jm}^{-3}) \)
**Magnetostatic Forces**

Force density on a uniformly magnetized body at constant temperature

\[ F_m = - \nabla G \]

\[ F_m = \nabla (\mu_0 H' \cdot M) \quad \nabla (H' \cdot M) = (H' \cdot \nabla) M + (M \cdot \nabla) H' \]

**Kelvin force**

\[ F_m = \mu_0 (M \cdot \nabla) H' \]

General expression, for when \( M \) is dependent on \( H \) is

\[ F_m = -\mu_0 \nabla \left[ \int_0^H \left( \frac{\partial M \nu}{\partial \nu} \right)_{H,T} \, dH \right] + \mu_0 (M \cdot \nabla) H. \]

\( \nu = 1/d \quad d \) is the density
Anisotropy.

*shape, magnetocrystalline, induced, strain*

The ferromagnetic axis lies in some particular direction determined by shape or some intrinsic anisotropy related to crystal or atomic structure.

$$E_a = K_1 \sin^2 \theta + K_2 \sin^4 \theta + .....$$

The *shape* contribution is derived from the energy expression

$$E_m = (1/2)\mu_0 \mathcal{N}M^2V$$

The magnetization lies along the direction for which $\mathcal{N}$ is smallest - the axis of a long bar.

The magnetocrystalline contribution for uniaxial crystals is given by a similar expression with different $K_1$ ......
Shape anisotropy.

The *shape* contribution is derived from the energy expression

\[ E_m = \frac{1}{2}\mu_0 \mathcal{N} M^2 V \]

\( \mathcal{N} \) is the demagnetizing factor for the easy direction - the axis of a long bar. \((1/2)[1 - \mathcal{N}]\) is the demagnetizing factor for the perpendicular direction (assuming an ellipsoid)

Hence \( \Delta E_m = \frac{1}{2} \mu_0 M^2 V \{(1/2)(1 - \mathcal{N}) - \mathcal{N}\} \)

so \( K_{sh} = \frac{1}{4}\mu_0 M^2 (1 - 3\mathcal{N}) \)

The biggest it can be is \((1/4)\mu_0 M^2 J \text{ m}^{-3}\)

\(~ 2 \times 10^5 J \text{ m}^{-3}\) for \(\mu_0 M = 1 \text{ T}\)
Magnetocrystalline anisotropy

The magnetocrystalline contribution is ultimately caused by spin-orbit coupling which connects the crystal structure (electron orbits) and magnetic moment direction.

\[ E_a = K_1 \sin^2 \theta + K_2 \sin^4 \theta + K_3 \sin^6 \theta + K'_3 \sin^6 \theta \sin 6\phi, \]

\[ \text{Tetragonal:} \quad E_a = K_1 \sin^2 \theta + K_2 \sin^4 \theta + K'_2 \sin^4 \theta \cos 4\phi + K_3 \sin^6 \theta \]
\[ + K'_3 \sin^6 \theta \sin 4\phi, \]

\[ \text{Cubic:} \quad E_a = K_{1c} (\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2) + K_{2c} (\alpha_1^2 \alpha_2^2 \alpha_3^2), \]
Anisotropy due to texture

Directional order of atomic constituents in a binary alloy can be induced by deposition on a magnetic field or by post depositional annealing.
Domains

Micromagnetic energy, wall width and structure

Domains form to minimize the dipolar energy $E_d$

$$E_d = -\frac{1}{2} \int \mu_0 \mathbf{H}_d \cdot \mathbf{M} \, d^3 r$$
Micromagnetic energy

\[ E_{tot} = E_{ex} + E_K + E_d + E_m + E_{stress} + E_{ms} \]

\[ E_{tot} = \int \left\{ A \left( \nabla \mathbf{M} / M_s \right)^2 - K_1 (\mathbf{e}_m \cdot \mathbf{e}_n)^2 - \cdots - \frac{1}{2} \mu_0 \mathbf{M} \cdot \mathbf{H}_d - \mu_0 \mathbf{M} \cdot \mathbf{H} \right\} d^3 r \]

exchange  anisotropy (leading term)  magnetostatic  Zeeman

Minimizing \( E_{tot} \) gives the mesoscale magnetic structure of the sample (monodomain, multidomain, or vortex)
Domain walls

