PY2T10 Electricity and Magnetism

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2.48 Lloyd Building

Course Outline

Course text: *Electromagnetism*, 2nd Edn. Grant and Phillips (Wiley)

Online at:

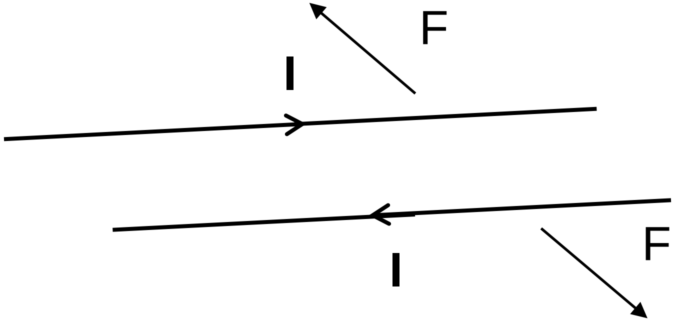
www.tcd.ie/Physics/People/Charles.Patterson/Teaching/SF/PY2T10/

Topics:

- Vector Operators and Vector Analysis
- Gauss' Law
- Ampere's Law
- Faraday's Law
- Maxwell's Equations in Vacuum
- Dielectrics
- Magnetism in Matter
- Maxwell's Equations in Matter

Sources and Forces

- Current **I** 1 ampère (amp) := current flowing in two parallel wires which produce repulsive force $F = 2 \times 10^{-7}$ N per metre of wire



- Charge q 1 coulomb (C) := amount of charge which must pass a point on a wire per second when a current of 1 amp flows

- Fundamental charge *e* 1.602×10^{-19} C

  1 amp = 6.242×10^{18} e s^{-1}

Charge and current densities

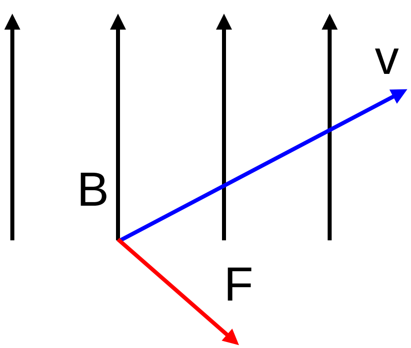
- Charge density $\rho(\mathbf{r},t)$ Cm^{-3} Scalar function of position and time. The *source* of electrostatic potential

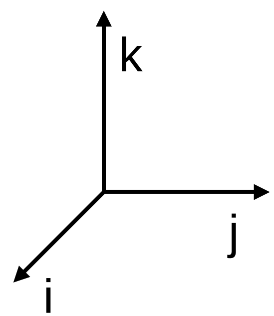
- Current density $\mathbf{j}(\mathbf{r},t)$ $Cm^{-2}\,s^{-1}$ Vector function of position and time. The *source* of electric vector potential

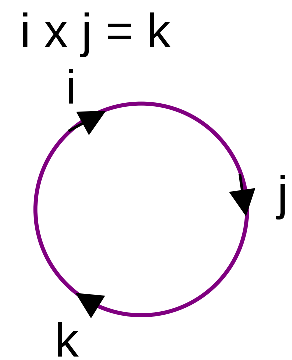
Force on charge due to electric and magnetic fields

- Lorentz force on single charge q

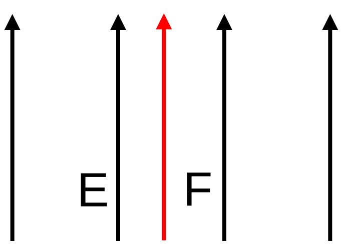
$\mathbf{F}_B = q\,\mathbf{v} \times \mathbf{B}$ **B** magnetic induction (tesla, T)







$\mathbf{F}_E = q\,\mathbf{E}$ **E** electric field strength (volts/m)



