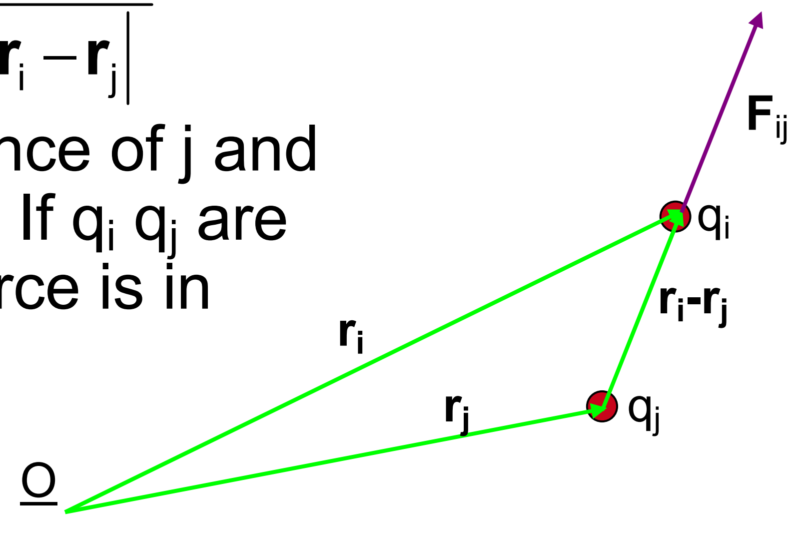
Coulomb's Law

- **Coulomb's Law**: force on charge i due to charge j is

$$\mathbf{F}_{ij} = \frac{1}{4\pi\varepsilon_o}\frac{q_i q_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}\left(\mathbf{r}_i - \mathbf{r}_j\right) = \frac{1}{4\pi\varepsilon_o}\frac{q_i q_j}{r_{ij}^{\;2}}\hat{\mathbf{r}}_{ij}$$

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \qquad r_{ij} = \left|\mathbf{r}_i - \mathbf{r}_j\right| \qquad \hat{\mathbf{r}}_{ij} = \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

- $\mathbf{F}_{ij}$ is force on i due to presence of j and acts along line of centres $\mathbf{r}_{ij}$. If q_i q_j are same sign then repulsive force is in direction shown



- Inverse square law of force

Principle of Superposition

- Total force on one charge i is

$$\mathbf{F}_i = q_i \sum_{j \neq i} \frac{1}{4\pi\varepsilon_o} \frac{q_j}{r_{ij}^{\ 2}} \hat{\mathbf{r}}_{ij}$$

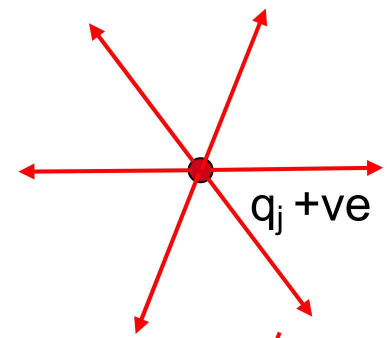
- i.e. linear superposition of forces due to all other charges

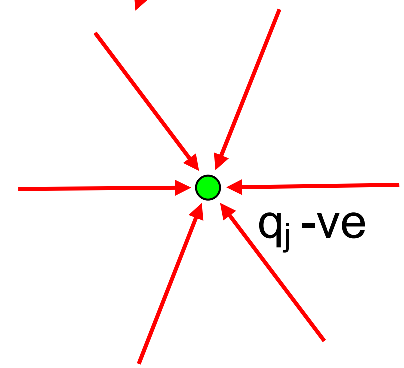
- **Test charge**: one which does not influence other 'real charges' – samples the electric field, potential
- Electric field experienced by a test charge q_i ar $\mathbf{r}_i$ is

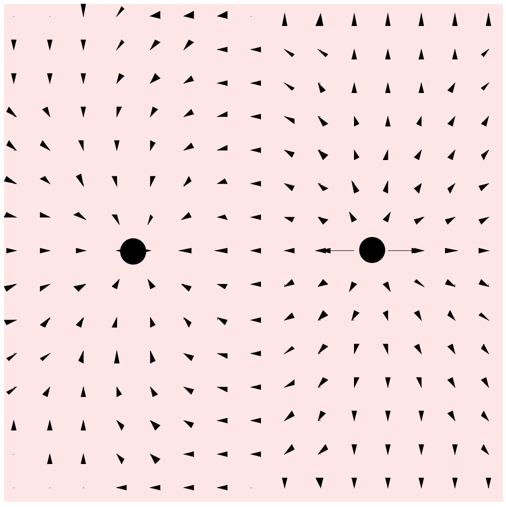
$$\mathbf{E}_i(\mathbf{r}_i) = \frac{\mathbf{F}_i}{q_i} = \sum_{j \neq i} \frac{1}{4\pi\varepsilon_o} \frac{q_j}{r_{ij}^{\ 2}} \hat{\mathbf{r}}_{ij}$$

Electric Field

- Field lines give local direction of field
- Field around *positive charge* directed *away from* charge
- Field around *negative charge* directed *towards* charge
- Principle of superposition used for field due to a dipole (+ve –ve charge combination). Which is which?







