Introduction to Plasma Physics (PY5012)
Lecture 8: Multiple-fluid theory of plasmas

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Hierarchy of plasma phenomena

Plasma Phenomena
- Single-particle Motion
- Distribution Function
  - Boltzmann Equation
    - Moments of Boltzmann Equation
      - Single fluid MHD
      - Multiple fluids
Fluid Approach to Plasmas

Fluid approach describes bulk properties of plasma. We do not attempt to solve unique trajectories of all particles in a plasma. This simplification works very well for majority of plasmas, despite gross simplifications made.

Fluid theory follows directly from moments of the Boltzmann equation (Lecture 7).

Each of the moments of the Boltzmann equation is a transport equation describing the dynamics of a quantity associated with a given power of \( v \).

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0 \quad \text{Continuity of mass or charge transport}
\]

\[
mn \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] = qn(E + u \times B) - \nabla \cdot P + P_{ij} \quad \text{Momentum Transport}
\]

\[
\frac{\partial}{\partial t} \left[ \frac{n}{2} m u^2 \right] + \nabla \cdot \left[ \frac{n}{2} m (u^2 u) \right] - n q(E \cdot u) = \frac{m}{2} \int u^2 \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \, du \quad \text{Energy Transport}
\]

Cold-Plasma Model

Simplest set of macroscopic equations can be obtained by simplifying the momentum transfer equation and neglect thermal motions of particles.

Here, set kinetic pressure tensor to zero, i.e., \( P = mn <ww> = 0 \) as \( w = 0 \).

Remaining macroscopic variables are then \( n \) and \( u \), described by

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0
\]

\[
mn \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] = qn(E + u \times B) + P_{ij}
\]

Collision term \( P_{ij} \) can be approximated by an “effective” collision frequency.

Assumed that collisions cause a rate of decrease in momentum:

\[
P_{ij} = -mnv_{eff} u
\]
Warm-Plasma Model

- Alternative set of macroscopic equation is obtained by truncating energy conservation equation.

- Consider pressure tensor: \( \mathbf{P} = mn\langle\mathbf{ww}\rangle = \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix} \)

- Components represent transport of momentum. Diagonal elements represent pressure, while off-diagonal represent shearing stresses.

- In warm-plasma model, only consider diagonal pressure elements, so
  \[ \nabla \cdot \mathbf{P} = \nabla p \]

- That is, viscous forces are neglected. We then have
  \[ \frac{\partial n}{\partial t} + \mathbf{V} \cdot (n \mathbf{u}) = 0 \]
  \[ mn \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p + \mathbf{P}_{ij} \]

Warm-Plasma Model

- The previous system of equations does not form a closed set, since scalar pressure is now a third variable. Usually determined by energy equation.

- If plasma is isothermal, assume equation of state of form:
  \[ p = nk_{B}T \quad \text{and} \quad \nabla p = k_{B}T \nabla n \]

- Holds for slow time variations, allowing temperatures to reach equilibrium.

- If plasma does not exchange energy with its surrounds, assume it is adiabatic:
  \[ p n^{\gamma} = \text{constant} \quad \text{and} \quad \frac{\nabla p}{p} = \gamma \frac{\nabla n}{n} \]

  where \( \gamma \) is the ratio of the specific heats at constant pressure.
Note, the energy equation can be written

\[ \frac{\partial \left[ \frac{1}{2} m \langle w^2 \rangle \right]}{\partial t} + \nabla \cdot \left( \frac{1}{2} nm \langle w^2 \rangle \mathbf{u} \right) + (\mathbf{P} \cdot \nabla) \mathbf{u} + \nabla \cdot \mathbf{q} = P_j \]

where \( \mathbf{q} \) is the heat flow vector. For electrons, commonly used approximation for \( \mathbf{q} \) is

\[ \mathbf{q} = K \nabla T \]

where \( K \) is the thermal Spitzer conductivity.

As average energy of plasma is \( 1/2m \langle \mathbf{w w} \rangle = 3/2 k_B T \) and using \( p = n k_B T \)

\[ \Rightarrow \frac{3}{2} p = 1/2nm \langle \mathbf{w w} \rangle. \]

Energy equation can then be written

\[ \frac{\partial \left( \frac{3}{2} p \right)}{\partial t} + \nabla \cdot \left( \frac{3}{2} pu \right) - p \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q} = P_j \]

The quantity \( 3/2pu \) represents the flow of energy density at the fluid velocity.

