Chapter 2: Magnetostatics

1. The Magnetic Dipole Moment
2. Magnetic Fields
3. Maxwell's Equations
4. Magnetic Field Calculations
5. Magnetostatic Energy and Forces

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Further Reading:

• David Jiles *Introduction to Magnetism and Magnetic Materials,* Chapman and Hall 1991; 1997
A detailed introduction, written in a question and answer format.

• Stephen Blundell *Magnetism in Condensed Matter*, Oxford 2001
A new book providing a good treatment of the basics

• Amikam Aharoni *Theory of Ferromagnetism*, Oxford 2003
Readable, opinionated phenomenological theory of magnetism

• William Fuller Brown *Micromagnetism*, 1949
The classic text

1. The Magnetic Dipole Moment

The magnetic moment *m* is the elementary quantity in solid state magnetism.

Define a local moment density - magnetization - M(r,t) which fluctuates wildly on a sub-nanometer and a sub-nanosecond scale.
Define a mesoscopic average magnetization

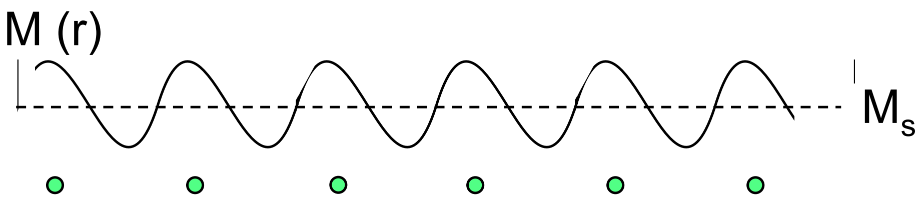
$$\delta m = M\delta V$$

The continuous medium approximation

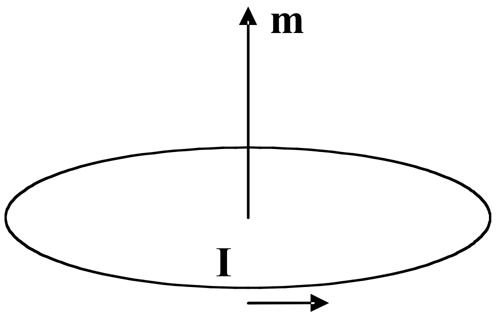
M can be the spontaneous magnetization M_s within a ferromagnetic domain

A macroscopic average magnetization is the domain average

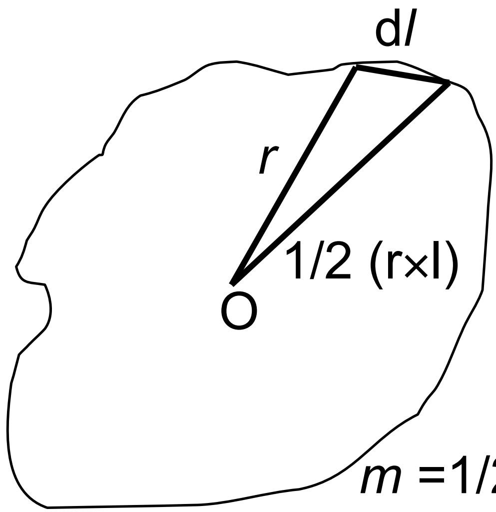
$$M = \Sigma_i M_i V_i / \Sigma_i V_i$$



The mesoscopic average magnetization



$m = IA$



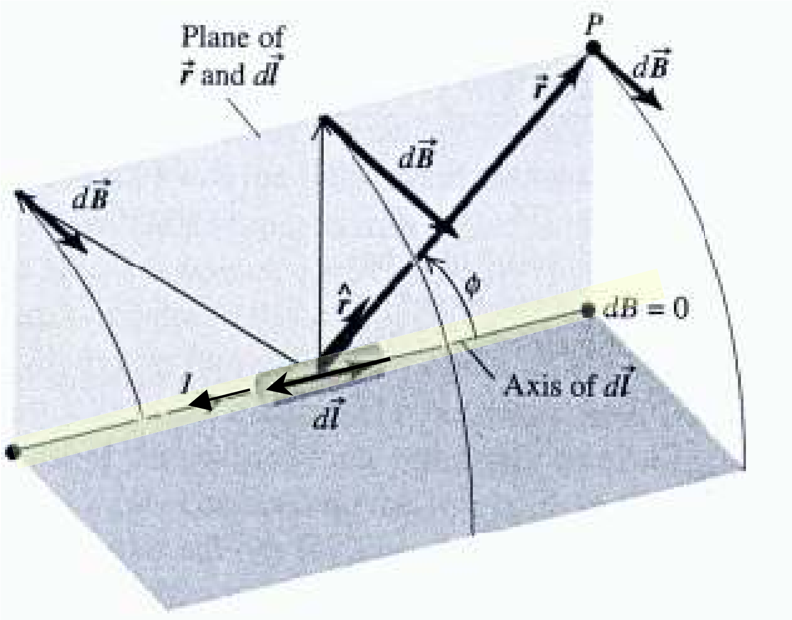
$m = 1/2 \int r \times j(r) d^3 r$

A magnetic moment m is equivalent to a current loop

$m = 1/2 \int r \times j(r) d^3 r = 1/2 \int r \times I dl = I \int dA = m$

Inversion	Space	Time
Polar vector	-j	j
Axial vector	M	-M

1.1 Field due to electric currents and magnetic moments



Biot-Savart Law

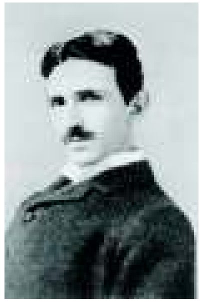
$$\delta \mathbf{B} = -\frac{\mu_0}{4\pi}\frac{\mathbf{r}\times\mathbf{j}}{|r^3|}\delta V$$

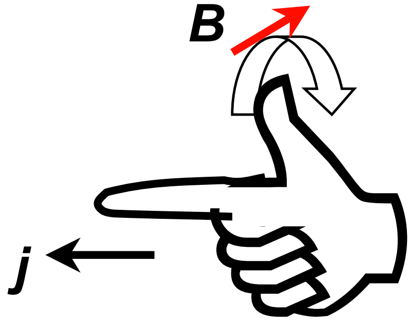
$$\delta \mathbf{B} = -\frac{\mu_0}{4\pi}I\frac{\mathbf{r}\times\delta\ell}{|r^3|}$$

Unit of B - Tesla

Unit of μ_0 T/Am^{-1}

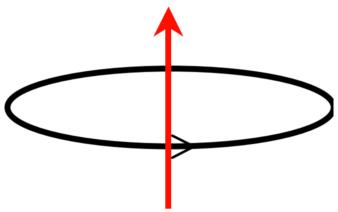
$\mu_0 = 4\pi\ 10^{-7}$ T/Am^{-1}





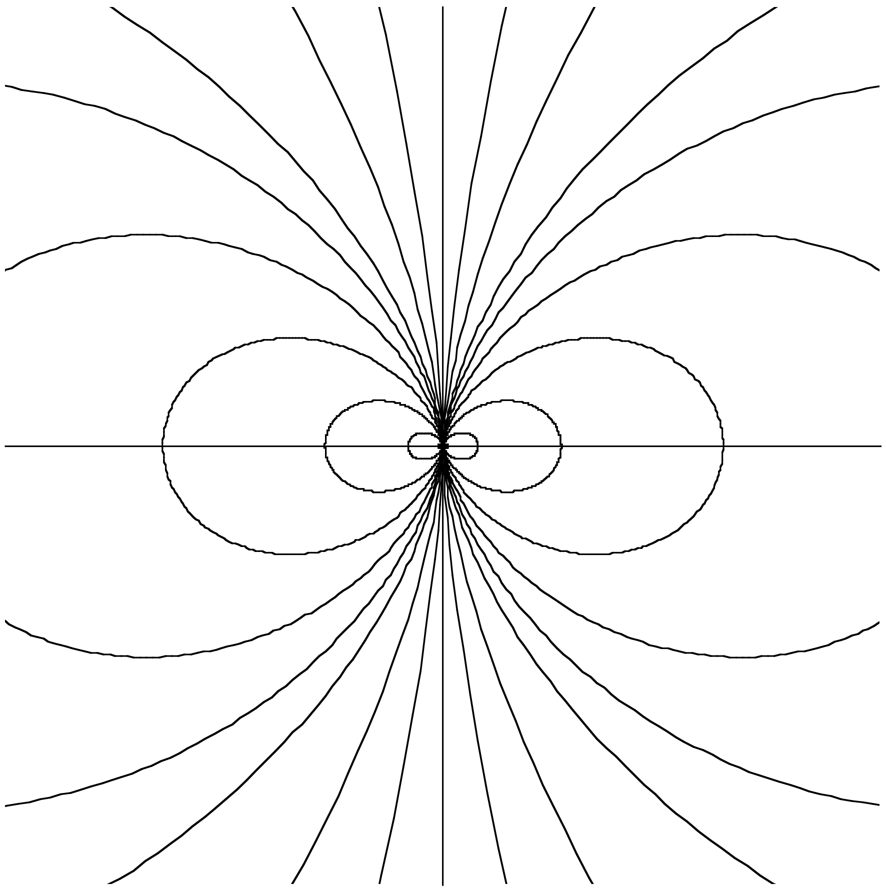
Right-hand corkscrew

1.1 Field due to electric currents and magnetic moments



$$B_0 = \frac{\mu_0 I}{2a}$$

Field at center of current loop



Dipole field far from current loop

- lines of force

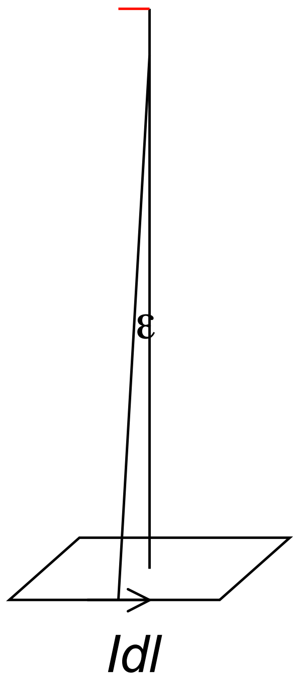
1.1 Field due to electric currents and magnetic moments