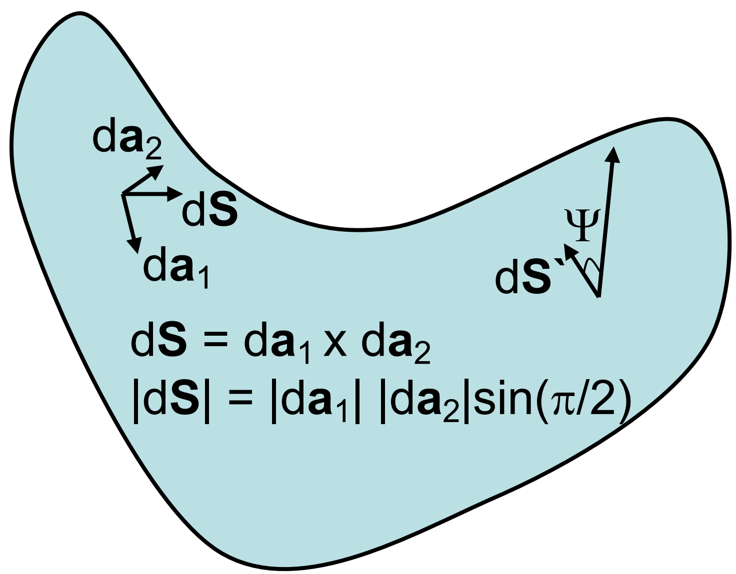
Flux of a Vector Field

- *Normal component* of vector field transports fluid across element of surface area

- Define surface area element as $d\mathbf{S} = d\mathbf{a}_1 \times d\mathbf{a}_2$

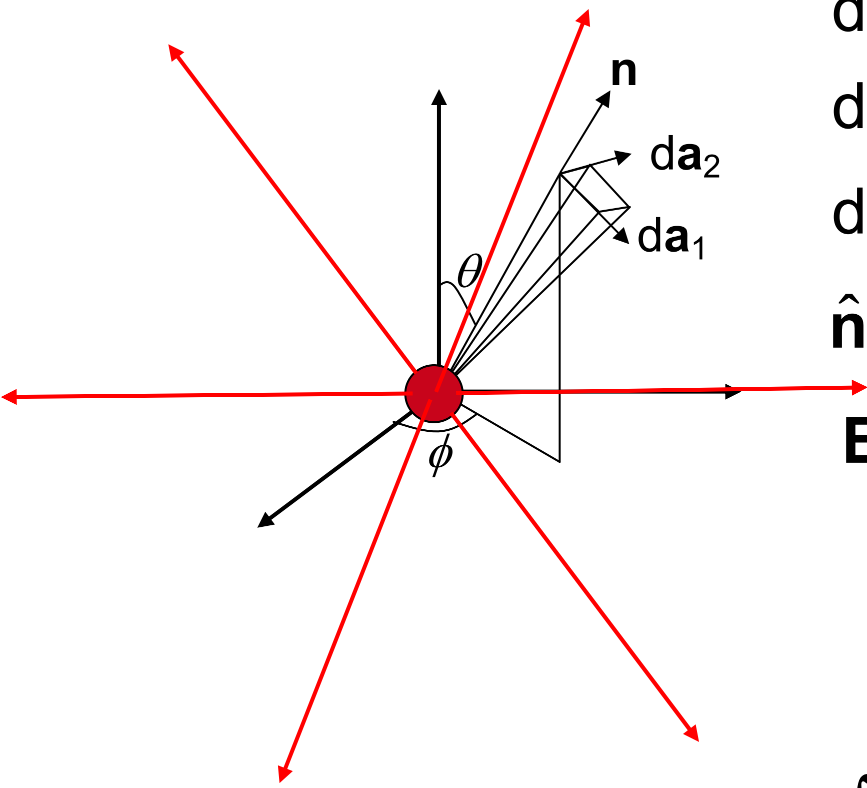
- Magnitude of normal component of vector field **V** is
  $\mathbf{V}.d\mathbf{S} = |\mathbf{V}||d\mathbf{S}| \cos(\Psi)$

- For current density **j**
  flux through surface S is
  $\int \mathbf{j}.d\mathbf{S}$ Cs^{-1}
  closed surface S



Flux of Electric Field

- Electric field is vector field (c.f. fluid velocity x density)
- Element of flux of electric field over closed surface **E**.d**S**



$$d\mathbf{a}_1 = r\, d\theta\, \hat{\boldsymbol{\theta}}$$

$$d\mathbf{a}_2 = r \sin\theta\, d\varphi\, \hat{\varphi}$$

$$d\mathbf{S} = d\mathbf{a}_1 \times d\mathbf{a}_2 = r^2 \sin\theta\, d\theta\ \ d\varphi\, \hat{\mathbf{n}}$$

$$\hat{\mathbf{n}} = \hat{\boldsymbol{\theta}} \times \hat{\varphi}$$

$$\mathbf{E}.d\mathbf{S} = \frac{q}{4\pi\varepsilon_o}\frac{\hat{\mathbf{r}}}{r^2}.\, r^2 \sin\theta\, d\theta\, d\varphi\, \hat{\mathbf{n}} \quad \hat{\mathbf{r}}.\hat{\mathbf{n}} = 1$$