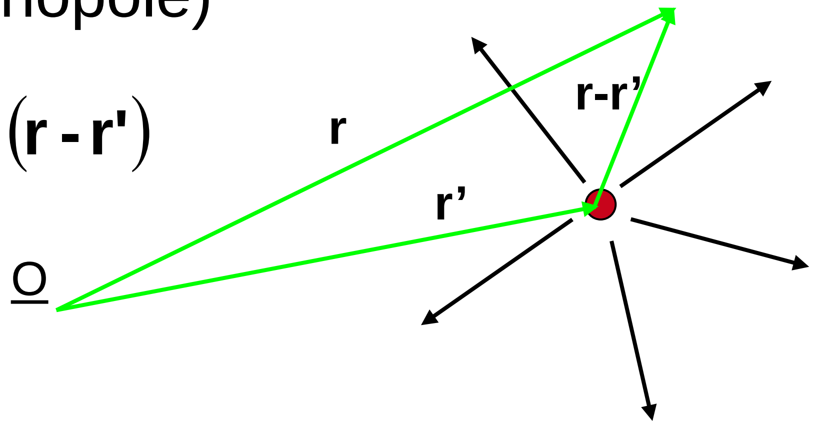
Sense of **F** depends on sign of q

Electric Fields

- Electric field strength **E**(**r**,t) volts m^{-1} or NC^{-1} Vector field of position and time

- Field at *field point* **r** due to single point charge at *source point* **r'** (electric monopole)

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')$$



Note **r-r'** vector directed away from source point when q is positive. Electric field lines point away from (towards) a positive (negative) charge

Electric Fields

- ε_o ‘epsilon nought’ is the permittivity of vacuum (free space)

- Its value is 8.854×10^{-12} $J^{-1}C^{2}m^{-1}$ (farad m^{-1})

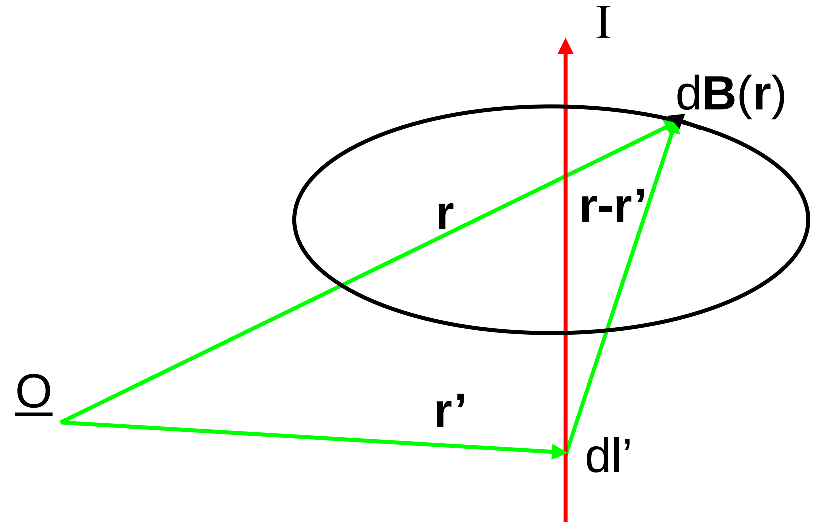
- The factor $\dfrac{1}{4\pi\varepsilon_o}$ has a value of 8.987×10^{9} $JC^{-2}m$

- High voltage equipment in laboratories, etc may be at kilovolts or hundreds of kilovolts (or higher) potential, separated from zero volts (laboratory floor etc) by distances of order 1 m. The corresponding field strength is $\sim 10^{5}$ Vm^{-1}
- Electric field strength in laser beam may be of order 10^{7} Vm^{-1}
- Electric field strength at the bohr radius in the H atom is $\sim 10^{11}$ Vm^{-1}

Magnetic Fields

- Magnetic Induction (Magnetic flux density) **B**(**r**,t) tesla (T) Vector field of position and time
- Field at *field point* **r** due to current element at *source point* **r'** is given by Biot-Savart Law

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi} \frac{dl' \, x (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3}$$



