Neutron Diffusion

- **1D Diffusion Equation**
  \[ J_z = -D \frac{\partial n}{\partial z} \]

- **3D Diffusion Equation**
  \[ \mathbf{J} = iJ_x + jJ_y + kJ_z = -\hat{D}\nabla n \]

- **Relation of the diffusion coefficient to the scattering length**
  \[ J_z = \frac{v}{6} n(z - \lambda_s) - \frac{v}{6} n(z + \lambda_s) = -\frac{v\lambda_s}{3} \frac{\partial n}{\partial z} = -D \frac{\partial n}{\partial z} \]
  \[ D = \frac{v\lambda_s}{3} \]

- **1st Fick’s Law**
Continuity Equation

- **Second Fick's Law in 3D:**
  \[ \frac{\partial n}{\partial t} = - \left[ \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right] = - \nabla \cdot \mathbf{J} \]

- The same in 1D (attention on the \(D\) not being constant):
  \[ \frac{\partial n}{\partial t} = D \nabla^2 n = \Delta Dn \]

- Time-dependent continuity equation for the neutron flux:
  \[ \frac{\partial n(\mathbf{r}, t)}{\partial t} = D \nabla^2 n(\mathbf{r}, t) - \Sigma_a \Phi(\mathbf{r}, t) + S(\mathbf{r}, t) \]

- Total macroscopic absorption cross section for nuclear density \(N\):
  \[ \Sigma_a = N \sigma_a \]
Constant or Infinite Source

\[ n(x, 0) \]
\[ n(x, 0.1 \cdot t_{\text{max}}) \]
\[ n(x, 0.01 \cdot t_{\text{max}}) \]
\[ n(x, t_{\text{max}}) \]
Finite Source

\begin{align*}
n(x, 0) \\
n\left(x, 0.1 \cdot t_{\text{max}}\right) \\
n\left(x, 0.01 \cdot t_{\text{max}}\right) \\
n\left(x, t_{\text{max}}\right)
\end{align*}
Constant and Finite Source Solutions (only diffusion) – Side by Side
Continuity Equation with Absorption: Constant Source Case
Continuity Equation with Absorption: Finite Source Case
Constant and Finite Source Solutions (with absorption) – Side by Side
Steady-state Continuity Equation

With no sources, other than at the boundary \( r=0 \), within the moderator \( M \):

\[
\Delta \Phi(r) = \frac{\nu \Sigma_a(M)}{D} \Phi(r) = \frac{\Phi(r)}{L^2}
\]

Where \( L \) is nothing but the diffusion length in the moderator \( M \):

\[
L^2 = \frac{D}{\nu \Sigma_a(M)}
\]

Another relation can be established with the neutron ‘lifetime’ in the moderator \( \tau \):

\[
\sqrt{\frac{D}{\nu \Sigma_a(M)}} = \sqrt{D\tau} \quad \tau = \frac{1}{\nu \Sigma_a(M)}
\]
The Gradient Operator in Different Coordinate Systems

- **Cartesian:**
  \[ \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \]

- **Cylindrical:**
  \[ \nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi + \frac{\partial f}{\partial z} \mathbf{e}_z \]

- **Spherical:**
  \[ \nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi \]
The Laplace Operator in Different Coordinate Systems

- **Cartesian:**
  \[ \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

- **Cylindrical:**
  \[ \Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \]

- **Spherical:**
  \[ \Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \]
Stationery Isotropic Diffusion in Spherical Coordinates

The Laplacian equation becomes:

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} = \frac{\Phi}{L^2}$$

Substituting in with a transformed flux:

$$\frac{\partial^2 u(r)}{\partial r^2} + \frac{u(r)}{L^2} = 0$$

The general solution is:

$$u(r) = ae^{-r/L} + be^{+r/L}$$

$$\Phi(r) = \frac{ae^{-r/L}}{r} + \frac{be^{+r/L}}{r}$$

$$\lim_{r \to \infty} \Phi(r) = 0 \Rightarrow b = 0$$
Mean Square Distance Travelled by a Neutron in Medium

Performing the averaging:

\[
\langle r^2 \rangle = \frac{\int_0^\infty r^2 \Phi(r) dV}{\int_0^\infty \Phi(r) dV} = \frac{\int_0^\infty r^3 e^{-r/L} dr}{\int_0^\infty r e^{-r/L} dr} = 6L^2
\]

With a volume element:

\[dV = 4\pi r^2 dr\]

The root mean square distance is therefore:

\[r_{\text{rms}} = \sqrt{6L} \approx 2.5L\]