Homework Problems I for PY2T10

Stokes' Theorem

1. For the vector field $\mathbf{V} = 4y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$

(a) Show that $\int_S (\nabla x \boldsymbol{V}).d\boldsymbol{S} = -3\pi a^2$ where *S* is the surface of a hemisphere of radius a with z ≥ 0, i.e. $x^2 + y^2 + z^2 = a^2$.

(b) Show that $\oint_C \boldsymbol{V}.d\boldsymbol{r} = -3\pi a^2$ where *C* is the circle $x^2 + y^2 = a^2$. Why are both results the same?

Hint for both parts: Evaluate the curl and scalar product in Cartesian coordinates and then convert to spherical polar or polar coordinates to perform the integration.

Electrostatic field tensor for point dipole

2. The electrostatic potential of a point electric dipole, **p**, of magnitude, p, placed at the Cartesian coordinate origin is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p.r}}{r^3} \quad \text{where } r^2 = x^2 + y^2 + z^2$$

Use the relationship between electrostatic potential and electric field to show that the electric field of the dipole is given by

$$\mathbf{E} = \mathbf{T}.\mathbf{p} \qquad (\mathbf{T})_{ij} = \frac{1}{4\pi\varepsilon_o}\frac{3x_i x_j - r^2\delta_{ij}}{r^5} \quad \text{where } r^2 = x^2 + y^2 + z^2$$

Subscripts ij refer to Cartesian components of the matrix **T** or position vector, **r**.

Explain why you would expect the curl of **E** to vanish.

Gauss' Law applied to a non-uniformly charged sphere

3. The charge density, $\rho(r)$, in a spherical object varies with distance from the centre as $\rho(r) = (1 - r^2)$, where $0 < r < 1$.

(a) Find the total charge enclosed by a sphere of radius, R, centred on the spherical object.

(b) Find the electric field strength at R and show that the field obeys Gauss' law in its differential form, $\nabla.\mathbf{E} = \dfrac{\rho}{\varepsilon_o}$.

The divergence of **E** in spherical polar coordinates is

$$\nabla.\mathbf{E} = \left(\frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial(E_\theta \sin\theta)}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}\right)$$