Lecture 4 - Gravitational acceleration

• Todays Topics

• Gravitation
  - Gravity: the universal law of gravitation 5.6
  - Keplers Laws: 5.9
  - Gravitational potential: Lowrie 2.2
    • potential well ..
    • equipotentials ..
    • influence of non uniform bodies ..
  - Gravity: Variations in gravitational acceleration 5.7
    • Microgravity variations - causes
      - Non-uniformity
      - Distance from object
      - External influences
      - Centrifugal accelerations
      - Shape of the earth
    • Geophysical applications and consequences

PY2P30 - Universal law of Gravitation

Newton's law of Universal Gravitation

\[ F_{12} = G \frac{m_1 m_2}{r_{12}^2} \quad W = mg = G \frac{M_E m}{R_E^2} \quad g = G \frac{M_E}{R_E^2} \]

\[ r_{12} \approx 1 \text{AU} \quad M_E \approx 5.98 \times 10^{24} \text{kg} \quad R_E \approx 6384 \text{km} \]

\[ m_E = \frac{G r_E^2}{\left(9.80 \text{ m/s}^2\right) \left(6.38 \times 10^6 \text{ m}\right)^2} = \frac{6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2}{5.98 \times 10^{24}} \approx 5.98 \times 10^{24} \text{kg} \]

Although Gravitation is universal, the value of \( g \) is not!

Gravitational acceleration measured in milligals in geophysics and earth sciences

1 gal = 1cm sec\(^{-2}\) = 10mm sec\(^{-2}\) \quad 1 mgal = 10\(\mu\)m sec\(^{-2}\)

\[ g = 9.81 \text{ m/sec}^2 = 981 \text{ gal} = 981,000 \text{ mgal} \]
Kepler’s laws: inferred from Newton’s law of Gravitation and laws of motion

1. The orbit of each planet is an ellipse with the sun at one focus
2. An imaginary line drawn from each planet to the sun sweeps out equal areas in equal times
3. The ratio of the squares of the periods is equal to the ratio of the cubes of the mean distances

\[
\left( \frac{T_1}{T_2} \right)^2 = \left( \frac{r_1}{r_2} \right)^3 \quad \text{or} \quad \frac{r_1^3}{T_1^2} = \frac{r_2^3}{T_2^2}
\]

How? - equate gravitational force to force on object in circular motion which must experience necessary centripetal acceleration

\[
F = ma; \quad a = \frac{v^2}{r};
\]

\[
v = rw; \quad \omega = \frac{2\pi}{T}
\]

Using above can show for any object orbiting a body of mass \( M \): e.g. satellite

\[
\frac{r^3}{T^2} = \frac{4\pi^2}{GM}
\]

If a force \( F \) moves through a small distance \( dr \) in the direction of the force, the work done is equivalent to the change in potential energy between the two points.

We can give a Gravitational potential energy as follows:

\[
U = \int F(r)dr = \int \frac{GMm}{r^2}dr = \left[ -\frac{GMm}{r} + C \right] = -\frac{GMm}{r}
\]

\[= mgh \text{ when } h \ll R \]

Plot \( U \) vs \( r \)

This is why we speak of a gravitational potential well

Show:

\[
U = \int F(r)dr = \int_{r_1}^{r_2} \frac{GMm}{r^2}dr \approx mgh
\]

Here we implicitly assume SPHERICAL SYMMETRY… is this warranted?

Cormac McGuinness
PY2P30 - Gravitational Potential

Can consider surfaces of equal gravitational potential

Assuming SPHERICAL SYMMETRY

Even if density is not constant but only varies with $r$ - can still have spherical symmetry of gravitational potential

In fact observe variation of $g$ with $r$ as follows:

$g = \frac{GM}{r^2}$

$
\begin{array}{c}
\text{EARTH} \\
\text{I M R M R}
\end{array}$

$0.33 \quad 0.4$

$\Rightarrow$ denser core of earth

How do we get this information?

$J_{EARTH} \approx 0.33M_E R_E^2 < 0.4M_E R_E^2$

Fig. 2.6 (a) Equipotential surfaces of a spherical mass form a set of concentric spheres. (b) The normal to the equipotential surface defines the vertical direction; the tangential plane defines the horizontal.
Gravitational influence of a non-uniform body

For spherical symmetry, we can treat the extended object (the sphere) as if it was a point mass.

Spherically symmetry is broken if:
(i) the shape is not a sphere

or

(ii) the density is not uniform and depends upon more than the distance from the centre.