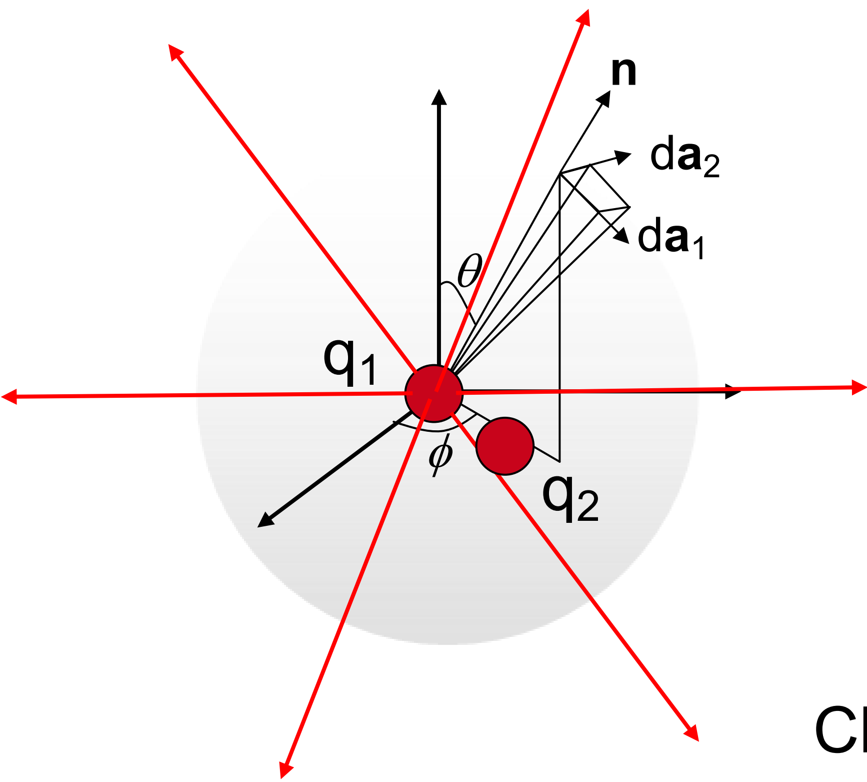
$$= \frac{q}{4\pi\varepsilon_o} \sin\theta\, d\theta\, d\varphi = \frac{q}{4\pi\varepsilon_o} d\Omega$$

$$\int_S \mathbf{E}.d\mathbf{S} = \frac{q}{\varepsilon_o}$$ Gauss' Law Integral Form

Integral form of Gauss' Law

- Factors of r^2 (area element) and $1/r^2$ (inverse square law) cancel in element of flux **E**.d**S**
- **E**.d**S** depends only on solid angle $d\Omega$

$$\mathbf{E}.d\mathbf{S} = \frac{q_1 + q_2}{4\pi\varepsilon_o} d\Omega$$



$$\int_S \mathbf{E}.d\mathbf{S} = \frac{\sum_i q_i}{\varepsilon_o}$$

Point charges: q_i *enclosed by S*

$$\int_S \mathbf{E}.d\mathbf{S} = \frac{\int_V \rho(\mathbf{r})dv}{\varepsilon_o}$$

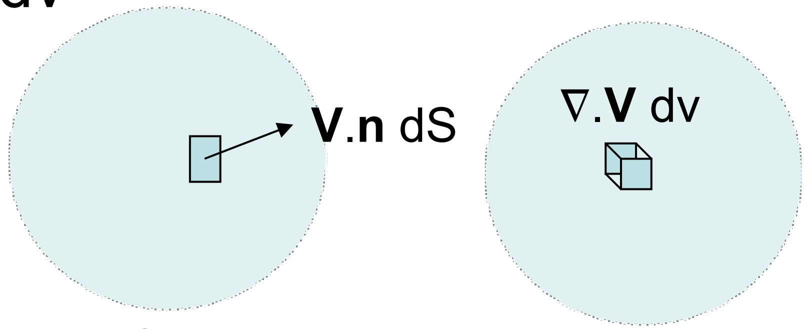
$$\int_V \rho(\mathbf{r})d\mathbf{v} = \text{total charge within v}$$

Charge distribution $\rho(\mathbf{r})$ *enclosed by S*

Differential form of Gauss' Law

- Integral form $\int_S \mathbf{E}.d\mathbf{S} = \frac{\int_V \rho(\mathbf{r})d\mathbf{r}}{\varepsilon_o}$

