Maxwell’s Equations in Vacuum

(1) \( \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \) \hspace{1cm} Poisson’s Equation

(2) \( \nabla \cdot \mathbf{B} = 0 \) \hspace{1cm} No magnetic monopoles

(3) \( \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \) \hspace{1cm} Faraday’s Law

(4) \( \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \partial \mathbf{E} / \partial t \) \hspace{1cm} Maxwell’s Displacement

Electric Field \( \mathbf{E} \) \( \text{Vm}^{-1} \)

Magnetic Induction \( \mathbf{B} \) \( \text{Tesla} \)

Charge density \( \rho \) \( \text{Cm}^{-3} \)

Current Density \( \mathbf{j} \) \( \text{Cm}^{-2}\text{s}^{-1} \)

Ohmic Conduction \( \mathbf{j} = \sigma \mathbf{E} \) \hspace{1cm} Electric Conductivity \( \text{Siemens} \) \( \text{(Mho)} \)
Constitutive Relations

(1) $D = \varepsilon_0 E + P$

Electric Displacement $D \ \text{Cm}^2$ and Polarization $P \ \text{Cm}^2$

(2) $P = \varepsilon_0 \chi E$

Electric Susceptibility $\chi$

(3) $\varepsilon = 1 + \chi$

Relative Permittivity (dielectric function) $\varepsilon$

(4) $D = \varepsilon_0 E$

(5) $H = B / \mu_o - M$

Magnetic Field $H \ \text{Am}^{-1}$ and Magnetization $M \ \text{Am}^{-1}$

(6) $M = \chi_B B / \mu_o$

Magnetic Susceptibility $\chi_B$

(7) $\mu = 1 / (1 - \chi_B)$

Relative Permeability $\mu$

(8) $H = B / \mu \mu_o$

$\mu \sim 1$ (non-magnetic materials), $\varepsilon \sim 1 - 50$
Electric Polarisation

- Apply Gauss’ Law to right and left ends of polarised dielectric

\[ \sigma_{\pm} \text{ surface charge density Cm}^{-2} \]
\[ \sigma_{\pm} = P \cdot n \quad n \text{ outward normal} \]
\[ E_{+2dA} = \sigma_{+}dA/\varepsilon_{0} \quad \text{Gauss’ Law} \]
\[ E_{+} = \sigma_{+}/2\varepsilon_{0} \]
\[ E_{-} = \sigma_{-}/2\varepsilon_{0} \]

- \( E_{Dep} = '\text{Depolarising field}' \)

\[ E_{Dep} = E_{+} + E_{-} = (\sigma_{+} + \sigma_{-})/2\varepsilon_{0} \]
\[ E_{Dep} = -P/\varepsilon_{0} \quad P = \sigma_{+} = \sigma_{-} \]

- Macroscopic electric field

\[ E_{Mac} = E + E_{Dep} = E - P/\varepsilon_{0} \]
Electric Polarisation

Define dimensionless dielectric susceptibility $\chi$ through

\[ P = \varepsilon_o \chi E_{Mac} \]

\[ E_{Mac} = E - P/\varepsilon_o \]
\[ \varepsilon_o E = \varepsilon_o E_{Mac} + P \]
\[ \varepsilon_o E = \varepsilon_o E_{Mac} + \varepsilon_o \chi E_{Mac} = \varepsilon_o (1 + \chi)E_{Mac} = \varepsilon_o \varepsilon E_{Mac} \]

Define dielectric constant (relative permittivity) $\varepsilon = 1 + \chi$

\[ E_{Mac} = E/\varepsilon \quad E = \varepsilon E_{Mac} \]

Typical values for $\varepsilon$: silicon 11.8, diamond 5.6, vacuum 1
Metal: $\varepsilon \rightarrow \infty$
Insulator: $\varepsilon^\infty$ (electronic part) small, $\sim$5, lattice part up to 20
Electric Polarisation

Rewrite $E_{Mac} = E - P/\varepsilon_0$ as

$$\varepsilon_0 E_{Mac} + P = \varepsilon_0 E$$

LHS contains only fields inside matter, RHS fields outside

Displacement field, $D$

$$D = \varepsilon_0 E_{Mac} + P = \varepsilon_0 \varepsilon E_{Mac} = \varepsilon_0 E$$

Displacement field defined in terms of $E_{Mac}$ (inside matter, relative permittivity $\varepsilon$) and $E$ (in vacuum, relative permittivity 1).