- Note d**B**(**r**) is the contribution to the circulating magnetic field which surrounds this infinite wire from the current element d**l**'

Magnetic Fields

- μ_o 'mu nought' is the permeability of vacuum (free space)
- Its value is defined as $4\pi \times 10^{-7}$ $Js^2C^{-2}m^{-1}$ (henry m^{-1})
- The factor $\mu_o/4\pi$ has a value of 10^{-7} $Js^2C^{-2}m^{-1}$ (henry m^{-1})
- Magnetic fields at the earth' surface 3 to 6 x 10^{-5} T (0.3 to 0.6 Gauss, G) 1 G = 10^{-4} T
- Magnetic fields in laboratory routinely ~ 1T
- MRI scanner in Lloyd building is 3T

Electric Fields in Matter

- External electric fields **E** cause electric polarisation in matter

- Polarisation **P** is a deformation of the electric charge density which depends (nearly) linearly on **E**

- **P** related to **E** by electric susceptibility, χ $\mathbf{P} = \varepsilon_o \chi \mathbf{E}$

- Introduce new fields polarisation, **P**, and electric displacement, **D** = **D**[**E**, **P**]

- This is a constitutive relation

Magnetic Fields in Matter

- External magnetic fields **B** cause currents **j** in matter
- Magnetisation, **M**, is related to current density in matter
- Introduce magnetic susceptibility χ_B $\mathbf{M} = \chi_B \mathbf{B}/\mu_o$
- Introduce new fields magnetisation, **M**, and magnetic field strength, $\mathbf{H} = \mathbf{H}[\mathbf{B}, \mathbf{M}]$
- The constitutive relation relates the magnetic field strength **H** to magnetic induction **B** and magnetisation **M**

Maxwell's Equations

$$\nabla.\mathbf{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla.\mathbf{B} = 0$$

$$\nabla x\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla x\mathbf{B} = \mu_o \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Vacuum

$$\nabla.\mathbf{D} = \rho_f$$

$$\nabla.\mathbf{B} = 0$$

$$\nabla x\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla x\mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Matter

- Expressed in integral or differential forms
- Simplest to derive integral form from physical principle
- Equations easier to use in differential form
- Forms related by vector field identities (Stokes' Theorem, Gauss' Divergence Theorem)
- Time-independent problems electrostatics, magnetostatics
- Time-dependent problems electromagnetic waves

Vector Operators and Analysis

- Div, Grad, Curl (and all that)

- Del or nabla operator
  - In Cartesian coordinates

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

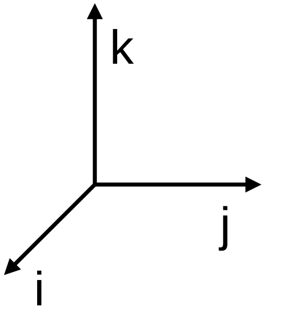
- Combining vectors in 3 ways
  - Scalar (inner) product **a**.**b** = **c** (scalar)
  - Cross (vector) product **a**x**b** = **c** (vector)
  - Outer product (dyad) **ab** = **c** (tensor)

Scalar Product - Divergence

- **r** is a Cartesian position vector **r=(x,y,z)**
- **A** is vector function of position **r** $\mathbf{A}(\mathbf{r}) = \left(A_x, A_y, A_z\right)$
- Div **A** = $\nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
- Scalar product of **del** with **A**
- Scalar function of position

Cross Product - Curl

- Curl **A** = $\nabla \mathrm{x}\mathbf{A} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$