- Divergence theorem applied to field **V**, volume v bounded by surface S $\oint_S \mathbf{V}.\mathbf{n}\, dS = \oint_S \mathbf{V}.d\mathbf{S} = \int_V \nabla.\mathbf{V}\, dv$



- Divergence theorem applied to electric field **E**

$$\int_S \mathbf{E}.d\mathbf{S} = \int_V \nabla.\mathbf{E}\, dv$$

$$\int_V \nabla.\mathbf{E}\, dv = \frac{1}{\varepsilon_o}\int_V \rho(\mathbf{r})dv$$

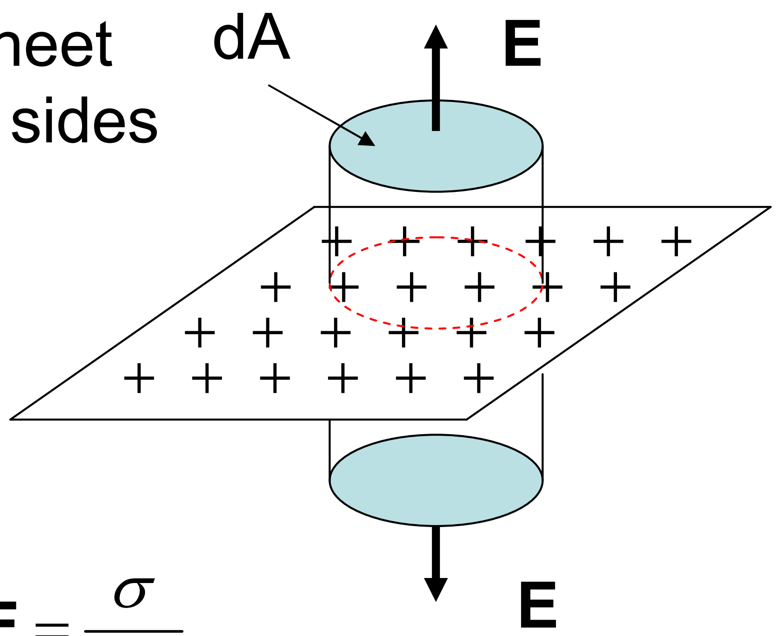
$$\nabla.\mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_o}$$

Differential form of Gauss' Law (Poisson's Equation)

Apply Gauss' Law to charge sheet

- ρ (C m^{-3}) is the 3D charge density, many applications make use of the 2D density σ (C m^{-2}):

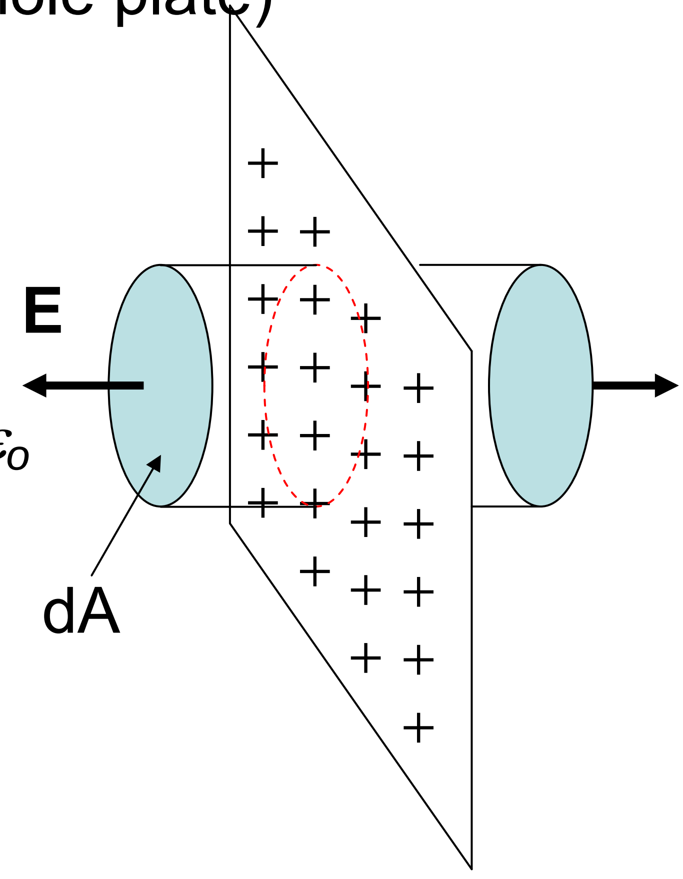
- Uniform *sheet* of charge density $\sigma =$ Q/A
- By symmetry, **E** is perpendicular to sheet
- Same everywhere, outwards on both sides
- Surface: cylinder sides + faces
- perp. to sheet, end faces of area dA
- Only end faces contribute to integral