Define

$$D = \varepsilon_0 \varepsilon E$$

where $\varepsilon$ is the relative permittivity and $E$ is the electric field
Gauss’ Law in Matter

- Uniform polarisation → induced surface charges only

- Non-uniform polarisation → induced bulk charges also

Displacements of positive charges  Accumulated charges
Gauss’ Law in Matter

Charge entering xz face at y = 0: $P_{x=0} \Delta y \Delta z$

Charge leaving xz face at y = $\Delta y$: $P_{x=\Delta x} \Delta y \Delta z$

$$= (P_{x=0} + \frac{\partial P_x}{\partial x} \Delta x) \Delta y \Delta z$$

Net charge entering cube: $(P_{x=0} - P_{x=\Delta x}) \Delta y \Delta z = -\frac{\partial P_x}{\partial x} \Delta x \Delta y \Delta z$

Charge entering cube via all faces:

$$-\left( \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) \Delta x \Delta y \Delta z = Q_{pol}$$

$$\rho_{pol} = \lim_{(\Delta x \Delta y \Delta z) \to 0} \frac{Q_{pol}}{\Delta x \Delta y \Delta z}$$

$$-\nabla \cdot P = \rho_{pol}$$
Gauss’ Law in Matter

Differentiate $-\nabla \cdot \mathbf{P} = \rho_{\text{pol}}$ wrt time

$\nabla \cdot \mathbf{P} + \partial \rho_{\text{pol}} / \partial t = 0$

Compare to continuity equation $\nabla \cdot \mathbf{j} + \partial \rho / \partial t = 0$

$\partial \mathbf{P} / \partial t = \mathbf{j}_{\text{pol}}$

Rate of change of polarisation is the polarisation-current density

Suppose that charges in matter can be divided into ‘bound’ or polarisation and ‘free’ or conduction charges

$\rho_{\text{tot}} = \rho_{\text{pol}} + \rho_{\text{free}}$
Gauss’ Law in Matter

Inside matter
\[ \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}_{\text{mac}} = \frac{\rho_{\text{tot}}}{\varepsilon_0} = \frac{\rho_{\text{pol}} + \rho_{\text{free}}}{\varepsilon_0} \]

Total (averaged) electric field is the macroscopic field
\[ -\nabla \cdot \mathbf{P} = \rho_{\text{pol}} \]
\[ \nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\text{free}} \]

Introduction of the displacement field, \( \mathbf{D} \), allows us to eliminate polarisation charges from any calculation. This is a form of Gauss’ Law suitable for application in matter.
Ampère’s Law in Matter

\[ \nabla \times \mathbf{B} = \mu_0 j \quad \Rightarrow \quad j = \frac{1}{\mu_0} \nabla \times \mathbf{B} \]

\[ \Rightarrow \nabla \cdot j = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \mathbf{B}) = 0 \quad \text{Problem!} \]

\[ \Rightarrow \nabla \cdot j = -\frac{\partial \rho}{\partial t} \neq 0 \quad \text{for non-steady currents} \quad \text{Continuity equation} \]

A steady current implies constant charge density, so Ampère’s law is consistent with the continuity equation for steady currents.

Ampère’s law is inconsistent with the continuity equation (conservation of charge) when the charge density is time dependent.
Ampère’s Law in Matter

Add term to LHS such that taking Div makes LHS also identically equal to zero:

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \varepsilon_0 \nabla \cdot \mathbf{E} = \rho \]

\[ \therefore \frac{\partial}{\partial t} (\varepsilon_0 \nabla \cdot \mathbf{E}) = \frac{\partial \rho}{\partial t} \Rightarrow \nabla \left( \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla \cdot \mathbf{j} \]

The extra term is in the bracket

\[ \mathbf{j} + \left( \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = \frac{1}{\mu_0} \nabla \times \mathbf{B} \]

Extended Ampère’s Law

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]