$B_A = 4(\mu_0 I dl/4\pi r^2)\sin\varepsilon$

$\sin\varepsilon = dl/2r$

A

$$B_A = 2\frac{\mu_0}{4\pi}\frac{m}{r^3}$$

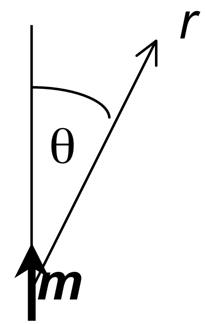


$$B_B = \frac{\mu_0}{4\pi} I\delta\ell\left\{\frac{1}{(r-\delta\ell/2)^2} - \frac{1}{(r+\delta\ell/2)^2} + \frac{2\sin\varepsilon}{r^2}\right\}$$

$$\approx -\frac{\mu_0}{4\pi}\frac{I\delta\ell}{r^2}\left\{\left(1-\frac{\delta\ell}{r}\right) - \left(1+\frac{\delta\ell}{r}\right) + \frac{\delta\ell}{r}\right\}$$

$$B_B = -\frac{\mu_0}{4\pi}\frac{m}{r^3}$$

B



At a general position, $B_r = 2\left(\frac{\mu_0 m}{4\pi r^3}\right)\cos\theta;\quad B_\theta = \left(\frac{\mu_0 m}{4\pi r^3}\right)\sin\theta;\quad B_\phi = 0$

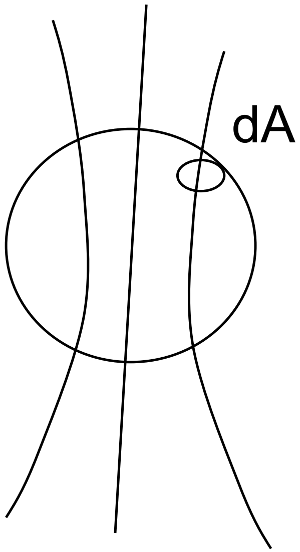
$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^5}\left\{3xz\,\mathbf{e}_x + 3yz\,\mathbf{e}_y + (3z^2 - r^2)\mathbf{e}_z\right\}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left\{3\frac{(\mathbf{m}.\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right\}$$

2. Magnetic Fields

2.1 The B-field

$\nabla . \boldsymbol{B} = 0$



$$\int_S \boldsymbol{B} . d\boldsymbol{A} = 0$$

Gauss's theorem

Flux: $d\Phi = BdA$

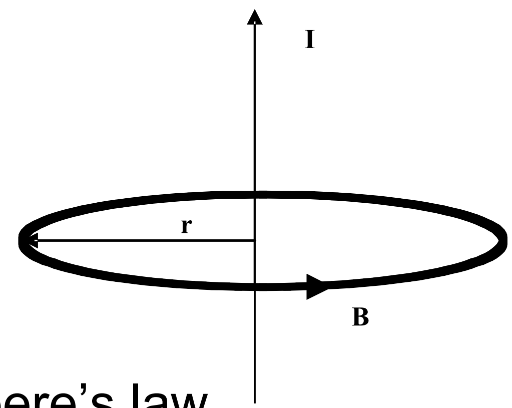
Unit Weber (Wb)

Flux quantum $\Phi_0 = 2.07\ 10^{15}$ Wb

The B-field

Sources of ***B***

- electric currents in conductors
- moving charges
- magnetic moments
- time-varying electric fields. Not in *magnetostatics*



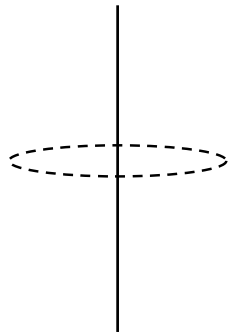
$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$

$$\oint \mathbf{B} \cdot d\ell = \mu_0 I$$

Ampere's law.

Good for very symmetric current paths.

$$\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_Y & B_Z \end{vmatrix}$$



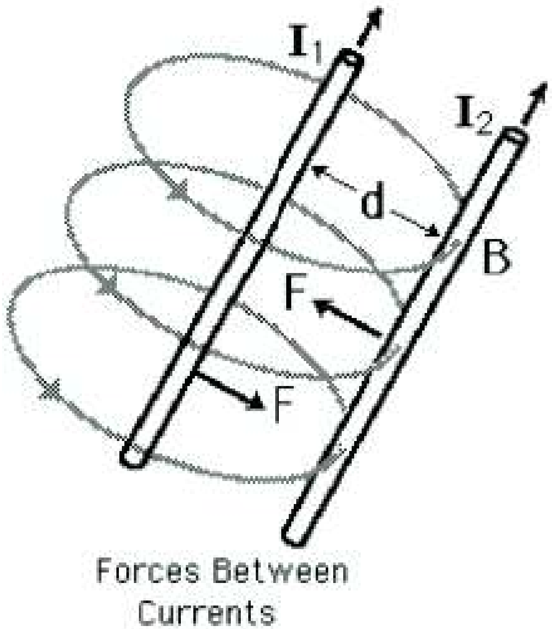
$B = \mu_0 I/2\pi r$

The B-field

Forces:

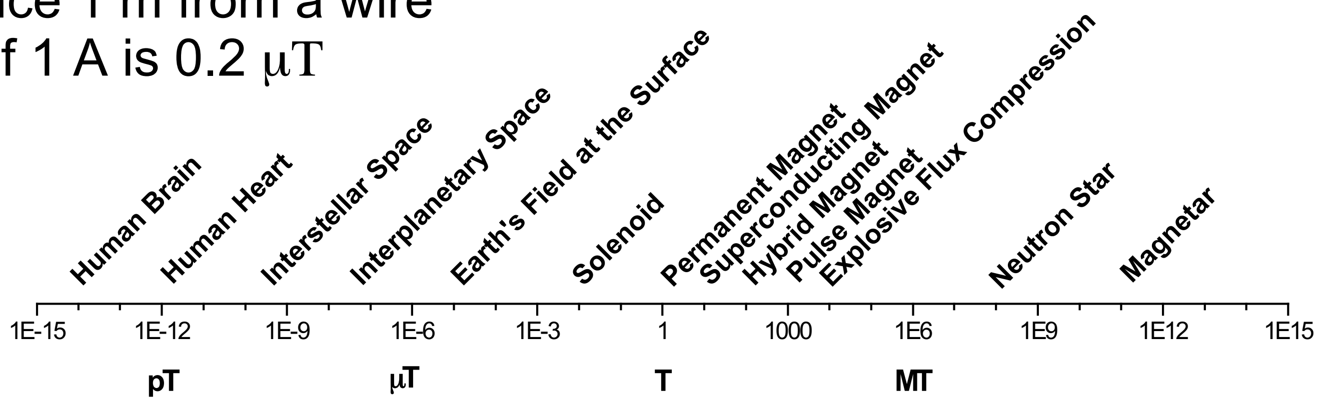
$\boldsymbol{F} = q(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B})$ Lorentz expression.

gives dimensions of *B* and *E.*

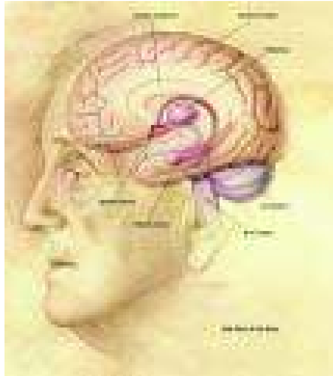


The force between two parallel wires each carrying one ampere is precisely $2\ 10^{-7}$ N m^{-1}.