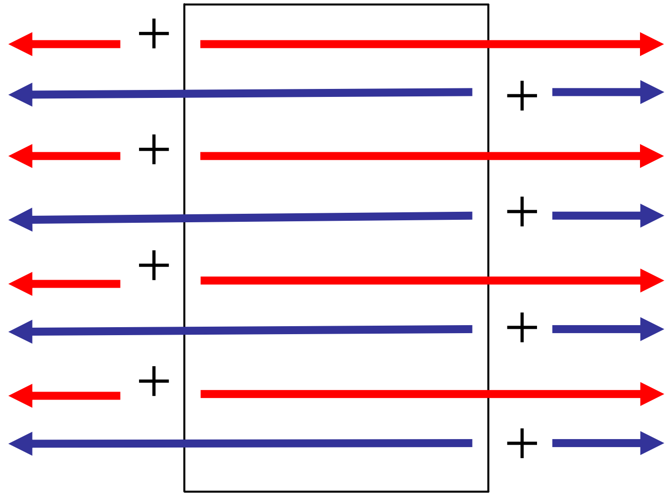
$$\int_{\mathbf{S}} \mathbf{E}.d\mathbf{S} = \frac{Q_{encl}}{\varepsilon_o} \Rightarrow E.2dA = \frac{\sigma.dA}{\varepsilon_o} \Rightarrow \mathbf{E} = \frac{\sigma}{2\varepsilon_o}$$

Apply Gauss' Law to charged plate

- $\sigma' = Q/2A$ surface charge density Cm^{-2} (c.f. Q/A for sheet)
- $E\ 2dA = 2\sigma'\ dA/\varepsilon_o$ (surface encloses whole plate)
- $E = \sigma'/\varepsilon_o$ (outside left surface shown)
- E = 0 (inside metal plate)
- why??



- *Outside* $E = \sigma'/2\varepsilon_o + \sigma'/2\varepsilon_o = \sigma'/\varepsilon_o = \sigma/2\varepsilon_o$
- *Inside* fields from opposite faces cancel



Work of moving charge in E field

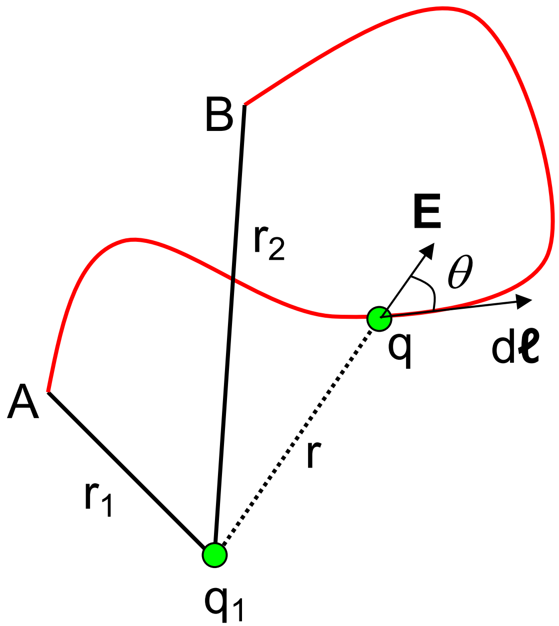
- $\mathbf{F}_{Coulomb} = q\mathbf{E}$
- Work done *on test charge* dW
- $dW = \mathbf{F}_{applied}.d\boldsymbol{\ell} = -\mathbf{F}_{Coulomb}.d\boldsymbol{\ell} = -q\mathbf{E}.d\boldsymbol{\ell} = -qEd\ell\cos\theta$
- $d\ell \cos\theta = dr$

$$dW = -q\frac{q_1}{4\pi\varepsilon_o}\frac{1}{r^2}dr$$

$$W = -q\frac{q_1}{4\pi\varepsilon_o}\int_{r_1}^{r_2}\frac{1}{r^2}dr$$

$$= -q\frac{q_1}{4\pi\varepsilon_o}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$= -q\int_A^B \mathbf{E}.d\ell$$



$$\oint \mathbf{E}.d\ell = 0$$

any closed path

- W is independent of the path (*electrostatic* **E** field is conservative)

Potential energy function

- Path independence of W leads to potential *and* potential energy functions

- Introduce electrostatic potential $$\phi(\mathbf{r}) = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{r}$$

- Work done on going from A to B = electrostatic potential energy difference
$$W_{BA} = PE(\mathbf{B}) - PE(\mathbf{A}) = q\left(\phi(\mathbf{B}) - \phi(\mathbf{A})\right)$$
$$= -q\int_A^B \mathbf{E}.d\mathbf{l}$$