Displacement current (vacuum)
Ampère’s Law in Matter

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ j_P = \frac{\partial \mathbf{P}}{\partial t} \]

\[ j = j_f + j_M + j_P \quad \text{Total current} \]

\[ j_M = \nabla \times \mathbf{M} \]

\[ \mathbf{M} = \sin(ay) \mathbf{k} \]

\[ j_M = \text{curl } \mathbf{M} = a \cos(ay) \mathbf{i} \]

- Polarisation current density from oscillation of charges in electric dipoles
- Magnetisation current density variation in magnitude of magnetic dipoles
Ampère's Law in Matter

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ \Rightarrow \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j}_f + \mathbf{j}_M + \mathbf{j}_P + \varepsilon_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t} \]

\[ = \mathbf{j}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ \nabla \times \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{j}_f + \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P}) \Rightarrow \nabla \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t} \]

\[ \frac{\partial \mathbf{D}}{\partial t} \text{ is the displacement current postulated by Maxwell (1862)} \]

In vacuum \( \mathbf{D} = \varepsilon_0 \mathbf{E} \quad \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \)

In matter \( \mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E} \quad \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t} \)

Displacement current exists throughout space in a changing electric field
Maxwell’s Equations

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Divergence Theorem 2-D 3-D

• From Green’s Theorem

\[ \int_{S} \mathbf{A} \cdot \mathbf{n} \, dA = \int_{V} \nabla \cdot \mathbf{A} \, dV \]

• In words - Integral of \( \mathbf{A} \cdot \mathbf{n} \, dA \) over surface contour equals integral of \( \nabla \cdot \mathbf{A} \) over surface area

• In 3-D
• Integral of \( \mathbf{A} \cdot \mathbf{n} \, dA \) over bounding surface \( S \) equals integral of \( \nabla \cdot \mathbf{V} \, dV \) within volume enclosed by surface \( S \)
• The area element \( \mathbf{n} \, dA \) is conveniently written as \( dS \)
Differential form of Gauss’ Law

\begin{align*}
\text{Integral form} & \quad \int_{\mathbf{V}} \rho(\mathbf{r}) \, d\mathbf{r} \\
\text{Divergence theorem applied to field } \mathbf{V}, \text{ volume } \mathbf{v} \text{ bounded by surface } \mathbf{S} & \quad \int_{\mathbf{S}} \mathbf{E} \cdot d\mathbf{S} = \frac{\int_{\mathbf{V}} \mathbf{V} \cdot d\mathbf{S} - \int_{\mathbf{S}} \mathbf{V} \cdot d\mathbf{S}}{\varepsilon_o} \\
\text{Divergence theorem applied to electric field } \mathbf{E} & \quad \int_{\mathbf{S}} \mathbf{V} \cdot n \, dA = \int_{\mathbf{S}} \mathbf{V} \cdot d\mathbf{S} = \int_{\mathbf{V}} \nabla \cdot \mathbf{V} \, dv \\
\text{Differential form of Gauss’ Law (Poisson’s Equation)} & \quad \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_o}
\end{align*}
Stokes’ Theorem 3-D

- In words - Integral of \((\nabla \times \mathbf{A}) \cdot \mathbf{n} \, dA\) over surface \(S\) equals integral of \(\mathbf{A} \cdot d\mathbf{r}\) over bounding contour \(C\).
- It doesn’t matter which surface (blue or hatched). Direction of \(d\mathbf{r}\) determined by right hand rule.

\[
\oint_{C} \mathbf{A} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, dA
\]
Faraday’s Law

Integral form of law:
Induced emf equals minus rate of change of magnetic flux

\[ \oint_{S} E \cdot d\ell = -\frac{d}{dt} \int_{S} B \cdot dS \]

\[ \oint_{C} E \cdot d\ell = \int_{S} (\nabla \times E) \cdot dS \quad \text{Stokes' Theorem} \]

\[ -\frac{d}{dt} \int_{S} B \cdot dS = \int_{S} (\nabla \times E) \cdot dS \]

\[ \int_{S} \left( \frac{\partial B}{\partial t} + \nabla \times E \right) \cdot dS = 0 \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad \text{Faraday's Law in differential form} \]
Differential form of Ampère’s Law