The field at a distance 1 m from a wire carrying a current of 1 A is 0.2 μT



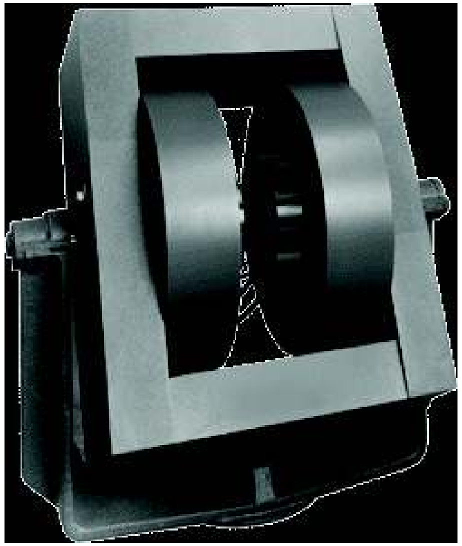
Typical values of *B*



Human brain 1 fT

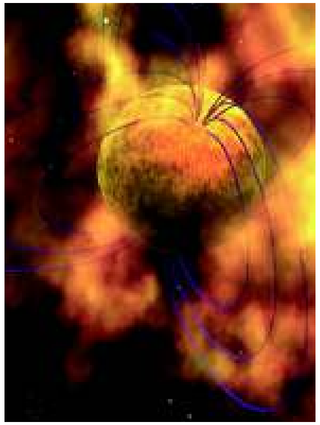
Earth 50 μT

Helmholtz coils 0.01 Am^{-}



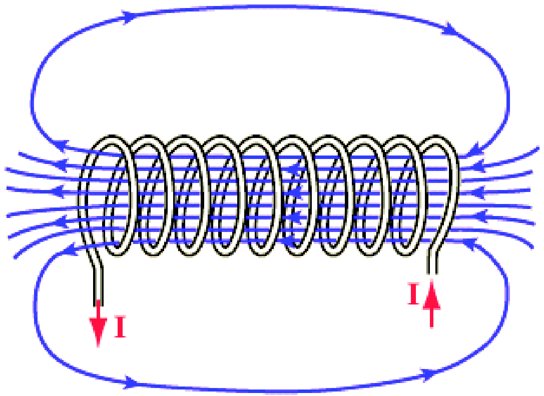
Electromagnet 1 T

Superconducting magnet 10 T

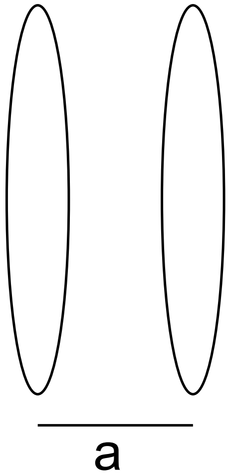


Magnetar 10^{12} T

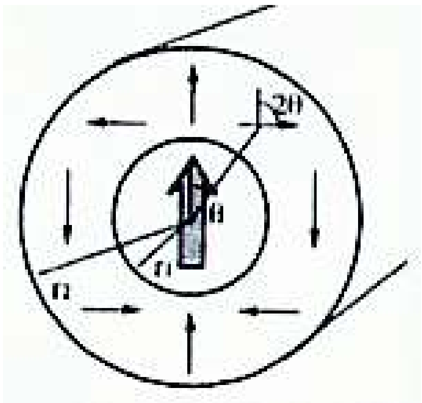
2.2 Uniform magnetic fields.



Long solenoid $B = \mu_0 n I$



Helmholtz coils $B = (4/5)^{3/2} \mu_0 N I / a$



Halbach cylinder $B = \mu_0 M \ln(r_2/r_1)$I

$$B_r = \left(\frac{\mu_0 \lambda}{4\pi r^2}\right)\cos\theta \; ; \; B_\theta = \left(\frac{\mu_0 \lambda}{4\pi r^2}\right)\sin\theta \; ; \; B_z = 0$$

2.3 The H- field.

In free space $B = \mu_0 H$

$$\nabla \times B = \mu_0(j_c + j_m)$$

The relation between $\boldsymbol{j}_m$ and $\boldsymbol{M}$ is simply

$$\boldsymbol{j}_m = \nabla \times \boldsymbol{M} \qquad (2.26)$$

In order to retain Ampère's law in a practically useful form, we will *define* a new field

$$\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M} \qquad (2.27)$$

so that $\nabla \times \boldsymbol{H} = \nabla \times \boldsymbol{B}/\mu_0 - \nabla \times \boldsymbol{M}$, and hence from (2.24) and (2.25)

$$\nabla \times \boldsymbol{H} = \boldsymbol{j}_c \qquad (2.28)$$

In integral form, Ampère's law for the H-field produced by conduction currents is

$$\oint \boldsymbol{H}.d\ell = I_c \qquad (2.29)$$

$\nabla . H = - \nabla . M$ *Coulomb approach to calculate H*

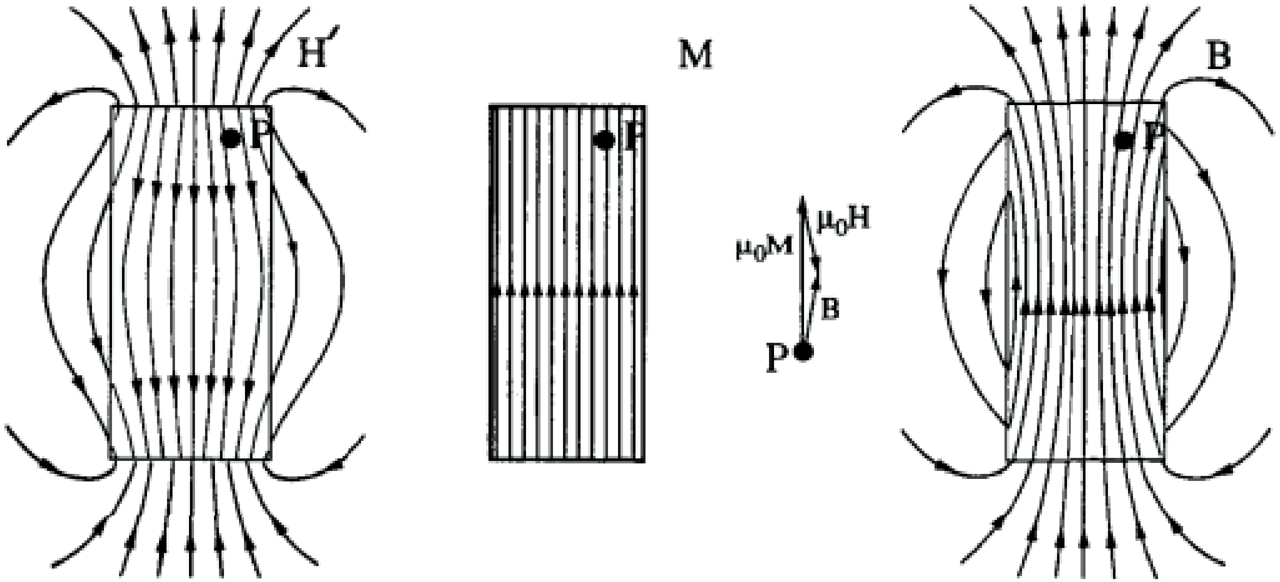
$\boldsymbol{H} = q_m \boldsymbol{r}/4\pi r^3$ q_m is magnetic charge

The H- field.

$$H = H_c + H_m$$

H_m is the *stray field* outside the magnet and the *demagnetizing field* inside it

$$B = \mu_0(H + M)$$



2.4 The demagnetizing field

The H-field in a magnet depends on the magnetization $\mathbf{M}(\mathbf{r})$ and on the *shape* of the magnet. $\mathbf{H}_d$ is uniform in the case of a *uniformly-magnetized ellipsoid*. Tensor relation:

$$\mathbf{H}_d = - N \mathbf{M}$$

A constraint on the values of N when $\mathbf{M}$ lies along one of the principal axes, x, y, z, is

$$N_x + N_y + N_z = 1$$

- It is common practice to use a demagnetizing factor to obtain approximate internal fields in samples of other shapes (bars, cylinders), which may not be quite uniformly magnetized.

- *Examples.*

	N
Long needle, **M** parallel to the long axis, a	0
Long needle, **M** perpendicular to the long axis	1/2

Sphere	1/3

Thin film, **M** parallel to plane	0
Thin film, **M** perpendicular to plane	1

Toroid, **M** perpendicular to **r**	0

General ellipsoid of revolution	$N_c = (1 - N_a)/2$

Some formulae for ellipsoids of revolution having major axes (a, a, c) with $\alpha = c/a$ are

$$\mathcal{N}_c = \frac{1}{(\alpha^2 - 1)}\left[\frac{\alpha}{\sqrt{\alpha^2 - 1}}\operatorname{arccosh}\alpha - 1\right] \qquad (2.36)$$

for prolate ellipsoids with $\alpha > 1$, and

$$\mathcal{N}_c = \frac{1}{(1 - \alpha^2)}\left[1 - \frac{\alpha}{\sqrt{1 - \alpha^2}}\arccos\alpha\right] \qquad (2.37)$$

for oblate ellipsoids with $\alpha < 1$. For nearly-spherical shapes with $\alpha \approx 1$, $\mathcal{N}_c = \frac{1}{3} - \frac{4}{15}(\alpha - 1)$.