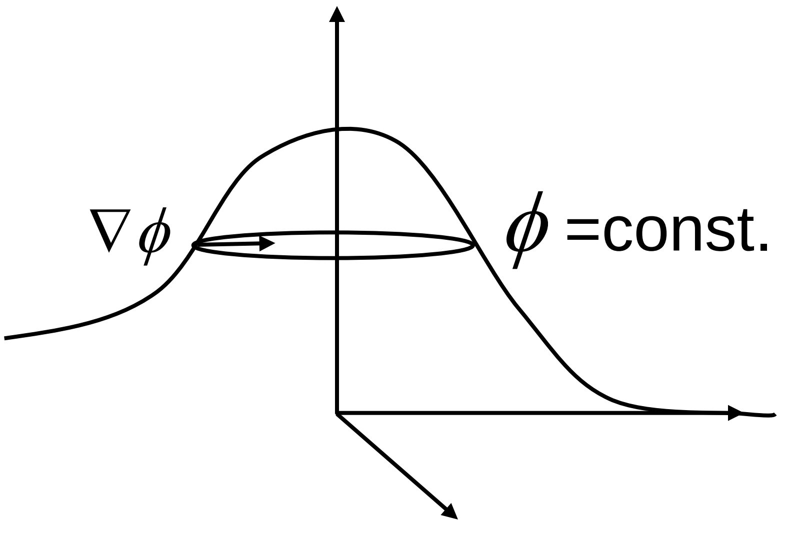
$$\nabla \mathrm{x}\mathbf{A} = \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

- Cross product of **del** with **A**

- Vector function of position

Gradient

- $\phi(x,y,z)$ is a scalar function of position
- Grad $\phi = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$
- Operation of del on scalar function
- Vector function of position



Div Grad – the Laplacian

- Inner product Del squared $\nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

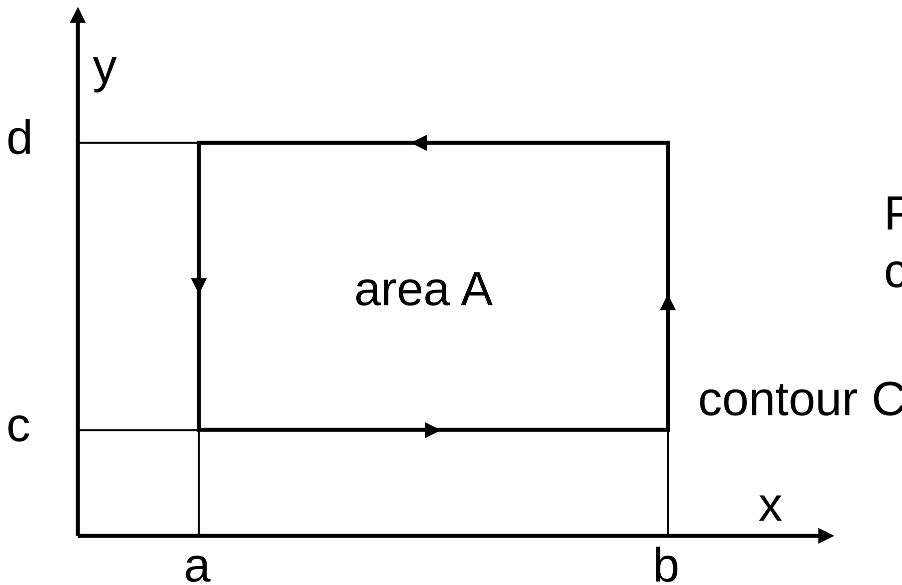
- Operates on a scalar function to produce a scalar function

- Outer product $\nabla\nabla = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2}{\partial z^2} \end{pmatrix}$

Green's Theorem on plane

- Leads to Divergence Theorem and Stokes' Theorem

- Fundamental theorem of calculus $$\int_a^b \frac{d}{dx} f(x)dx = f(b) - f(a)$$



P(x,y), Q(x,y) functions with continuous partial derivatives

- Green's Theorem $$\oint_C P(x,y)dx + Q(x,y)dy = \iint_A \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy$$