Integral form of law: enclosed current is integral $d\mathbf{S}$ of current density $\mathbf{j}$

$$\oint \mathbf{B} \cdot d\ell = \mu_0 I_{\text{encl}} = \mu_0 \int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{S}$$

Apply Stokes’ theorem

$$\oint_{\mathcal{S}} \mathbf{B} \cdot d\ell = \int_{\mathcal{S}} (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{S}$$

$$\int_{\mathcal{S}} (\nabla \times \mathbf{B} - \mu_0 \mathbf{j}) \cdot d\mathbf{S} = 0$$

Integration surface is arbitrary

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

Must be true point wise
Maxwell’s Equations in Vacuum

Take curl of both sides of 3’

\[
(3) \nabla \times (\nabla \times \mathbf{E}) = -\partial (\nabla \times \mathbf{B})/\partial t = -\partial (\mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t)/\partial t = -\mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2
\]

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad \text{vector identity}
\]

\[
- \nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2 \quad (\nabla \cdot \mathbf{E} = 0)
\]

\[
\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2 = 0 \quad \text{Vector wave equation}
\]
Maxwell’s Equations in Vacuum

Plane wave solution to wave equation

\[ E(r, t) = \text{Re} \{ E_0 e^{i(\omega t - k \cdot r)} \} \]

\( E_0 \) constant vector

\[ \nabla^2 E = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)E = -k^2 E \]

\[ \nabla \cdot E = \partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z = -i k \cdot E = -i k \cdot E_0 e^{i(\omega t - k \cdot r)} \]

If \( E_0 \parallel k \) then \( \nabla \cdot E \neq 0 \) and \( \nabla \times E = 0 \)

If \( E_0 \perp k \) then \( \nabla \cdot E = 0 \) and \( \nabla \times E \neq 0 \)

For light \( E_0 \perp k \) and \( E(r, t) \) is a transverse wave
Maxwell’s Equations in Vacuum

1-D waves $e^{i k z} \rightarrow$ 3-D waves $e^{i k \cdot r}$

$k \cdot r = k \cdot (r_\perp + r_\parallel) = k \cdot r_\parallel + 0$

$k \cdot r_\parallel = k r_\parallel$

$k r_\parallel = 2 \pi \Rightarrow r_\parallel = \lambda$

i.e. $k = \frac{2 \pi}{\lambda}$

Consecutive wave fronts

Plane waves travel parallel to wave vector $k$

Plane waves have wavelength $2 \pi / k$
Maxwell’s Equations in Vacuum

Plane wave solution to wave equation

\[ \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_o \, e^{i(\omega t - \mathbf{k}.\mathbf{r})} \quad \mathbf{E}_o \text{ constant vector} \]

\[ \mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2 = -\mu_0 \varepsilon_0 \omega^2 \mathbf{E} \quad \mu_0 \varepsilon_0 \omega^2 = k^2 \]

\[ \omega = \pm k/\left(\mu_0 \varepsilon_0\right)^{1/2} = \pm ck \quad \omega/k = c = (\mu_0 \varepsilon_0)^{-1/2} \text{ phase velocity} \]

\[ \omega = \pm ck \quad \text{Linear dispersion relationship} \]
Maxwell’s Equations in Vacuum

Magnetic component of the electromagnetic wave in vacuum

From Faraday’s law

$$\nabla \times (\nabla \times B) = \mu_0 \varepsilon_0 \frac{\partial (\nabla \times E)}{\partial t}$$

$$= \mu_0 \varepsilon_0 \frac{\partial (-\partial B/\partial t)}{\partial t}$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \nabla^2 B$$

$$- \nabla^2 B = -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} \quad (\nabla \cdot B = 0)$$

$$\nabla^2 B - \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} = 0 \quad \text{Same vector wave equation as for } E$$
Maxwell’s Equations in Vacuum

If \( \mathbf{E}(\mathbf{r}, t) = E_0 \mathbf{e}_x e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \) and \( \mathbf{k} \parallel \mathbf{e}_z \) and \( \mathbf{E} \parallel \mathbf{e}_x \) (\( \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \) unit vectors)

\[ \nabla \times \mathbf{E} = -i\mathbf{k} E_0 \mathbf{e}_y e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} = -\partial \mathbf{B}/\partial t \quad \text{From Faraday’s Law} \]

\[ \partial \mathbf{B}/\partial t = i\mathbf{k} E_0 \mathbf{e}_y e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \]

\[ \mathbf{B} = (k/\omega) E_0 \mathbf{e}_y e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} = (1/c) E_0 \mathbf{e}_y e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \]