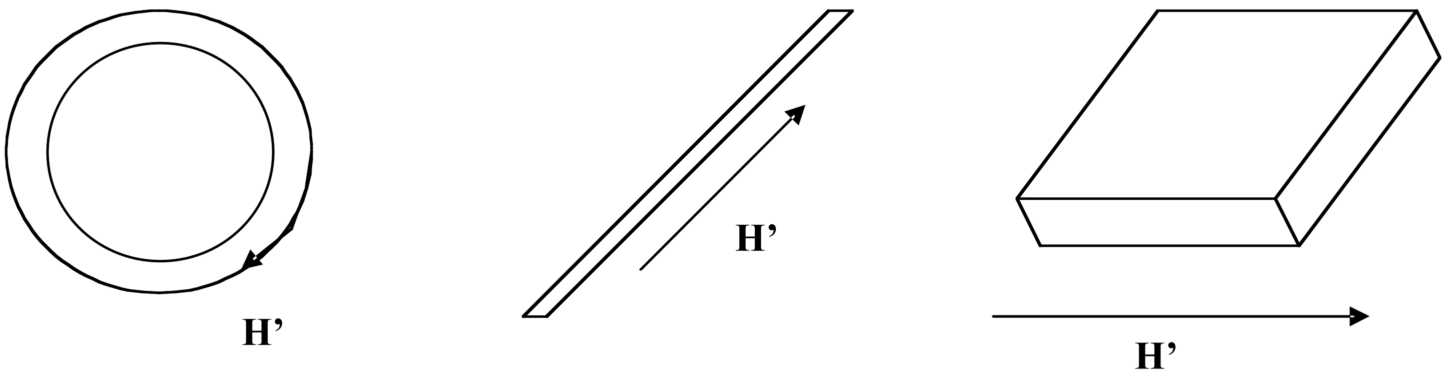
2.5 External and internal fields

$$H = H' + H_d$$

Inernal field   applied field   demag field

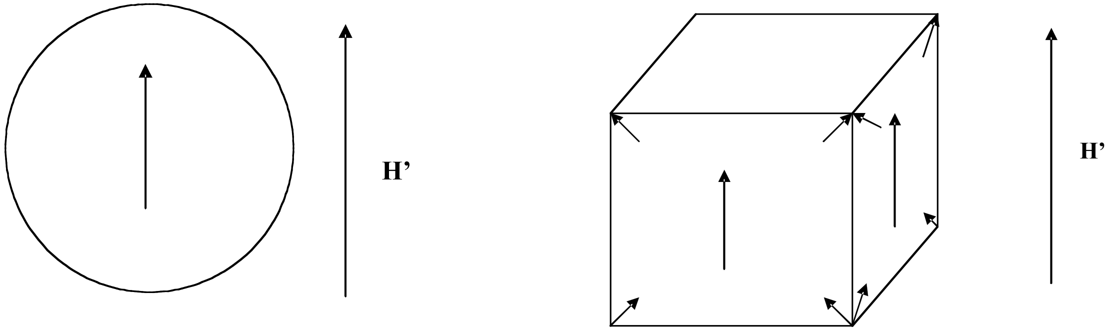
$$H \approx H' - N\,M$$

For a powder sample $N_p = (1/3) + f(N - 1/3)$ f is the packing fraction



Ways of measuring magnetization with no need for a demag correction

toroid   long rod   thin film



Magnetization of a sphere, and a cube

The state of magnetization of a sample depends on ***H***, ie **M** = ***M***(***H***). ***H*** is the independent variable.

2.6 Susceptibility and permeability

Simple materials are linear, isotropic and homogeneous (LIH)

$\boldsymbol{M} = \chi' \boldsymbol{H}'$ χ' is external susceptibility

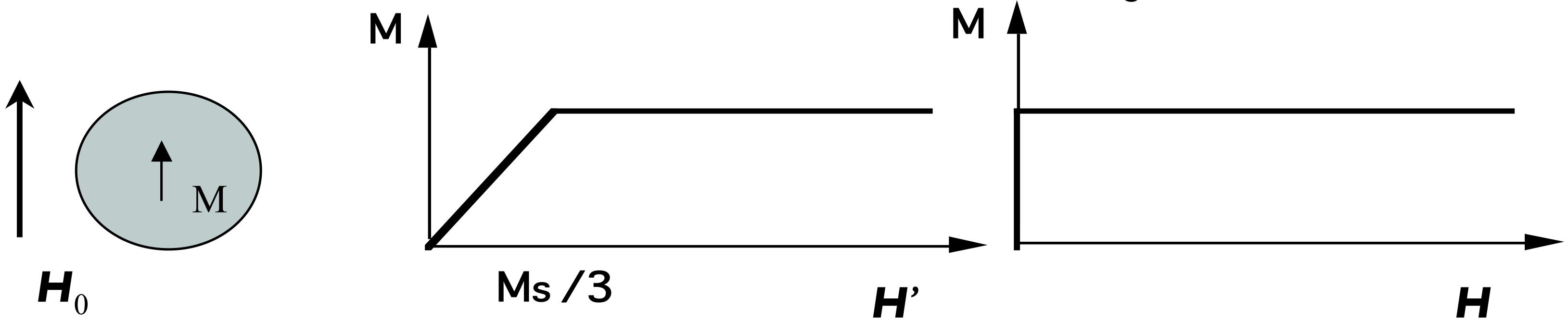
$\boldsymbol{M} = \chi \boldsymbol{H}$ χ is internal susceptibility

It follows that from $\boldsymbol{H} = \boldsymbol{H}' + \boldsymbol{H}_d$ that

$$1/\chi = 1/\chi' - N$$

For typical paramagnets and diamagnets $\chi \approx 10^{-5}$ to 10^{-3}, so the difference between χ and χ' can be neglected.

In ferromagnets, χ is much greater; it diverges as $T \rightarrow T_C$ but χ' never exceeds $1/N$.



Magnetization curves for a ferromagnetic sphere, versus the external and internal fields. $\chi'=3$

• A related quantity is the *permeability*, defined for a paramagnet, or a soft ferromagnet in small fields as

$$\mu = B/H.$$

Since $B = \mu_0(H + M)$, it follows that $\mu = \mu_0(1 + \chi_r)$.

The relative permeability $\mu_r = \mu/\mu_0 = (1 + \chi)$ μ_0 is the *permeability of free space.*

•In practice it is much easier to measure the mass of a sample than its volume. Measured magnetisation is usually $\sigma = M/\rho$, the magnetic moment per unit mass (ρ is the density).

Likewise the mass susceptibility is defined as $\chi_m = \chi/ \rho$

2. Maxwell's Equations

In electrostatics, there is also an auxiliary field, ***D***. $\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$

(J is defined as the 'magnetic polarization' $\boldsymbol{J} = \mu_0\boldsymbol{M}$)

Maxwell's equations in a material medium are expressed in terms of the four fields

$$\nabla.\mathbf{D} = \rho$$

$$\nabla.\boldsymbol{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$$

$$\nabla \times \boldsymbol{H} = j + \partial\mathbf{D}/\partial t$$

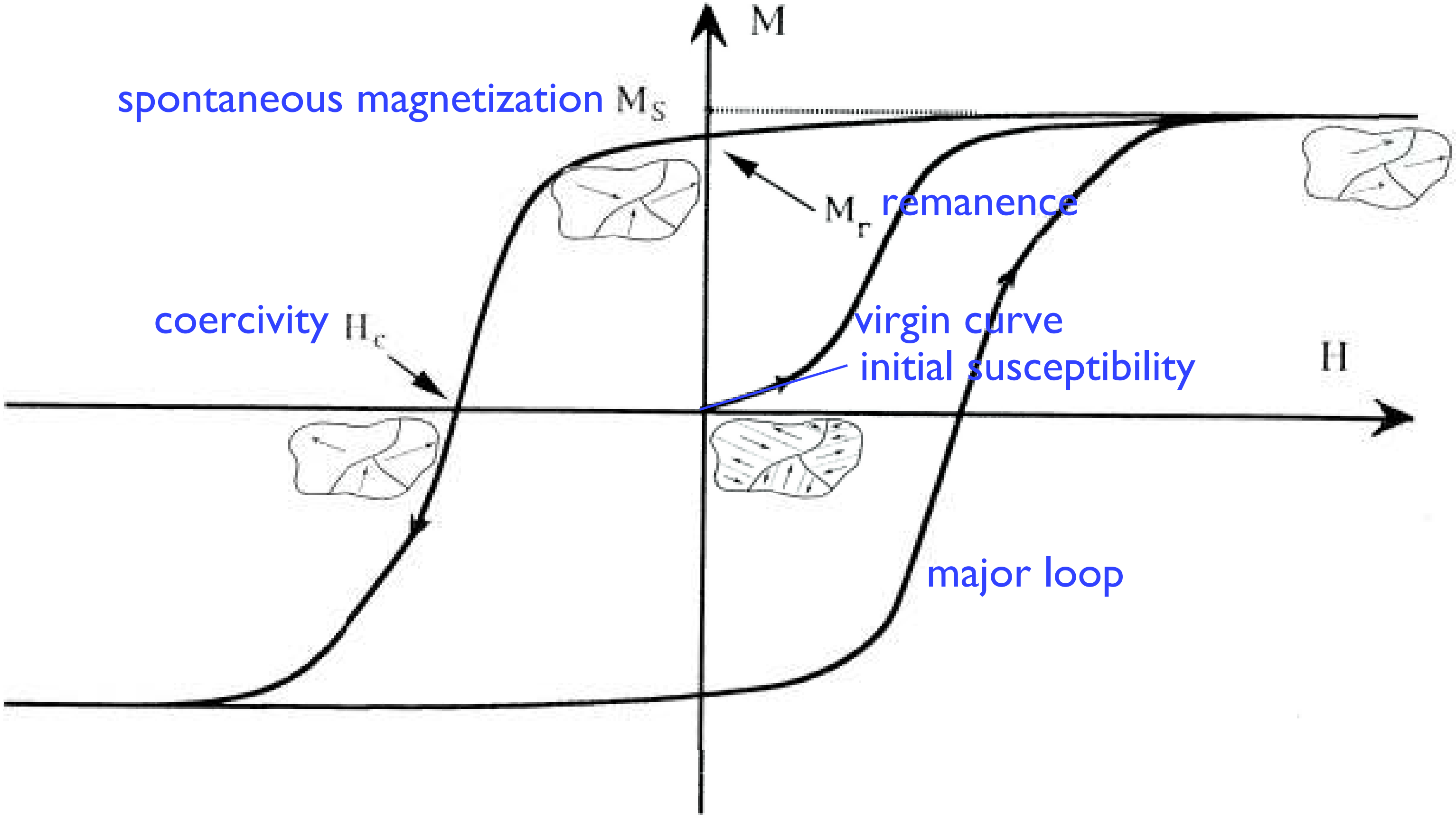
In m*agnetostatics* there is no time-dependence of *B. D* or ρ