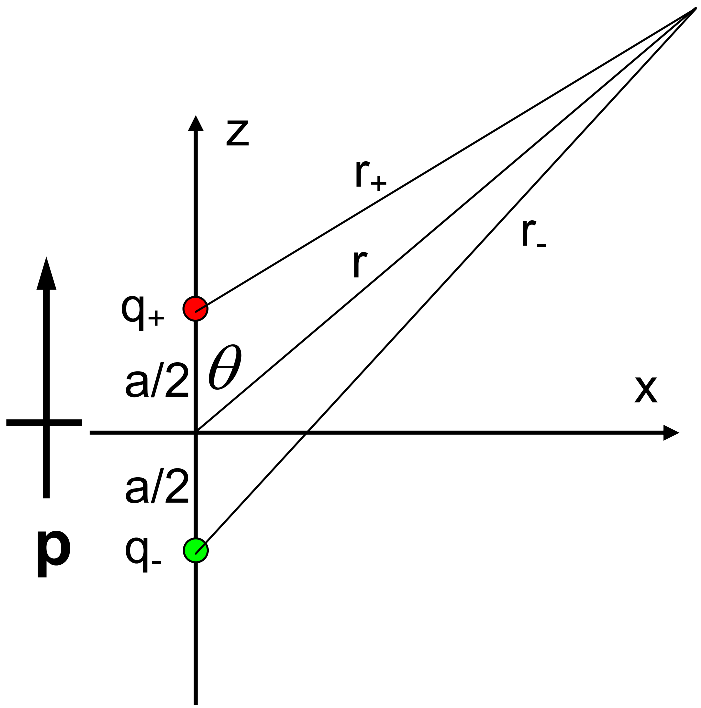
- Zero of potential energy is arbitrary
  - choose $\phi(r\to\infty)$ as zero of energy

Electric field from electrostatic potential

- Electric field created by q_1 at $\mathbf{r} = \mathbf{r}_B$ $$\mathbf{E} = \frac{q_1}{4\pi\varepsilon_o}\frac{\mathbf{r}}{r^3}$$
- Electric potential created by q_1 at $\mathbf{r}_B$ $$\phi(r_B) = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{r}$$
- Gradient of electric potential $$\nabla\phi(r_B) = -\frac{q_1}{4\pi\varepsilon_o}\frac{\mathbf{r}}{r^3}$$
- Electric field is therefore $\mathbf{E} = -\nabla\phi$

Electrostatic potential of point dipole

- +/- charges, equal magnitude, q, separation a
- axially symmetric potential (z axis)



$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$r_\pm^2 = r^2 + \left(\frac{a}{2}\right)^2 \mp a\,r\cos\theta$$

$$= r^2\left(1 + \left(\frac{a}{2r}\right)^2 \mp \frac{a}{r}\cos\theta\right)$$

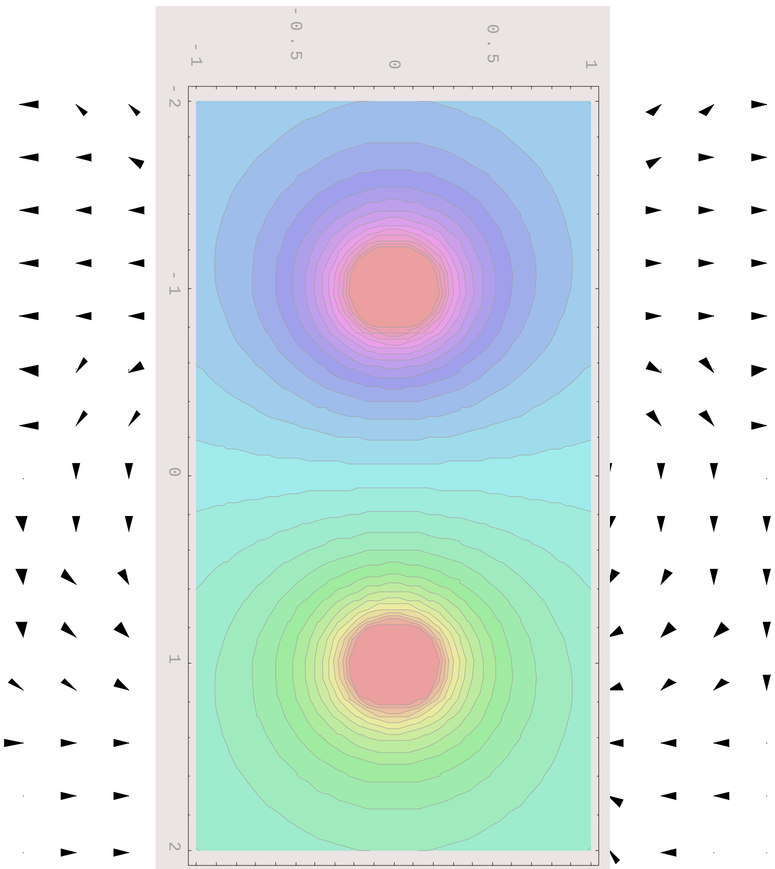
$$\frac{1}{r_\pm} = \frac{1}{r}\left(1 + \left(\frac{a}{2r}\right)^2 \mp \frac{a}{r}\cos\theta\right)^{-1/2}$$

$$= \frac{1}{r} \pm \frac{a}{2r^2}\cos\theta$$

$$\varphi(\mathbf{r}) = \frac{qa}{4\pi\varepsilon_o}\frac{\cos\theta}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_o\,r^2}$$

Electrostatic potential of point dipole

- Equipotential lines for an electric dipole || z axis
- Contours on which electric potential is constant