For this wave \( \mathbf{E} \parallel \mathbf{e}_x, \mathbf{B} \parallel \mathbf{e}_y, \mathbf{k} \parallel \mathbf{e}_z, c\mathbf{B}_0 = E_0 \)
Maxwell’s Equations in Matter

Solution of Maxwell’s equations in matter for $\mu = 1$, $\rho_{\text{free}} = 0$, $j_{\text{free}} = 0$

Maxwell’s equations become

$$\nabla \times E = -\partial B / \partial t$$

$$\nabla \times H = \partial D / \partial t \quad H = B / \mu_0 \quad D = \varepsilon_0 \varepsilon E$$

$$\nabla \times B = \mu_0 \varepsilon_0 \varepsilon \partial E / \partial t$$

$$\nabla \times \partial B / \partial t = \mu_0 \varepsilon_0 \varepsilon \partial^2 E / \partial t^2$$

$$\nabla \times (-\nabla \times E) = \nabla \times \partial B / \partial t = \mu_0 \varepsilon_0 \varepsilon \partial^2 E / \partial t^2$$

$$-\nabla(\nabla \cdot E) + \nabla^2 E = \mu_0 \varepsilon_0 \varepsilon \partial^2 E / \partial t^2 \quad \nabla \cdot \varepsilon E = \varepsilon \nabla \cdot E = 0 \text{ since } \rho_{\text{free}} = 0$$

$$\nabla^2 E - \mu_0 \varepsilon_0 \varepsilon \partial^2 E / \partial t^2 = 0$$
Maxwell’s Equations in Matter

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \mathbf{E}(r, t) = E_0 \mathbf{e}_x \text{Re}\{e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}\}$$

$$\nabla^2 \mathbf{E} = -k^2 \mathbf{E} \quad \mu_0 \varepsilon_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \varepsilon_0 \varepsilon \omega^2 \mathbf{E}$$

$$(-k^2 + \mu_0 \varepsilon_0 \varepsilon \omega^2) \mathbf{E} = 0$$

$$\omega^2 = k^2 / (\mu_0 \varepsilon_0 \varepsilon) \quad \mu_0 \varepsilon_0 \varepsilon \omega^2 = k^2 \quad k = \pm \omega \sqrt{\mu_0 \varepsilon_0 \varepsilon} \quad k = \pm \sqrt{\varepsilon} \omega / c$$

Let $$\varepsilon = \varepsilon_1 - i \varepsilon_2$$ be the real and imaginary parts of $$\varepsilon$$ and $$\varepsilon = (n - i \kappa)^2$$

We need $$\sqrt{\varepsilon} = n - i \kappa$$

$$\varepsilon = (n - i \kappa)^2 = n^2 - \kappa^2 - i \ 2n \kappa \quad \varepsilon_1 = n^2 - \kappa^2 \quad \varepsilon_2 = 2n \kappa$$

$$\mathbf{E}(r, t) = E_0 \mathbf{e}_x \text{Re}\{e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}\} = E_0 \mathbf{e}_x \text{Re}\{e^{i(\omega t - k z)}\} \quad \mathbf{k} \parallel \mathbf{e}_z$$

$$= E_0 \mathbf{e}_x \text{Re}\{e^{i(\omega t - (n - i \kappa) \omega z / c)}\} = E_0 \mathbf{e}_x \text{Re}\{e^{i(\omega t - n \omega z / c) e^{-\kappa \omega z / c}}\}$$

Attenuated wave with phase velocity $$v_p = c/n$$
Maxwell’s Equations in Matter

Solution of Maxwell’s equations in matter for \( \mu = 1, \rho_{\text{free}} = 0, j_{\text{free}} = \sigma(\omega)E \)