Either will give a variation in $g$ depending upon the position relative to the object.

Q: What of $g$ inside a hollow spherical shell?

Distance from center of the earth - e.g. high mountain ...

At equator - centrifugal force:
PY2P30 - Variations in observed $g$

Due to moon ...

$R_E \approx 6384\text{km}$

$M_E \approx 5.98 \times 10^{24}\text{kg}$

$M_m \approx 7.36 \times 10^{22}\text{kg}$

$R_{Em} \approx 3.844 \times 10^3\text{km}$

Due to sun ...

$M_s \approx 1.989 \times 10^{30}\text{kg}$

$R_{Es} \approx 1.49597 \times 10^8\text{km}$

Centrifugal acceleration varies with the angle of latitude $\lambda$.

$a_c = \frac{v^2}{x} = x\omega^2 = (r \cos \lambda)\omega^2$

Observed direction of $g$ may not point to center of earth

The physical equipotential surface of gravity is called the *geoid*.

Fig. 2.8 The outwardly directed centrifugal acceleration $a_c$ at latitude $\lambda$ on a sphere rotating at angular velocity $\omega$.

Fig. 2.20 Comparison of the dimensions of the International Reference Ellipsoid with a sphere of equal volume.
State of the art Gravity mapping by satellite - GRACE - Gravity Recovery and Climate experiment

Pair of satellites in polar orbit - position obtained precisely by GPS; distance between ~200km but measured to better than 10μm

The data shown as the lumpy bumpy contours of the Earth that GRACE observes actually shows a gravitational equipotential surface, with the colours depicting the difference from the average value of $g$ as would be expected from the geoid.

Plot of variations in gravity measured in milligals

1 gal = 1cm sec$^{-2}$

The pressure at the center due to both columns must be equal - but reduced pressure from equator can only support a reduced column height of water to pole.

Newton assumed constant density and obtained a flattening of 1:230

Actual flattening of earth ~ 1:298

Newton’s argument:
- assume you can drill wells filled with water to center of earth from both equator and a pole.
- At the equator the outward centrifugal acceleration opposes inward gravitational acceleration and pulls column of water upwards. No such action occurs at the pole. The pressure at the center due to the column of water from the equator is thus reduced.
- The pressure at the center due to both columns must be equal - but reduced pressure from equator can only support a reduced column height of water to pole.
PY2P30 – Variations in observed g

Due to nearby mountains ...

\[ V_{cone} = \frac{1}{3} \pi r^2 h \]

\[ \rho_{quartz} = 2.65 \times 10^3 \text{kg} \cdot \text{m}^{-3} = 2.65 \text{g} \cdot \text{cm}^{-3} \]

Due to variations of density underneath the ground ...

Simplest model for local micro gravity variations due to geology assumes a buried sphere of differing density - known as gravity anomalies

Buried sphere of high density

Buried sphere of low density

Distance from surface
Example

Measure $g$ over mountain range

Must correct for difference in height above sea level (actually correct for height above reference geoid) - modeling of anomaly gives information on composition

The physical equipotential surface of gravity is called the **geoid**.

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The physical equipotential surface of gravity is called the **geoid**.

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Fig. 2.42: Determination of the density of near-surface rocks by satellite methods. (a) Gravity measurements are made on a profile across a small hill. (b) The data are corrected for elevation with various test values of the density. The optimum density gives minimum correlation between the gravity anomaly ($\Delta g$) and the topography.
**Gravity mapping by satellite**

State of the art Gravity mapping by satellite - **GRACE** - Gravity Recovery and Climate experiment

Pair of satellites in polar orbit - position obtained precisely by GPS; distance between ~200km but measured to better than 10nm.

See: http://www.csr.utexas.edu/grace/

**Plot of variations in gravity measured in milligals**

1 gal = 1cm sec\(^{-2}\)

**Earth's Gravity Field Anomalies (milligal)**

-50 -10 -30 -50 0 10 20 30 40 50

Slide 17

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**Gravity mapping by satellite**

**GRACE** - Gravity Recovery and Climate experiment

High technology and simple physics gives needed insight into global problems …

... measurement of gravitational variations due to **geology, oceans, ice** and changes in distributions of all of these

Slide 18
<table>
<thead>
<tr>
<th>PY2P30 - Physics for Earth Sciences</th>
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<tbody>
<tr>
<td>• Next three lectures in mechanics stream</td>
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