Green's Theorem on plane

- Integral of derivative over A

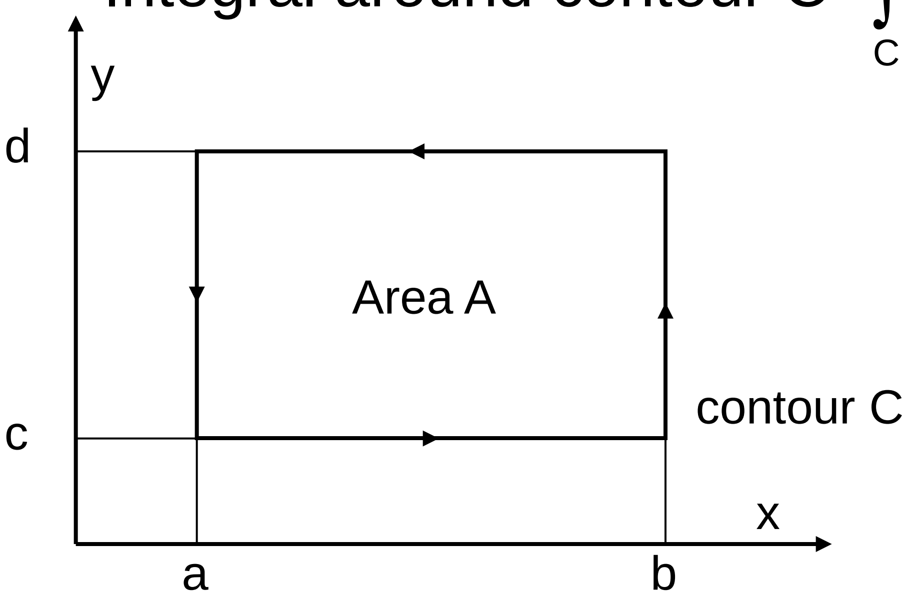
$$\iint_A \frac{\partial Q(x,y)}{\partial x} dxdy = \int_c^d dy \int_a^b \frac{\partial Q(x,y)}{\partial x} dx$$

$$= \int_c^d \left(Q(b,y) - Q(a,y)\right) dy$$

- Integral around contour C

$$\oint_C Q(x,y)dy = \int_c^d Q(b,y)dy + \int_d^c Q(a,y)dy$$

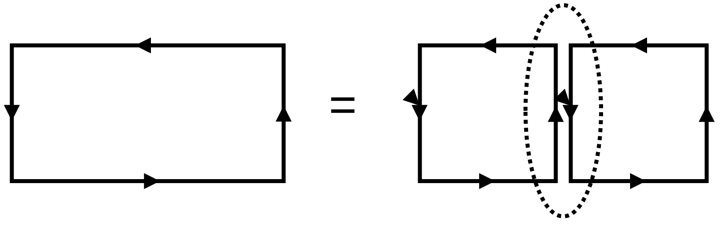
$$= \int_c^d \left(Q(b,y) - Q(a,y)\right) dy$$



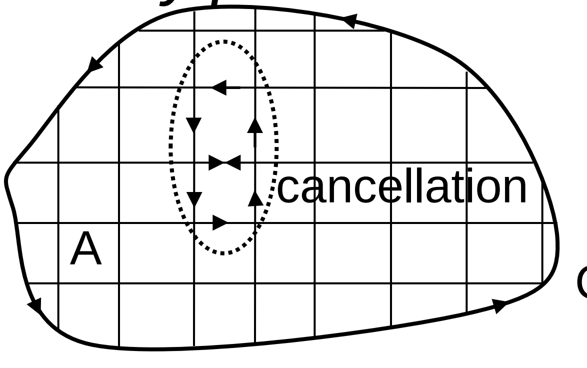
$$\oint_C Q(x,y)dy = \iint_A \frac{\partial Q(x,y)}{\partial x} dxdy$$

Green’s Theorem on plane

- Similarly $$\oint_C P(x,y)dx = -\iint_A \frac{\partial P(x,y)}{\partial y} dxdy$$
- Green’s Theorem relates an integral along a *closed* contour C to an area integral over the *enclosed* area A
- QED for a rectangular area (previous slide)
- Consider *two* rectangles and then *arbitrary planar surface*