Maxwell’s equations become

\[
\nabla \times E = -\partial B/\partial t
\]

\[
\nabla \times H = j_{\text{free}} + \partial D/\partial t \quad H = B / \mu_0 \quad D = \varepsilon_0 \varepsilon E
\]

\[
\nabla \times B = \mu_0 j_{\text{free}} + \mu_0 \varepsilon_0 \varepsilon \partial E/\partial t
\]

\[
\nabla \times \partial B/\partial t = \mu_0 \sigma \partial E/\partial t + \mu_0 \varepsilon_0 \varepsilon \partial^2 E/\partial t^2
\]

\[
\nabla \times (-\nabla \times E) = \nabla \times \partial B/\partial t = \mu_0 \sigma \partial E/\partial t + \mu_0 \varepsilon_0 \varepsilon \partial^2 E/\partial t^2
\]

\[
-\nabla(\nabla \cdot E) + \nabla^2 E = \mu_0 \sigma \partial E/\partial t + \mu_0 \varepsilon_0 \varepsilon \partial^2 E/\partial t^2 \quad \nabla \cdot \varepsilon E = \varepsilon \nabla \cdot E = 0 \text{ since } \rho_{\text{free}} = 0
\]

\[
\nabla^2 E - \mu_0 \sigma \partial E/\partial t - \mu_0 \varepsilon_0 \varepsilon \partial^2 E/\partial t^2 = 0
\]
Maxwell’s Equations in Matter

\[ \nabla^2 E - \mu_0 \sigma \frac{\partial E}{\partial t} - \mu_0 \varepsilon_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \quad E(r, t) = E_0 e_x \text{Re}\{e^{i(\omega t - k \cdot r)}\} \quad k \parallel e_z \]

\[ \nabla^2 E = -k^2 E \quad \mu_0 \sigma \frac{\partial E}{\partial t} = \mu_0 i \omega E \quad \mu_0 \varepsilon_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = -\mu_0 \varepsilon_0 \varepsilon \omega^2 E \]

\[ (-k^2 - \mu_0 \sigma \omega + \mu_0 \varepsilon_0 \varepsilon \omega^2)E = 0 \quad \sigma \gg \varepsilon_0 \varepsilon \omega \quad \text{for a good conductor} \]

\[ k^2 = -\omega \sigma \mu \quad k = \sqrt{-i \sqrt{\omega \sigma \mu}} = \frac{1}{\sqrt{2}}(1-i)\sqrt{\omega \sigma \mu} \]

\[ E(r, t) = E_0 e_x \text{Re}\{e^{i(\omega t - \sqrt{(\omega \sigma \mu_0/2)z})}e^{-\sqrt{(\omega \sigma \mu_0/2)z}}\} \]

NB wave travels in +z direction and is attenuated

The skin depth \( \delta = \sqrt{(2/\omega \sigma \mu_0)} \) is the thickness over which incident radiation is attenuated. For example, Cu metal DC conductivity is \( 5.7 \times 10^7 \) (\( \Omega \text{m} \))^{-1}

At 50 Hz \( \delta = 9 \) mm and at 10 kHz \( \delta = 0.7 \) mm
Bound and Free Charges

Bound charges
All valence electrons in insulators (materials with a ‘band gap’)
Bound valence electrons in metals or semiconductors (band gap absent/small)

Free charges
Conduction electrons in metals or semiconductors

Resonance frequency \( \omega_o \sim (k/M)^{1/2} \) or \( \sim (k/m)^{1/2} \)
Ions: heavy, resonance in infra-red \( \sim 10^{13} \text{Hz} \)
Bound electrons: light, resonance in visible \( \sim 10^{15} \text{Hz} \)
Free electrons: no restoring force, no resonance
Bound and Free Charges

Bound charges
Resonance model for uncoupled electron pairs

\[
\begin{align*}
\ddot{x} + m \dot{x} + k x &= q E_o \text{Re}\{e^{+i\omega t}\} \\
\dot{x} + \Gamma \dot{x} + \frac{k}{m} x &= \frac{q}{m} E_o \text{Re}\{e^{+i\omega t}\}
\end{align*}
\]

\[x(t) = \text{Re}\{A(\omega) e^{+i\omega t}\} \text{ trial solution}
\]

\[\dot{x}(t) = +i\omega x(t) \quad \ddot{x}(t) = -\omega^2 x(t) \quad (\text{Re}\{\}) \text{ assumed hereafter}
\]

\[
\begin{pmatrix}
-\omega^2 + i\Gamma \omega + \frac{k}{m}
\end{pmatrix} x(t) = \frac{q}{m} E_o e^{+i\omega t}
\]

\[
\begin{pmatrix}
-\omega^2 + i\Gamma \omega + \frac{k}{m}
\end{pmatrix} A(\omega) e^{+i\omega t} = \frac{q}{m} E_o e^{+i\omega t}
\]
Bound and Free Charges