Consider plasma of two species; ions and electrons, in which fluid is fully ionised, isotropic and collisionless. The charge and current densities are

\[ \sigma = n_i q_i + n_e q_e \]
\[ \mathbf{j} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e \]

Using \( \mathbf{v} = \mathbf{u} \), complete set of electrodynamics equations are then \((j = i \text{ or } e)\)

\[ m_j n_j \left[ \frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla) \mathbf{v}_j \right] = -\nabla p_j - q_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) \]

\[ \frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0 \]

\[ \varepsilon_0 \nabla \cdot \mathbf{E} = \sigma \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ p_j = C_j n_j^{\gamma_j} \]
Fluids drifts perpendicular to B

- Since a fluid element is composed of many individual particles, expect drifts perpendicular to \( \mathbf{B} \). But, the \( \text{grad} \ (p) \) term results in a fluid drift called diamagnetic drift.

- Consider momentum equation for each species:

\[
\frac{mn}{\partial t} \left[ \nabla \right] + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - qn(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

(1) \quad (2) \quad (3)

- Consider ratio of terms (1) to (3):

\[
\frac{(1)}{(3)} \approx \frac{m n i \omega_i}{q n v_B} = \frac{\omega}{\omega_c}
\]

- Here we have used \( \frac{\partial}{\partial t} = i \omega \). If only consider slow drifts compared to time-scale of the gyrofrequency, can set (1) to zero.

\[
\text{Fluids drifts perpendicular to } \mathbf{B}
\]

- Therefore can write

\[
0 = -qn(\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B}) - \nabla p
\]

where \( \mathbf{v} \times \mathbf{B} = (\mathbf{v}_\perp + \mathbf{v}_\parallel) \times \mathbf{B} = \mathbf{v}_\perp \times \mathbf{B} \)

- Taking the cross-product

\[
0 = qn[\mathbf{E} \times \mathbf{B} + (\mathbf{v}_\perp \times \mathbf{B}) \times \mathbf{B}] - \nabla p \times \mathbf{B}
\]

- Using the identity \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \) we can write

\[
0 = qn[\mathbf{E} \times \mathbf{B} + (\mathbf{v}_\perp \cdot \mathbf{B})\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\mathbf{v}_\perp] - \nabla p \times \mathbf{B}
\]

- As \( \mathbf{v}_\perp \) is perpendicular to \( \mathbf{B} \), \( \mathbf{v}_\perp \cdot \mathbf{B} = 0 \). Therefore

\[
\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\nabla p \times \mathbf{B}}{qnB^2}
\]

\[
= \mathbf{v}_E + \mathbf{v}_D
\]
Fluids Drifts perpendicular to B

- In previous equation is \( \mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \) \( \mathbf{E} \times \mathbf{B} \) drift

and \( \mathbf{v}_D = -\frac{\mathbf{v}_p \times \mathbf{B}}{q \mathbf{n} B^2} \) is diamagnetic drift.

- The \( \mathbf{v}_E \) drift is same as for guiding centres, but there is now a new drift, called the diamagnetic drift. Is in opposite directions for ions and electrons.

- Gives currents in plasma that reduce magnetic field in plasma. More ions moving to left in shaded area that to the right (Inan & Golkowski, Page 111).

**Diamagnetic Drift in Q-machines**

- Diamagnetic drift first measured in \( Q \)-machines

- See http://www.physics.uiowa.edu/xplasma/Qmachine.html
Fluid drifts parallel to B

- Consider the component of fluid equation of motion:

\[ mn \left( \frac{\partial v_z}{\partial t} + (\mathbf{v} \cdot \nabla)v_z \right) = -q n E_z - \frac{\partial p}{\partial z} \]

- The convective term can be neglected as it is much smaller than \( \partial v_z/\partial t \)

- Using \( p = n k_B T \) or \( \frac{\partial p}{\partial z} = k_B T \frac{\partial n}{\partial z} \) we can write

\[ \frac{\partial v_z}{\partial t} = \frac{q}{m} E_z - \frac{\gamma k_B T}{mn} \frac{\partial n}{\partial z} \]

- This shows that the fluid is accelerated along \( \mathbf{B} \) under the combined electrostatic and pressure gradient forces.

Fluid drifts parallel to B

- Taking the limit as \( m \to 0 \) and \( q = -e \) and \( E_z = -\partial \phi/\partial z \) we have

\[ eE_z = e \frac{\partial \phi}{\partial z} = \frac{\gamma k_B T}{n} \frac{\partial n}{\partial z} \]

- Electrons are so mobile that their heat conductivity is almost infinite.

- Assuming isothermal electrons and taking \( \gamma = 1 \), we can integrate to get

\[ e\phi = k_B T \ln(n) + C \]

- We can therefore write

\[ n = n_0 \exp(e\phi/k_B T) \]

- This is called the Boltzmann relation or Boltzmann factor for electrons.

- Implies that electrons have a tendency to move rapidly in response to and external force (i.e., electrostatic potential gradient).