Minimizing $E_{\text{tot}}$ for two oppositely magnetized regions gives the domain wall width

$$\delta_w = \pi \sqrt{A/K_1}$$

$A \sim 10 \text{ pJ m}^{-1}$; $K \sim 10^5 \text{ Jm}^{-3}$; $\delta_w \sim 30 \text{ nm}$. 
Superparamagnetism

Fine particles, blocking

\[ E_a = K \sin^2 \theta \]
\[ \Delta = KV \]

the ratio \( \Delta/kT \) is critical

Néel relaxation \( \tau = \tau_0 \exp(\Delta/kT) \)

Here \( \tau_0 \) is an inverse attempt frequency, \( 10^{-9} \text{s}^{-1} \)

If \( \Delta/kT = 25 \), \( \tau = 70 \text{ s} \).
2.2 Sensor Principles

Flux - Faraday’s law
MR - Lorentz force
Hall effect - Lorentz force
AMR - Spin-orbit scattering
GMR - spin accumulation
TMR - spin-dependent tunelling
MO - Faraday effect
SQUID - Flux quantization
NMR - magnetic resonance
GMI - high-frequency permeability
<table>
<thead>
<tr>
<th>Sensor</th>
<th>Principle</th>
<th>Detects</th>
<th>Frequency</th>
<th>Field (T)</th>
<th>Noise</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coil</td>
<td>Faraday’s law</td>
<td>dΦ/dt</td>
<td>10^{-3} - 10^{9}</td>
<td>10^{-10} - 10^{-2}</td>
<td>100 nT</td>
<td>bulky, absolute</td>
</tr>
<tr>
<td>Fluxgate</td>
<td>saturation</td>
<td>H</td>
<td>dc - 10^{3}</td>
<td>10^{-10} - 10^{-3}</td>
<td>10 pT</td>
<td>bulky</td>
</tr>
<tr>
<td>Hall probe</td>
<td>Lorentz f’ce</td>
<td>B</td>
<td>dc - 10^{5}</td>
<td>10^{-5} - 10</td>
<td>100 nT</td>
<td>thin film</td>
</tr>
<tr>
<td>MR</td>
<td>Lorentz f’ce</td>
<td>B^2</td>
<td>dc - 10^{5}</td>
<td>10^{-2} - 10</td>
<td>10 nT</td>
<td>thin film</td>
</tr>
<tr>
<td>AMR</td>
<td>spin-orbit int</td>
<td>H</td>
<td>dc - 10^{7}</td>
<td>10^{-9} - 10^{-3}</td>
<td>10 nT</td>
<td>thin film</td>
</tr>
<tr>
<td>GMR</td>
<td>spin accum. n</td>
<td>H</td>
<td>dc - 10^{9}</td>
<td>10^{-9} - 10^{-3}</td>
<td>10 nT</td>
<td>thin film</td>
</tr>
<tr>
<td>TMR</td>
<td>tunelling</td>
<td>H</td>
<td>dc - 10^{9}</td>
<td>10^{-9} - 10^{-3}</td>
<td>1 nT</td>
<td>thin film</td>
</tr>
<tr>
<td>GMI</td>
<td>permability</td>
<td>H</td>
<td>dc - 10^{4}</td>
<td>10^{-9} - 10^{-2}</td>
<td></td>
<td>wire</td>
</tr>
<tr>
<td>MO</td>
<td>Kerr/Faraday</td>
<td>M</td>
<td>dc - 10^{5}</td>
<td>10^{-9} - 10^{-2}</td>
<td>1 pT</td>
<td>bulky</td>
</tr>
<tr>
<td>SQUID lt</td>
<td>flux quanta</td>
<td>Φ</td>
<td>dc - 10^{9}</td>
<td>10^{-15} - 10^{-2}</td>
<td>1 fT</td>
<td>cryogenic</td>
</tr>
<tr>
<td>SQUID ht</td>
<td>flux quanta</td>
<td>Φ</td>
<td>dc - 10^{4}</td>
<td>10^{-15} - 10^{-2}</td>
<td>30 fT</td>
<td>cryogenic</td>
</tr>
<tr>
<td>NMR</td>
<td>resonance</td>
<td>B</td>
<td>dc - 10^{3}</td>
<td>10^{-10} - 10</td>
<td>1 nT</td>
<td>Very precise</td>
</tr>
</tbody>
</table>
2.2.1 Inductive sensors

Inductive sensors detect an emf in a coil proportional to the rate of change of flux, according to Faraday’s law:

\[ E = -\frac{d\Phi}{dt} \]

They provided an absolute measurement of

\[ B = \frac{\Phi}{nA} \]

as in a search coil with an integrating voltmeter, or the rotating coil gaussmeter.