Conservation of charge $\nabla.\boldsymbol{j} = -\partial\rho/\partial t$. In a steady state $\partial\rho/\partial t = 0$

Magnetostatics: $\nabla.\boldsymbol{j} = 0$; $\nabla.\boldsymbol{B} = 0$ $\nabla\times\boldsymbol{H} = \boldsymbol{j}$

Constituent relations: $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{E})$; $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{E})$; $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{H})$

Hysteresis



The hysteresis loop shows the irreversible, nonlinear response of a ferromagnet to an internal magnetic field ***M*** = ***M***(***H***). It reflects the arrangement of the magnetization in ferromagnetic *domains*.

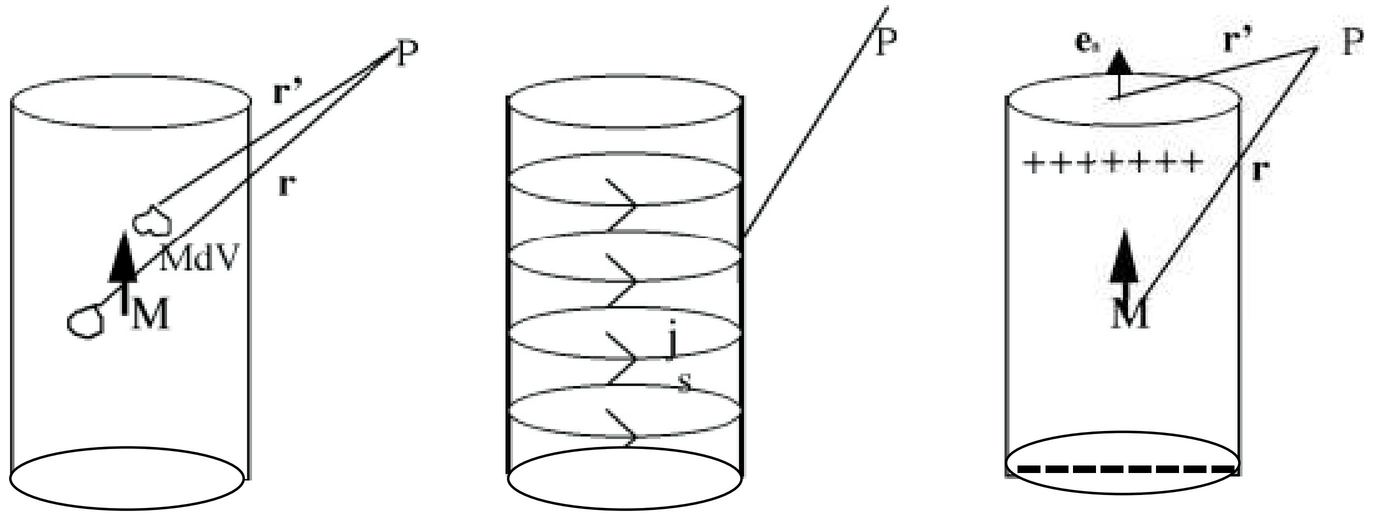
The ***B*** = ***B***(***H***) loop is deduced from the relation ***B*** = μ_0(***H*** + ***M***).

3 Magnetic Field Calculations

In magnetostatics, the sources of magnetic field are
i) current-carrying conductors and
ii) magnetic material

Biot-Savart law

$$\delta \boldsymbol{B} = -\frac{\mu_0}{4\pi}\frac{\boldsymbol{r}\times\boldsymbol{j}}{|r^3|}\delta V$$



Dipole sum | Amperian approach-currents | Coulomb approach-magnetic charge

a) Dipole integral Integrate over the magnetization distribution ***M*(*r*)**

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left\{3\frac{(\mathbf{m}.\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right\}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\left[\int\left\{\frac{3\mathbf{M}(\mathbf{r}').(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5}(\mathbf{r}-\mathbf{r}') - \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{2}{3}\mu_o\mathbf{M}(\mathbf{r}')\delta(0)d^3r'\right\}\right]$$

Compensates the divergence at the origin

a) Amperian approach Integrate over the equivalent currents $\boldsymbol{j}(\boldsymbol{r})$

$$\boldsymbol{j}_m = \nabla \times \boldsymbol{M} \qquad \textit{and} \qquad \boldsymbol{j}_{ms} = \boldsymbol{M} \times \boldsymbol{e_n}$$

Evaluate from the Biot-Savart law.

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{(\nabla \times \boldsymbol{M}) \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d^3 r' + \int \frac{(\boldsymbol{M} \times \mathbf{e}_n) \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} d^2 r'\right\}$$

Zero for a uniform distribution of M

a) Coulomb approach Use the equivalent distribution of magnetic charge

ρ_m = -∇.$\boldsymbol{M}$ *and* ρ_{ms} = $\boldsymbol{M}.\boldsymbol{e_n}$

Evaluate from the Biot-Savart law.

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi}\left\{-\int_V \frac{(\nabla.\mathbf{M})(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3r' + \int_S \frac{\mathbf{M}.\mathbf{e}_n(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^2r'\right\}$$

Zero for a uniform distribution of M

4.1 The magnetic potentials

a) Vector potential for $\mathbf{B}$

Maxwell's =n $\nabla.\mathbf{B} = 0$

Now $\nabla.(\nabla \times \mathbf{A}) = 0$ hence $\mathbf{B} = \nabla \times \mathbf{A}$

$\mathbf{A}$ is the magnetic vector potential. Units T m.

Latitude in the choice of $\mathbf{A}$: $(0, 0, B_z)$ can be represented by (0, xB,0), (-yB, 0, 0) or (1/2yB, 1/2xB, 0)

The gradient of any scalar f(r) can be added to $\mathbf{A}$ since $\nabla \times \nabla f = 0$

$\mathbf{B}$ is unchanged by any transformation $\mathbf{A} \rightarrow \mathbf{A}'$ known as a gauge transformation.

Coulomb gauge: choose f(r) so that $\nabla.\mathbf{A}$

then $\mathbf{A} = (1/2)\mathbf{B} \times \mathbf{r}$

Vector potential for *B*

Since $(\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^3 = -\nabla[1/(\boldsymbol{r}-\boldsymbol{r}')]$, the expression for the field due to a distribution of currents obtained by integrating the Biot-Savart law (2.5) over the variable $\boldsymbol{r}'$,

$$\boldsymbol{B}(\boldsymbol{r}) = (\mu_0/4\pi)\int(\boldsymbol{j}(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}'))/|\boldsymbol{r}-\boldsymbol{r}'|^3 d^3r' \tag{2.59}$$

may be written in the form $\boldsymbol{B}(\boldsymbol{r}) = (\mu_0/4\pi)\nabla \times \int(\boldsymbol{j}(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|d^3r'$. Hence

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r \tag{2.60}$$

$\boldsymbol{A}$, like $\boldsymbol{j}$ is a polar vector. Ampère's law $\nabla \times \boldsymbol{B} = \mu_0\boldsymbol{j}$ can be written in terms of $\boldsymbol{A}$ as $\nabla \times \nabla \times \boldsymbol{A} = \mu_0\boldsymbol{j}$. Since $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla.\boldsymbol{A}) - \nabla^2\boldsymbol{A}$, we see that in the Coulomb gauge, the vector potential satisfies Poisson's equation

$$\nabla^2\boldsymbol{A} = \mu_0\boldsymbol{j} \tag{2.61}$$

By expanding $1/|\boldsymbol{r}-\boldsymbol{r}'|$ as $1/r + \ldots\ldots$, it follows that at large distances the vector potential for the current loop equivalent to a magnetic moment $\mathfrak{m}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\mathfrak{m} \times \boldsymbol{r}}{r^3} \tag{2.62}$$

The expression for a distribution of magnetization $\boldsymbol{M}(\boldsymbol{r}')$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int(\boldsymbol{M}(\boldsymbol{r}) \times (\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^3)\, d^3r' \tag{2.63}$$

The above two equations (2.58) and (2.60) will give the vector potential, and hence the magnetic flux density for any given distribution of magnetisation or electric current. For example, the field due to a dipole $\boldsymbol{B}(\boldsymbol{r}) = (\mu_0/4\pi)\nabla \times (\mathfrak{m} \times \boldsymbol{r})/r^3 = -(\mu_0/4\pi)\nabla[(\mathfrak{m}.\boldsymbol{r})/r^3]$ which is equivalent to (2.12).

b) scalar potential for ***H***

When the H-field is produced only by magnets, and *not* by conduction currents, it can be expressed in terms of a potential.