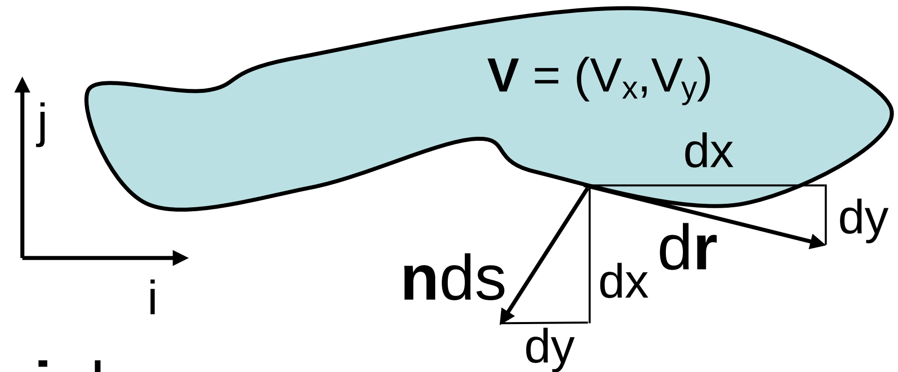
Contributions from boundaries cancel



No cancellation on boundary

- *Green’s Theorem applies to arbitrary, bounded surfaces*

Divergence Theorem



- Tangent d**r** = **i** dx + **j** dy
- Outward normal **n** ds = **i** dy – **j** dx
- **n** unit vector along outward normal
- ds = $(dx^2+dy^2)^{1/2}$
- $P(x,y) = -V_y$ $Q(x,y) = V_x$

  Cartesian components of the same vector field **V**
- $Pdx + Qdy = -V_y dx + V_x dy$
- $(\mathbf{i}\ V_x + \mathbf{j}\ V_y).(\mathbf{i}\ dy - \mathbf{j}\ dx) = -V_y\ dx + V_x\ dy = \mathbf{V}.\mathbf{n}\ ds$

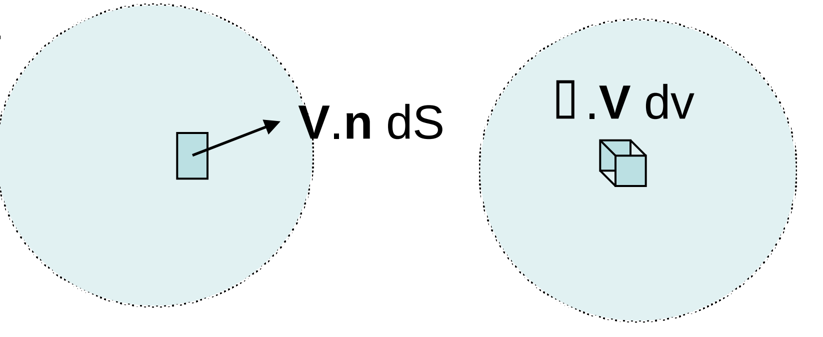
Divergence Theorem 2-D 3-D

- Apply Green's Theorem

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_A \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy$$

$$\oint_C \mathbf{V}.\mathbf{n}\, ds = \iint_A \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) dxdy = \iint_A \nabla.\mathbf{V}\, dxdy$$

- *In words* - Integral of **V**.**n** ds over surface contour equals integral of div **V** over surface area



- In 3-D $\oint_S \mathbf{V}.\mathbf{n}\, dS = \int_V \nabla.\mathbf{V}\, dv$
- Integral of **V**.**n** dS over bounding surface S equals integral of div **V** dv within volume enclosed by surface S