Inductive read/write heads were widely used until 1990 in magnetic recording.
Magnetic circuits

\[ \nabla B = 0 \]

Assuming no flux leakage

\[ B_m A_m = B_g A_g \]  \hspace{1cm} (1)

Assuming ideal soft material \( \mu = \infty \)

Ampere’s law \( \int H \cdot dl = 0 \)

\[ \frac{H_m}{H_g} = -\frac{H_g}{H_g} \]  \hspace{1cm} (2)

Multiplying \( B_m H_m V_m = -B_g^2 V_g / \mu_0 \)

Dividing \(-B_m / H_m = \mu_0 A_g l_m / A_m l_g \)

The permeance coefficient

\[ P = 1/R_m \]

reluctance

Figure: A simple magnetic circuit, and its electrical equivalent, with and without losses.
Ideal $M(H)$ loop

Ideal $B(H)$ loop

$B = \mu_0(H + M)$

$BH = \mu_0(H^2 - MH)$

$= \mu_0M^2(\mathcal{N}^2 - \mathcal{N})$

Maximizing $(BH)_{\text{max}} \rightarrow \mathcal{N} = 1/2$

Second-quadrant $\mu_0M(H)$ and $B(H)$ for an ideal permanent magnet. The working point and load line for maximum energy product are indicated.
<table>
<thead>
<tr>
<th><strong>Table</strong></th>
<th>Analogy between electric and magnetic circuits</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Field</strong></td>
<td>Electric</td>
</tr>
<tr>
<td>Potential</td>
<td>$E$ (V m$^{-1}$)</td>
</tr>
<tr>
<td>Current/flux density</td>
<td>$\varphi_e$ (V)</td>
</tr>
<tr>
<td>Potential difference</td>
<td>$j$ (A m$^{-2}$)</td>
</tr>
<tr>
<td>Continuity condition</td>
<td>$\nabla \cdot j = 0$</td>
</tr>
<tr>
<td>Linear response law</td>
<td>$j = \sigma E$</td>
</tr>
<tr>
<td>Current/flux</td>
<td>$I$ (A)</td>
</tr>
<tr>
<td>Resistance/reluctance</td>
<td>$R = \varphi_e / I$ (Ω)</td>
</tr>
<tr>
<td>for a cylinder of section $A$ and length $l$</td>
<td>$R = l / A\sigma$</td>
</tr>
<tr>
<td>Conductance/permeance</td>
<td>$G = 1 / R$</td>
</tr>
</tbody>
</table>
2.2.2 Fluxgates

Fluxgates depend on the nonlinear saturation of the magnetization of a soft magnetic core. Two identical cores (or a single toroidal core) have oppositely wound ac field windings. A parallel applied field leads to saturation of one of the cores, producing an ac signal linear in H.

Fluxgates are bulky but sensitive, reliable and impervious to radiation. Used, for example, in space.
2.2.3 Hall sensors

Effect discovered by Edwin Hall in 1879

Hall voltages linear in field are produced in semiconductor plates, especially Si and in 2-deg GaAs/GaAlAs structures. These are four-terminal devices, and the current source and high-gain amplifier are often integrated on a chip. Used for secondary field measurements — each probe must be calibrated — and as proximity sensors. About a billion are produced each year.
2.2.4 Classical magnetoresistance

The simplest Lorentz force device is a semiconductor or semimetal which exhibits classical positive $B^2$ magnetoresistance. High-mobility semiconductors such as InAs and InSb show large effects ($\sim 100 \% \ T^{-1}$). Field is applied perpendicular to the semiconductor slab, and it is possible to achieve a desired resistance by patterning a series of metallic contacts.