Bound charges
In and out of phase components of $x(t)$ relative to $E_o \cos(\omega t)$

$A(\omega) = \frac{qE_o}{m} \left( -\omega^2 + i\Gamma \omega + \omega_o^2 \right)$

$\omega_o^2 = \frac{k}{m}$

$A(\omega) = \frac{qE_o}{m} \left( \omega_o^2 - \omega^2 + i\Gamma \omega \right)$

$\text{Re}\{A(\omega)\} = \frac{qE_o}{m} \left( \frac{\omega^2 - \omega_o^2}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\Gamma \omega\right)^2} \right)$

$\text{Im}\{A(\omega)\} = \frac{qE_o}{m} \left( \frac{-\Gamma \omega}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\Gamma \omega\right)^2} \right)$

$x(t) = \text{Re}\{A(\omega)e^{+i\omega t}\} = \text{Re}\{A(\omega)\} \text{Re}\{e^{+i\omega t}\} - \text{Im}\{A(\omega)\} \text{Im}\{e^{+i\omega t}\}$

$= \frac{qE_o}{m} \left[ \frac{\left(\omega_o^2 - \omega^2\right)\cos(\omega t)}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\Gamma \omega\right)^2} + \frac{\Gamma \omega \sin(\omega t)}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\Gamma \omega\right)^2} \right]$

_in phase_ \hspace{1cm} _out of phase_
Bound and Free Charges

Bound charges
Connection to $\chi$ and $\varepsilon$

Polarisation $\equiv$ dipole moment per unit volume $= qx(t)/V$

$$P(t) = \frac{q^2}{mV} \text{Re}\left\{ \frac{\left(\omega_o^2 - \omega^2\right)}{\left(\omega_o^2 - \omega^2\right)^2 + (\Gamma \omega)^2} \right\} e^{-i\omega t}$$

$$\chi(\omega) = 1 + \frac{q^2}{m\varepsilon_o V} \frac{\omega_o^2 - \omega^2 - i\Gamma \omega}{\left(\omega_o^2 - \omega^2\right)^2 + (\Gamma \omega)^2}$$

$$\varepsilon(\omega) = 1 + \chi(\omega) = 1 + \frac{q^2}{m\varepsilon_o V} \frac{\omega_o^2 - \omega^2 - i\Gamma \omega}{\left(\omega_o^2 - \omega^2\right)^2 + (\Gamma \omega)^2}$$

- $\varepsilon(\omega)$
- $\text{Im}\{\varepsilon(\omega)\}$
- $\text{Re}\{\varepsilon(\omega)\}$

model dielectric function
Bound and Free Charges

Free charges
Let $\omega_0 \rightarrow 0$ in $\chi$ and $\varepsilon$  

$$j_{\text{pol}} = \frac{\partial P}{\partial t}$$

Current density  

$$j(t) = \sigma E(t) \quad \sigma \equiv \text{conductivity}$$

$$P(t) = \varepsilon_o \frac{q^2}{m\varepsilon_o V} \frac{1}{\omega^2 - \omega^2 + i\Gamma \omega} E_o e^{+i\omega t}$$

$$j_{\text{pol}}(t) = \frac{\partial P(t)}{\partial t} = \varepsilon_o \frac{q^2}{m\varepsilon_o V} \frac{+i\omega}{\omega^2 - \omega^2 + i\Gamma \omega} E_o e^{+i\omega t}$$

$$j_{\text{free}}(t) = \frac{\partial P(t)}{\partial t} = \varepsilon_o \frac{q^2}{m\varepsilon_o V} \left( -\omega^2 + i\Gamma \omega \right) E_o e^{+i\omega t}$$