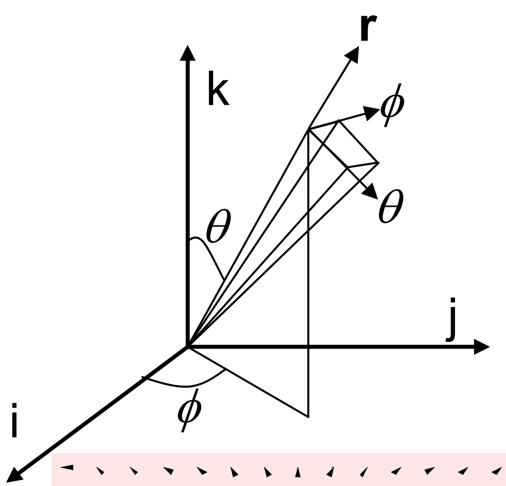


$$\varphi(\mathbf{r}) = \frac{p\cos\theta}{4\pi\varepsilon_o r^2}$$

- Equipotential lines perp. to field lines

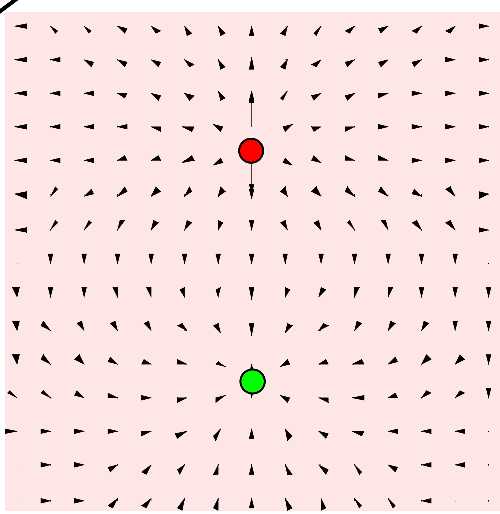
Electrostatic potential of point dipole

- Field lines for a point dipole || z axis
- Generated from **E** = $-\nabla \phi$



$$\nabla_{\text{Cart.}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \qquad (i, j, k)$$

$$\nabla_{\text{Sph.Pol.}} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi} \right) \quad (\hat{r}, \hat{\theta}, \hat{\phi})$$



$$\varphi(r) = \frac{p}{4\pi\varepsilon_o}\frac{\cos\theta}{r^2}$$

$$\mathbf{E}(r,\theta) = -\nabla\varphi(r,\theta) = \frac{p}{4\pi\varepsilon_o}\left(\frac{2\cos\theta}{r^3}, \frac{\sin\theta}{r^3}, 0 \right)$$

← NB diagram not a point dipole

Electrostatic potential of point dipole

- More generally, dipole direction given by **p** vector

$$\varphi(\mathrm{r}) = \frac{1}{4\pi\varepsilon_o r^3}\mathbf{p.r} \quad \mathbf{p} = (p_x, p_y, p_z) \;\; \mathbf{r} = (x, y, z)$$

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{p.r}}{r^3}\right) = \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)p_x - \frac{3xy}{r^5}p_y - \frac{3xz}{r^5}p_z$$

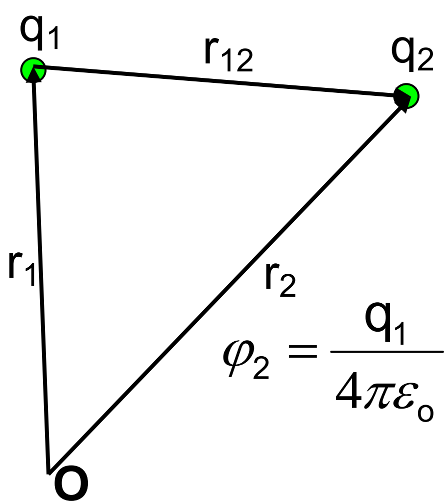
$$\mathbf{E}(r,\theta) = -\nabla\varphi(x, y, z) = \frac{1}{4\pi\varepsilon_o}\mathbf{T.p}$$

$$(\mathbf{T})_{ij} = \frac{3x_i x_j - r^2\delta_{ij}}{r^5}$$

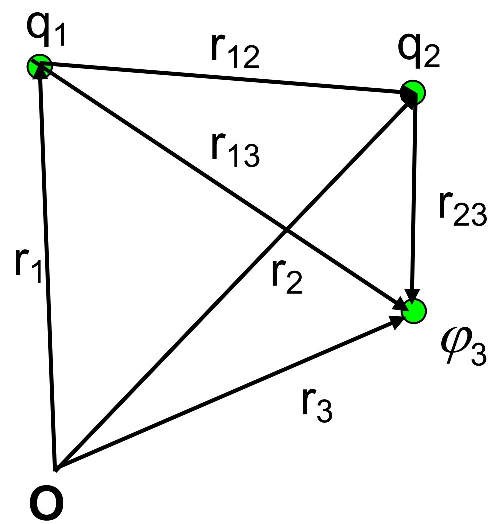
Dipole field propagation tensor

Electrostatic energy of point charges

- Work to bring charge q_2 to $\mathbf{r}_2$ from ∞ when q_1 is at $\mathbf{r}_1$ $W_2 = q_2\,\phi_2$



$$\varphi_2 = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{r_{12}}$$



$$\varphi_3 = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{r_{13}} + \frac{q_2}{4\pi\varepsilon_o}\frac{1}{r_{23}}$$