The field is conservative, $\nabla \times \boldsymbol{H} = 0$

Since $\nabla \times \nabla f(r) = 0$ for any scalar, we can express ***H*** as

$$\boldsymbol{H} = -\nabla\varphi_m$$

Units of φ_m are Amps. $\nabla.(\boldsymbol{H} + \boldsymbol{M}) = 0$

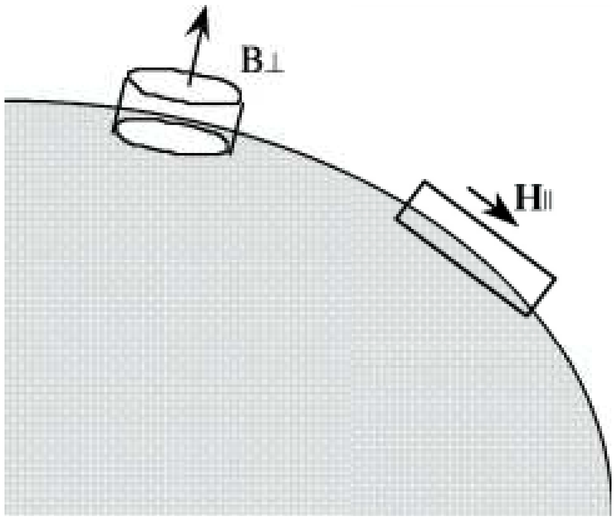
Hence $\nabla^2 \varphi_m = -\rho_m$ where $\rho_m = -\nabla.\boldsymbol{M}$

The potential due to a charge q_m is $\varphi_m = q_m/4\pi r$

$$\varphi_m(\mathbf{r}) = \frac{1}{4\pi}\left\{-\int_V \frac{\nabla.\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \int_S \frac{\mathbf{e}_n.\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^2r'\right\}$$

A dipole ***m*** has potential $\boldsymbol{m}.\boldsymbol{r}/4\pi r^3$

4.2 Boundary conditions



At any interface, it follows from Gauss's law

$$\int_S \boldsymbol{B} \cdot d\mathcal{A} = 0$$

that the *perpendicular component of* ***B*** *is continuous*.

It follows from from Ampère's law

$$\int_{loop} \boldsymbol{H} \cdot dl = I_0 = 0$$

(there are no conduction currrents on the surface)

that the *parallel component of* ***H*** *is continuous*.

Since ***B*** = ∇ x ***A***

$$\int_S \boldsymbol{B} \cdot d\mathcal{A} = \int_{loop} \boldsymbol{A} \cdot dl \quad \text{(Stoke's theorem)}$$

If dollows that the *parallel component of* ***A*** *is continuous*.

The scalar potential is continuous $\varphi_{m1} = \varphi_{m2}$

Boundary conditions

In LIH media, $\boldsymbol{B} = \mu_0 \mu_r \boldsymbol{H}$

$\boldsymbol{B}_1\boldsymbol{e}_n = \boldsymbol{B}_2\boldsymbol{e}_n$

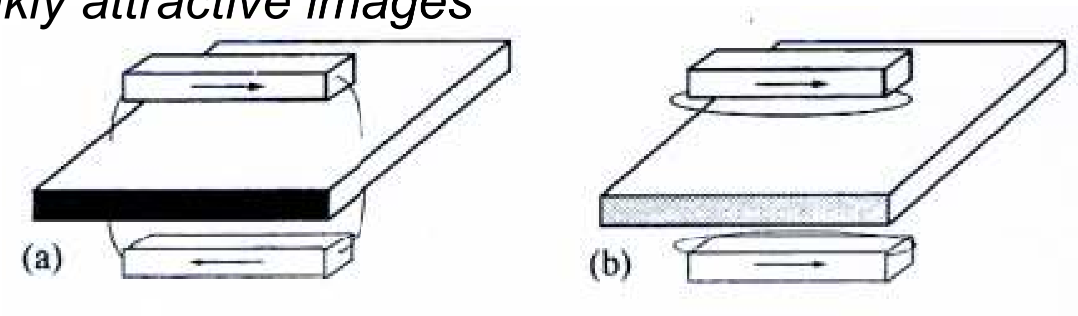
$\boldsymbol{H}_1\boldsymbol{e}_n = \mu_{r2}/\mu_{r1} \boldsymbol{H}_2\boldsymbol{e}_n$

Hence field lies ≈ perpendicular to the surface of soft iron

but parallel to the surface of a superconductor.

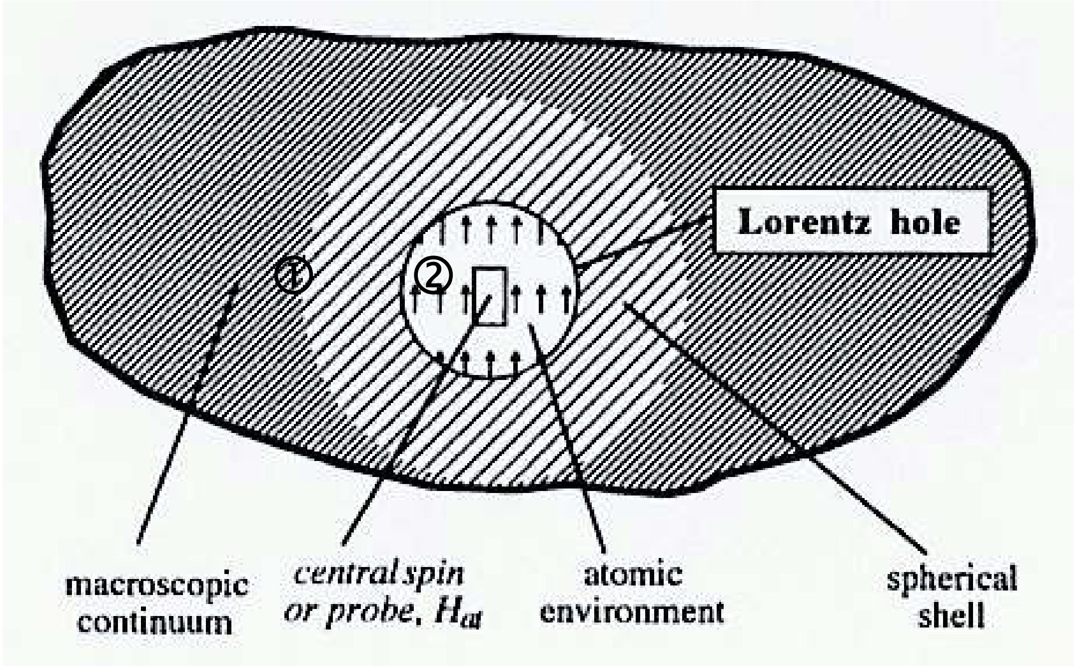
Diamagnets produce weakly repulsive images

Paramagnets produce weakly attractive images



Images in a ferromagnet (a) and a superconductor (b)

4.3 Local magnetic fields



$$\boldsymbol{H}_{loc} = \boldsymbol{H}_1 + \boldsymbol{H}_2$$

$$\boldsymbol{H}_1 = -N\boldsymbol{M} + (1/3)\boldsymbol{M}_2$$

$\boldsymbol{H}_2$ is evaluated as a dipole sum.

$$H_2 = \sum \frac{1}{4\pi}\left\{3\frac{(\mathbf{m}.\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right\}$$

Generally H_2 =f M

Here f ≈ 1; it depends on the crystal lattice

f = 0 for a cubic lattice.

Dipole interactions are source of an intrinsic anisotropy contribution.

5. Magnetostatic Energy and Forces

Magnetostatic (dipole-dipole) forces are long-ranged, but weak. They determine the magnetic microstructure.

$M \approx 1$ MA m^{-1}, $\mu_0 H_d \approx 1$ T, hence $\mu_0 H_d M \approx 10^6$ J m^{-3}

Atomic volume $\approx (0.2$ nm$)^3$; equivalent temperature ≈ 1 K.

Products BH, BM, $\mu_0 H^2$, $\mu_0 M^2$ are all energies per unit volume.

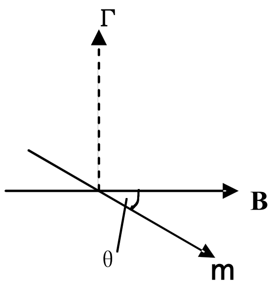
Magnetic forces *do no work* on moving charges F = q(**v**x**B**) or currents F = **j** x **B**)

No potential energy associated with the magnetic force.