Let $\omega_0 \rightarrow 0$

$$\sigma_{\text{free}}(\omega) = \frac{q^2}{mV} \frac{-i\omega}{-\omega^2 + i\Gamma \omega} = \frac{q^2}{mV} \frac{-i\omega^3 + \Gamma \omega^2}{\omega^4 + (\Gamma \omega)^2} = \frac{q^2}{mV} \frac{-i\omega + \Gamma}{\omega^2 + \Gamma^2}$$

Let $\omega_0 \rightarrow 0$

$$\sigma_{\text{free}}(0) = \frac{q^2}{mV\Gamma} = \frac{Ne^2}{m} \quad \tau \equiv \frac{1}{\Gamma}$$

Drude conductivity
Energy in Electromagnetic Waves

Rate of doing work on a moving charge

\[ W = \frac{d}{dt}(F \cdot dr) \quad F = qE + qv \times B \]

\[ = \frac{d}{dt}((qE + qv \times B) \cdot dr) = \frac{d}{dt}(qE \cdot dr) = qv \cdot E = \int j \cdot E \, d^3r \]

\[ j(r) = q \, v \, \delta(r - r') \]

\[ W = \int j \cdot E \, d^3r \quad \text{fields do work on currents in integration volume} \]

Eliminate \( j \) using modified Ampère’s law

\[ \nabla \times H = j_{\text{free}} + \frac{\partial D}{\partial t} \]

\[ W = \int (\nabla \times H - \frac{\partial D}{\partial t}) \cdot E \, d^3r \]
Energy in Electromagnetic Waves

Vector identity \( \nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \)

\[ W = \int (\nabla \times H - \partial D/\partial t) \cdot E \, d^3r \]

becomes

\[ W = \int (\nabla \cdot (H \times E) + H \cdot (\nabla \times E) - E \cdot \partial D/\partial t) \, d^3r \]

\[ = \int (\nabla \cdot (H \times E) - H \cdot \partial B/\partial t - E \cdot \partial D/\partial t) \, d^3r \]

\[ \nabla \times E = - \partial B/\partial t \]

\[ W = \int j \cdot E \, d^3r = -\int \nabla \cdot (E \times H) \, d^3r - \frac{d}{dt} \int \frac{1}{2} (H \cdot B + D \cdot E) \, d^3r \]

\[ U = \frac{1}{2} (H \cdot B + D \cdot E) \]  
Local energy density

\[ N = E \times H \]  
Poynting vector

\[
\frac{\partial U}{\partial t} + \nabla \cdot N = -j \cdot E \]  
Energy conservation
Energy in Electromagnetic Waves

Energy density in plane electromagnetic waves in vacuum

\[ U = \frac{1}{2} (\mathbf{B} \cdot \mathbf{H} + \mathbf{D} \cdot \mathbf{E}) \]

\[ \mathbf{E} = E_0 \mathbf{e}_x \text{Re} \left( e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right) \]
\[ \mathbf{H} = H_0 \mathbf{e}_y \text{Re} \left( e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right) \quad \mathbf{k} \parallel \mathbf{e}_z \]

\[ U = \frac{1}{2} (\varepsilon_0 \mathbf{E} \cdot \mathbf{E} + \mu_0 \mathbf{H} \cdot \mathbf{H}) \cos^2(\omega t - kz) \]

\[ U = \frac{1}{2} \left( \varepsilon_0 \varepsilon_\mu \mu_0 H \mu_0 H \frac{E}{\mu_0 c} \right) \cos^2(\omega t - kz) \quad E = B \quad c = \mu_0 H \quad c \]

\[ U = \frac{\mathbf{E} \cdot \mathbf{H}}{c} \cos^2(\omega t - kz) \quad \varepsilon_\mu = c^{-2} \]

\[ \overline{U} = \frac{\mathbf{E} \cdot \mathbf{H}}{c} \frac{1}{2} ab \ c \tau \quad \langle \cos^2(\omega t - kz) \rangle = \frac{1}{2} \quad \text{mean energy density} \]

\[ \overline{U_c} = \frac{1}{2} \mathbf{E} \cdot \mathbf{H} \quad \text{mean energy flux} \quad \text{c.f. Poynting} \quad \mathbf{N} = \mathbf{E} \times \mathbf{H} \]