Curl and Stokes' Theorem

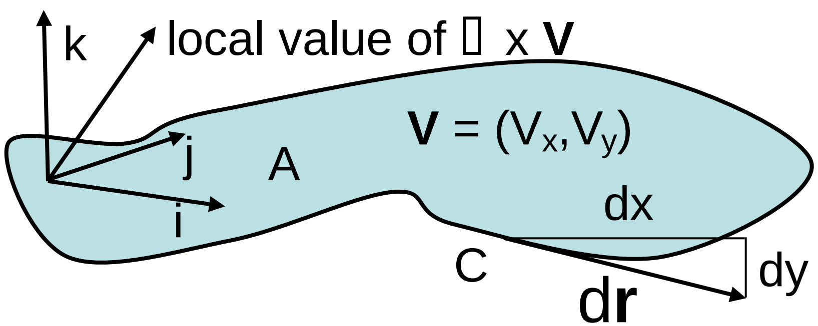
- For divergence theorem $P(x,y) = -V_y \quad Q(x,y) = V_x$
- Instead choose $P(x,y) = V_x \quad Q(x,y) = V_y$
- $Pdx + Qdy = V_x\,dx + V_y\,dy$
- $\mathbf{V} = \mathbf{i}\,V_x + \mathbf{j}\,V_y + 0\,\mathbf{k}$

$$P(x,y)dx + Q(x,y) = V_x\ dx + V_y\ dy$$

$$P(x,y)dx + Q(x,y)dy = \left(\mathbf{i}\,V_x + \mathbf{j}\,V_y\right).\left(\mathbf{i}\,dx + \mathbf{j}\,dy\right) = \mathbf{V}\,.\,d\mathbf{r}$$

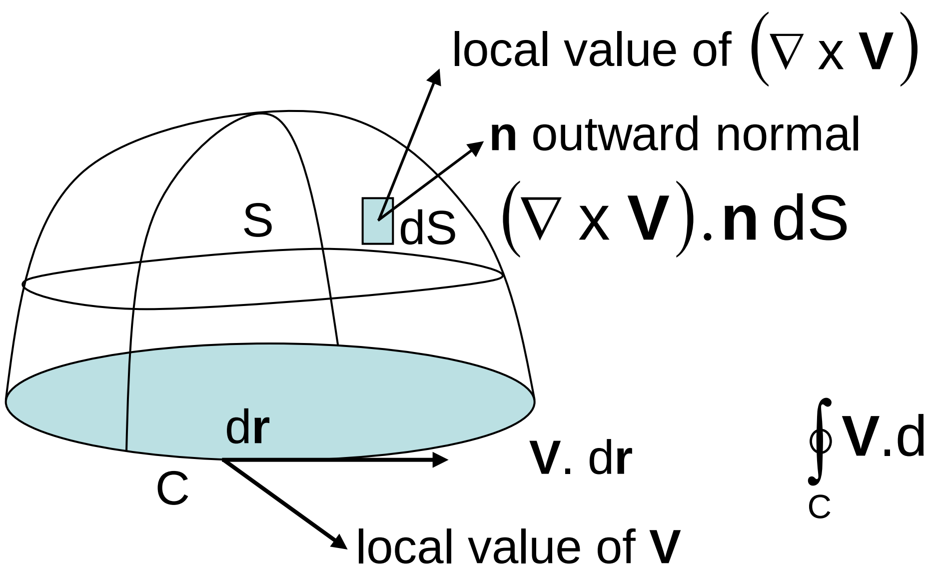
$$\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \left(\nabla \text{ x } \mathbf{V}\right).\mathbf{k}$$

$$\oint_C \mathbf{V}.d\mathbf{r} = \iint_A \left(\nabla \text{ x } \mathbf{V}\right).\mathbf{k}\ dxdy$$



Stokes' Theorem 3-D

- *In words* - Integral of $(\nabla \times \mathbf{V}).\mathbf{n}\,dS$ over surface S equals integral of **V.**d**r** over bounding contour C
- It doesn't matter which surface (blue or hatched). Direction of d**r** determined by right hand rule.



$$\oint_C \mathbf{V}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{V}).\mathbf{n}\,dS$$

Summary

- Green's Theorem

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_A \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy$$

- Divergence theorem

$$\oint_S \mathbf{V}.\mathbf{n}\, dS = \int_V \nabla.\mathbf{A}\, dv$$

- Stokes' Theorem

$$\oint_C \mathbf{V}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{V}).\, \mathbf{n}\, dS$$

- Continuity equation

$$\nabla.\mathbf{j}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0$$