$\boldsymbol{\Gamma} = \mathbf{m} \times \boldsymbol{B}$

$U = \int mB\sin\theta' d\theta'$

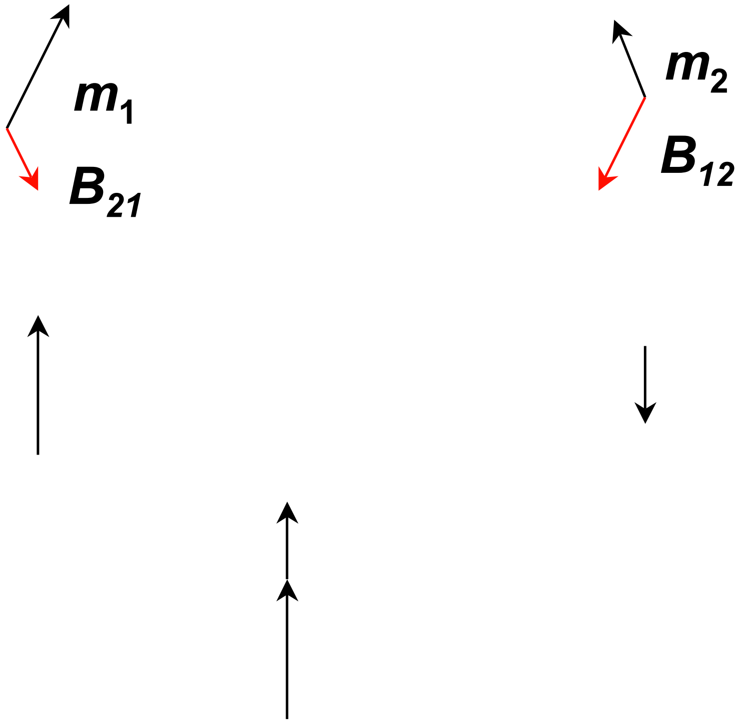
$U_m = -\mathbf{m}.\mathbf{B}$



In a non-uniform field, $F = -\nabla U_m$

$\mathbf{F} = \mathbf{m}.\nabla\mathbf{B}$

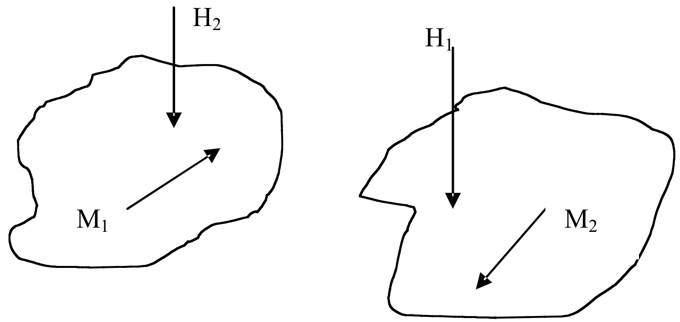
Interaction of two dipoles:



$U_p = -m_1 B_{21} = -m_2 B_{12}$

$U_p = -(1/2)(m_1 B_{21} + m_2 B_{12})$

Reciprocity theorem:



$$U = -\mu_0 \int \mathbf{M}_1 . \mathbf{H}_2 d^3r = -\mu_0 \int \mathbf{M}_2 . \mathbf{H}_1 d^3r$$

5.1 Self-energy   Energy of a body in the field H_d it creates itself.

magnetised body. The energy needed to bring it into position is $\delta U = -\mu_0 \delta \mathrm{m}.\boldsymbol{H}_{loc}$.
We neglect H_2 in (2.71) in the mesoscopic approximation where

$$\boldsymbol{H}_{loc} = \boldsymbol{H}_d + \boldsymbol{H}_L \tag{2.76}$$

Then, $\delta U = -\mu_0 \delta \mathrm{m}.(\boldsymbol{H}_d + \boldsymbol{H}_L)$. Since $\boldsymbol{H}_L = \frac{1}{3}\boldsymbol{M}$, integration over the whole sample gives

$$U = -\frac{1}{2}\int_v \mu_0 \boldsymbol{H}_d.\boldsymbol{M}\, d^3r - \frac{1}{6}\int_v \mu_0 M^2 d^3r \tag{2.77}$$

The factor of $\frac{1}{2}$ which always appears in expressions for the self-energy is needed to avoid double counting because each element δm contributes as a field source and as a moment. This energy is plotted in Fig 2.17 for a uniformly-magnetized ellipsoid of revolution, for which $U = \frac{1}{2}\mu_0 V(\mathcal{N} - \frac{1}{3})\boldsymbol{M}^2$. The second term is actually unimportant since it tends to align the moments all in the same direction, but it is much smaller than the exchange energy which has the same form. The magnetostatic self energy is conventionally defined as $U_m = U + (\mu_0/6)\int_v M^2 d^3r$ so that

$$U_m = -\frac{1}{2}\int_v \mu_0 \boldsymbol{H}_d.\boldsymbol{M}\, d^3r \tag{2.78}$$

Since $\boldsymbol{M} = \boldsymbol{B}/\mu_0 - \boldsymbol{H}$, the integral may be written in an equivalent form

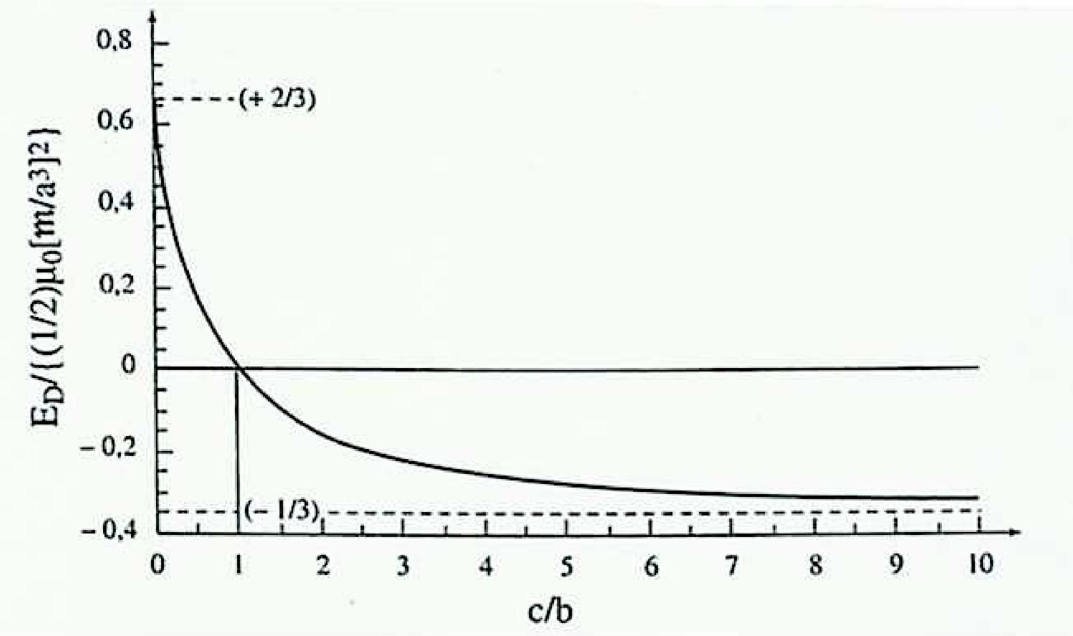
$$U_m = \frac{1}{2}\int \mu_0 H_d^2 d^3 r \tag{2.79}$$

where the integral is now over all space. We have used the handy result for a magnet in its own field, when no currents are present, that

$$\int \boldsymbol{B}.\boldsymbol{H}_d d^3 r = 0 \tag{2.80}$$

, where the integral is over *all space*. In the case of a uniformly magnetised ellipsoid (2.78) gives

$$U_m = \frac{1}{2}\mu_0 V \mathcal{N} M^2. \tag{2.81}$$



5.2 Energy associated with a magnetic field

An expression for the energy associated with a static magnetic field may be obtained by considering an inductor L consisting of a current loop which creates a flux $\Phi = LI$. By Faraday's law, $\mathcal{E} = -d\Phi/dt$ where $\mathcal{E}$ is the emf developed in a circuit and Φ is the flux threading it, so the power needed to maintain a current I in the inductor is $\mathcal{E}I = -LI dI/dt$. Integrating from 0 to I gives an expression for the energy associated with the inductor $U = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I$. The same energy can be associated with the field in space created by the inductor. First the flux is expressed in terms of the vector potential $\boldsymbol{A}$ using Stokes theorem $\int_s \boldsymbol{B}.d\boldsymbol{A} = \oint \boldsymbol{A}.d\boldsymbol{\ell}$ and the energy is written as $U = \frac{1}{2}\oint I\boldsymbol{A}.d\boldsymbol{\ell}$. This can be generalised from a single current loop to a continuous current density by replacing $Id\boldsymbol{\ell}$ by $\boldsymbol{j}dr^3$ since $\boldsymbol{j}d\boldsymbol{A}$ is the current dI in a tube of cross section $d\boldsymbol{A}$. The general expression for the energy associated with a magnetic field distribution is therefore

$$U = \frac{1}{2}\int \boldsymbol{j}.\boldsymbol{A}d^3r \tag{2.82}$$

This is analogous to the expression for the electrostatic energy $U = \int \rho\varphi_e d^3r$.