- NB $q_2\,\phi_2 = q_1\,\phi_1$ (Could equally well bring charge q_1 from ∞)
- Work to bring charge q_3 to $\mathbf{r}_3$ from ∞ when q_1 is at $\mathbf{r}_1$ and q_2 is at $\mathbf{r}_2$ $W_3 = q_3\,\phi_3$
- Total potential energy of 3 charges = $W_2 + W_3$
- In general $W = \frac{1}{4\pi\varepsilon_o}\sum_{i<j} q_i \sum_j \frac{q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\varepsilon_o}\sum_{i\neq j} q_i \sum_j \frac{q_j}{r_{ij}}$

Electrostatic energy of charge distribution

- For a continuous distribution of charge

$$W = \frac{1}{2} \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r})\phi(\mathbf{r})$$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_{\text{all space}} d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$W = \frac{1}{2}\frac{1}{4\pi\varepsilon_o} \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r}) \int_{\text{all space}} d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Energy in vacuum in terms of **E**

- Gauss' law relates ρ to electric field and potential
- Replace ρ in energy expression using Gauss' law

$$\nabla.\mathbf{E} = \frac{\rho}{\varepsilon_o} \text{ and } \mathbf{E} = -\nabla\phi$$

$$\Rightarrow \nabla^2\phi = -\frac{\rho}{\varepsilon_o} \Rightarrow \rho = -\varepsilon_o \nabla^2\phi$$

$$\therefore \mathrm{W} = \tfrac{1}{2}\int_v \phi\,\rho\,\mathrm{dv} = -\frac{\varepsilon_o}{2}\int_v \phi\,\nabla^2\phi\,\mathrm{dv}$$

- Expand integrand using identity:
  $\nabla.\psi\mathbf{F} = \psi\nabla.\mathbf{F} + \mathbf{F}.\nabla\psi$

Exercise: write $\psi = \phi$ and $\mathbf{F} = \nabla\phi$ to show:

$$\nabla.\phi\nabla\phi = \phi\nabla^2\phi + (\nabla\phi)^2$$

$$\Rightarrow \phi\nabla^2\phi = \nabla.\phi\nabla\phi - (\nabla\phi)^2$$

Energy in vacuum in terms of **E**

$$W = -\frac{\varepsilon_o}{2}\left[\int_v \nabla.\phi\nabla\phi\,dv - \int_v (\nabla\phi)^2 dv\right]$$

$$= -\frac{\varepsilon_o}{2}\left[\int_S (\phi\nabla\phi).d\mathbf{S} - \int_v (\nabla\phi)^2 dv\right] (\text{Green's first identity})$$

Surface integral replaces volume integral (Divergence theorem)

For pair of point charges, contribution of surface term

$\phi \sim 1/r$ $\nabla\phi \sim -1/r^2$ $dA \sim r^2$ overall $\sim -1/r$

Let $r \rightarrow \infty$ and only the volume term is non-zero

$$W = \frac{\varepsilon_o}{2}\int_{\text{all space}} (\nabla\phi)^2 dv = \frac{\varepsilon_o}{2}\int_{\text{all space}} E^2\, dv$$

Energy density $$\frac{dW}{dv} = \frac{\varepsilon_o}{2}E^2(\mathbf{r})$$

Energy of charge distribution in external potential

- Energy of localised charge distribution $\rho(\mathbf{r})$ in external potential $\phi(\mathbf{r})$

$$U = \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r})\phi(\mathbf{r})$$

$$f(a+h) = f(a) + hf'(a) + \ldots \quad \text{1-D Taylor expansion}$$

$$\phi(\mathbf{r}) = \phi(\mathbf{0}) + \mathbf{r}.\nabla\phi(\mathbf{0}) + \ldots \quad \text{3-D Taylor expansion about origin}$$

$$U = \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r})\big(\phi(\mathbf{0}) + \mathbf{r}.\nabla\phi(\mathbf{0}) + \ldots\big)$$

$$= \phi(\mathbf{0}) \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r}) + \nabla\phi(\mathbf{0}). \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r})\mathbf{r} + \ldots$$

$$U = Q\phi(\mathbf{0}) + \mathbf{p}.\nabla\phi(\mathbf{0}) + \ldots = Q\phi(\mathbf{0}) - \mathbf{p}.\mathbf{E}(\mathbf{0}) + \ldots$$

$$Q = \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r}) \qquad \mathbf{p} = \int_{\text{all space}} d\mathbf{r}\ \rho(\mathbf{r})\mathbf{r}$$

Energy of electric dipole in electric field $U_p = \mathbf{-p.E}$