The sensors are nonlinear, two-terminal devices providing a good response in large fields. They are used as position sensors in brushless dc permanent magnet motors.
2.2.5 Anisotropic magnetoresistance (AMR)

Discovered by W. Thompson in 1857

\[ \rho = \rho_0 + \Delta \rho \cos^2 \theta \]

Magnitude of the effect \( \Delta \rho / \rho < 3\% \)
The effect is usually positive; \( \rho_\parallel > \rho_\perp \)

AMR is due to spin-orbit s-d scattering

High field sensitivity is achieved in thin films of soft ferromagnetic films such as permalloy (Fe\textsubscript{20}Ni\textsubscript{80}).
Planar Hall effect

Planar Hall effect is a variant of AMR; \( \rho_\parallel \neq \rho_\perp \) \( \rho_\parallel \) is when \( j \parallel M \) ….

\[ E_\parallel = \rho_\parallel j_x \cos \theta \quad E_\perp = \rho_\perp j_x \sin \theta \]

Components of electric field parallel and perpendicular to the current are
\[ E_x = E_\parallel \cos \theta + E_\perp \sin \theta, \quad E_y = E_\parallel \sin \theta - E_\perp \cos \theta \]
\[ E_x = j(E_\parallel \rho_\perp + \Delta \rho \cos^2 \theta) \quad E_y = j \Delta \rho \sin \theta \cos \theta \]

Hence
\[ V_{pH} = j w \Delta \rho \sin \theta \cos \theta \]

The biggest effect is when \( \theta \) changes from 45 to 135 degrees.
2.2.6 Giant magnetoresistance.

Peter Grunberg and Albert Fert;

$10^9$ GMR sensors per year

Discovery of GMR 1988
Implementation in hard disk drives 1998
Nobel Prize 2007
GMR spin valve
Exchange-biased stack

Exchange biased GMR spin valve
TMR spin valve
Exchange-biased stack

I

Free
pinned

Af F1 Cu F2

Superparamagnetic Effect

TMR
GMR
AMR

Travelstar 6GN
Ultrastar 2XP
Spitfire
Allicat
Corsair
3390K
3380K
3380E
3390/3
3390/2
3380
3350
3370
2314
1311
2330
3340
RAMAC

Area Density, MBits/m²

10^0
10^1
10^2
10^3
10^4
10^5


1 µm²
2.2.7 Tunnel magnetoresistance.

Magnetoresistance is > 100%, 10 times as great as for GMR spin valves

\[ \text{MgO=2.50nm} \]
\[ \text{TMR=229\% (RT)} \]
\[ R=1587\Omega \ (RT) \]
\[ S=30\times30\mu m^2 \]
2.2.8 Giant magnetoimpedance.

GMI sensors are soft ferromagnetic wires (sometimes permalloy-plated copper wires) or films. An ac current is passed along the wire, and \( L \) is measured as a function of applied field. At high frequency, the skin depth < wire diameter. Permeability depends on \( f \) and \( H \).

Very high field sensitivity, \( 10^4 \% \text{ mT}^{-1} \) is achievable. Used in Wii games, three-axis compasses ….
2.2.9 Magneto-optic sensors.

Optical fibre field sensors, based on magneto-optic Faraday effect. They are bulky, and used for large fields.

Rotation sensors based on Sagnac effect.
2.2.10 Superconducting quantum interference devices.

SQUIDs detect the change of flux threading a flux-locked loop. The flux is generally coupled to the SQUID via a superconducting flux transformer. The device is sensitive to a small fraction of a flux quantum. SQUIDSSs offer ultimate field sensitivity. They generally operate with a flux-locked loop.

\[ \Phi_0 = 2 \times 10^{-15} \text{T m}^2 \]
2.2.11 Nuclear magnetic resonance.

Protons (in water for example) can be polarized by a field pulse, and then allowed to precess freely (fid) at the Larmor frequency in the field to be measured. $f_L$ in the Earth’s field is $\sim 2 \text{ kHz}$.

Rb or Cs vapor can be magnetized by optical pumping with circularly-polarized light, and the nuclear precession measured. The Rb-vapour magnetometer provides an extremely precise, absolute value of the magnitude of the field. These magnetometers have been packaged on a chip.