Finally, (2.82) is expressed in terms of the magnetic field. Since $\boldsymbol{j} = \nabla \times \boldsymbol{H}$ and $(\nabla \times \boldsymbol{H}).\boldsymbol{A} = \boldsymbol{\nabla}.(\boldsymbol{H} \times \boldsymbol{A}) + \boldsymbol{H}.(\nabla \times \boldsymbol{A})$, there are two terms in the integral. The first is zero for a localised field source since by Stokes's theorem the integral is equal to the flux of $\boldsymbol{H} \times \boldsymbol{A}$ through a surface at infinity. Hence, the only remaining term is $U = \frac{1}{2}\int \boldsymbol{H}.\boldsymbol{B}d^3r$.

$$U = \frac{1}{2}\int \mu_0 H^2 d^3r \tag{2.83}$$

The local energy density associated with the magnetic field is $\frac{1}{2}\mu_0 H^2$. Note that that this a quite general statement irrespective of whether the field is created by electric currents or magnetic material.

An equivalent expression for magnetic material can be deduced from (2.80) and (2.33). From these equations we find that

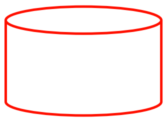
$$\frac{1}{2}\int \mu_0 H^2 d^3r = -\frac{1}{2}\int \mu_0 \boldsymbol{H}.\boldsymbol{M} d^3r \qquad (2.84)$$

This shows that the energy associated with a permanent magnet can be associated with a permanent magnet can be either associated with the integral of H^2 over all space (2.33), or with the integral of $-\boldsymbol{H}_d.\boldsymbol{M}$ over the magnet (2.78), but not both. These are alternative ways of regarding the same energy term.

When designing magnetic circuits with permanent magnets, the aim is usually to maximize the energy associated with the field created by the magnet in the space around it. Rewriting (2.84)

$$\frac{1}{2}\int_o \mu_0 H_d^2 d^3r = -\frac{1}{2}\int_i \mu_0 H_d^2 d^3r - \frac{1}{2}\int_i \mu_0 \boldsymbol{M}.\boldsymbol{H}_d d^3r \qquad (2.85)$$

where the indices o and i indicate integrals over space outside and inside the magnet. The integral on the left is the one to be maximized. For a uniformly-magnetized ellipsoid, the sum of the two integrals on the right over space inside the magnet is $-\frac{1}{2}\mu_0 M^2(\mathcal{N}^2 - \mathcal{N})$; which is maximum when the demagnetizing factor $\mathcal{N} = \frac{1}{2}$. The ideal shape for a permanent magnet is therefore an ellipsoid of revolution with c/a = 0.55 (Fig 2.17). A squat cylinder with height equal to radius is almost as good.



Energy product $-\int_i \mu_0 \mathbf{B}.\mathbf{H}_d\, d^3r$

5.2 Energy in an external field

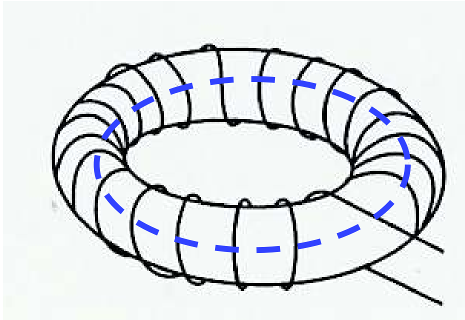
For hysteretic material, $B \neq \mu H$. The energy needed to prepare a state depends on the path followed.

The work done to produce a small flux change is

$\delta W = -\varepsilon I \delta t = I \delta \Phi$. By Ampere's law, $I = \int_{loop} H dl$.

$\delta W = \int_{loop} \delta \Phi H dl$.

$$\delta W = \int \delta B H d^3 r$$



It would be better to have an expression for the energy of M(r) in the external, applied field H', because we don't know what H(r) is like throughout the body. The real H-field is the one in Maxwell's equations

$$H = H' + H_d$$

The constitutive relation is M = M(H) nor M = M(H')

Energy in an external field

The applied field H' is created by some current distribution j'

$\nabla.H' = 0$ $\nabla \times H' = j'$

The field created by the body satisfies

$\nabla.H_d = - \nabla.M$ $\nabla \times H_d = 0$

$B = \mu_0(H + M) = \mu_0(H' + H_d + M)$

The magnetic work $\delta W' = \int \delta B(H' + H_d)\, d^3r$ Subtract the term $\mu_0 \int \delta H' H' d^3r$ for space

Energy change due to the body is $\delta W' = \int(\delta B\, H' - \mu_0\, \delta H' H)\, d^3r$

Using the expressions (2.38) and (2.33) for H and B, $H\,\delta B = \mu_0(H' + H_d)(\delta H' + \delta H_d + \delta M)$. Hence

$$\delta W' = \mu_0 \left\{ \int \delta(H'.H_d) d^3r + \int H_d \delta H_d d^3r + \int H \delta M d^3r \right\} \qquad (2.88)$$

= 0

$$\delta W' = \delta U_m + \mu_0 \int \boldsymbol{H}.\delta \boldsymbol{M}\ d^3r \tag{2.89}$$

where the integral is over the volume of the magnet as $\boldsymbol{M} = 0$ elsewhere. This expression relates the magnetic energy to the self energy and the constitutive relation $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{H})$. From (2.78)

$$\delta U_m = -\frac{1}{2} \int \mu_0 (\boldsymbol{H}_d.\delta \boldsymbol{M} + \boldsymbol{M}.\delta \boldsymbol{H}_d) d^3r \tag{2.90}$$

$$\delta U_m = -\int \mu_0 \boldsymbol{H}_d.\delta \boldsymbol{M}\ d^3r \tag{2.91}$$

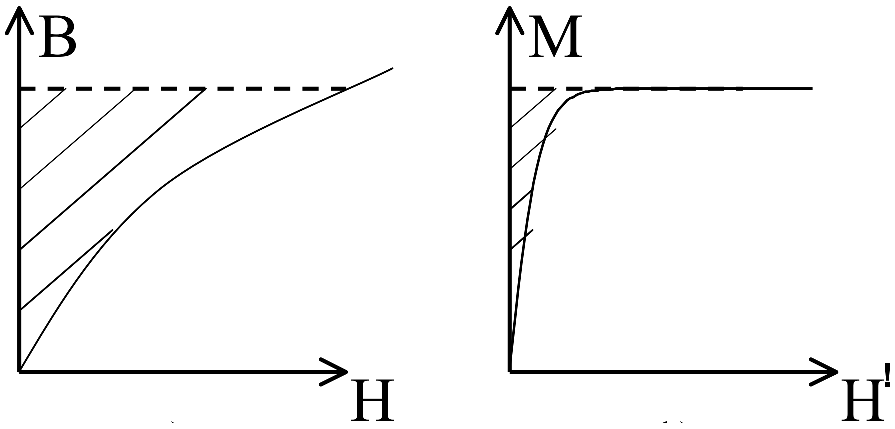
by reciprocity. From (2.38), (2.87) and (2.89)

$$\delta W' = \int \mu_0 \boldsymbol{H}'.\delta \boldsymbol{M}\ d^3r \tag{2.92}$$

Again, when the magnetization is uniform, this expression reduces to

$$\delta W' = \mu_0 H' \delta M \tag{2.93}$$

Energy in an external field



$\int H dB$ $\quad\quad$ $\int \mu_0 H' dM$

5.4 Thermodynamics of magnetic materials

$$dU = H_x dX + dQ$$

$dQ = TdS$

Four thermodynamic potentials

$U(X,S)$

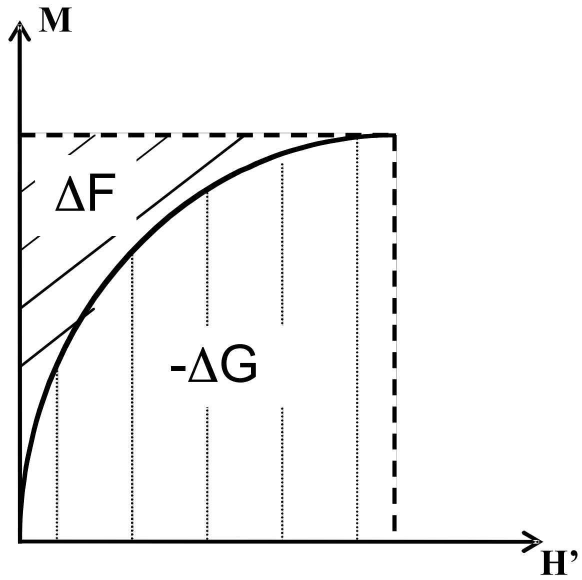
$E(H_X,S)$

$F(X,T) = U - TS$ $\quad$ $dF = HdX - SdT$

$G(H_X,T) = F - H_X X$ $\quad$ $dG = -XdH - SdT$

Magnetic work is $H\delta B$ or $\mu_0 H'\delta M$

$$dF = \mu_0 H'dM - SdT$$
$$dG = -\mu_0 MdH' - SdT$$



$S = -(\partial G/\partial T)_{H'}$ $\quad$ $\mu_0 M = -(\partial G/\partial H')_{T'}$

Maxwell relations

$(\partial S/\partial H')_{T'} = -\mu_0(\partial M/\partial T)_{H'}$ etc.