Great Philosophers – Gottlob Frege
Evening lecture series, Department of Philosophy

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Overview

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*Beg riffsschrift* (Conceptual Notation)

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Early Life and Education

• Friedrich Ludwig *Gottlob Frege*, born in Wismar, Germany, on the coast of the Baltic sea, 8th of November 1848.

• His father, Karl Alexander, founded and directed a private girls’ school, and his mother, Auguste, was a teacher and later a principal of the school. Like most families in the region, they were Lutheran.

• Frege’s father dies in 1866, while Frege was still attending the *Gymnasium* in Wismar.

• Frege graduated in spring 1869 and immediately entered Jena University.

• He spent four semesters at Jena until the winter of 1870 and then transferred to Göttingen.

• At Jena, he studied mathematics, but also chemistry and philosophy. He was at Göttingen for five semesters and studied mathematics, philosophy of religion, and physics.
Ph.D. in Mathematics, and Habilitation

- In 1873, Frege was awarded a Ph.D. in mathematics. His dissertation was entitled "Über eine geometrische Darstellung der imaginären Gebilde in der Ebene" ("On a Geometrical Representation of Imaginary Forms in the Plane").

- In 1874, Frege completed his Habilitationsschrift, entitled "Rechnungsmethoden, die sich auf eine Erweiterung des Größenbegriffes gründen" ("Methods of Calculation Based on an Extension of the Concept of Quantity").

- "The elements of all geometrical constructions are intuitions, and geometry refers to intuition as the source of its axioms. Since the object of arithmetic does not have an intuitive character, its fundamental propositions cannot stem from intuition." (Frege 1874, translation in McGuinness (ed.) 1984, 56)

- This means that our knowledge of geometry and of arithmetic stems from different sources.

- There is a seed here of Frege’s work on logicism; the claim that arithmetic stems from logic and is therefore part of logic. We will come back to this.
Teaching Mathematics

• On the recommendation of his former professor, Abbe, Frege is admitted as a Lecturer in Jena, and began teaching in the summer semester of 1874.

• He was highly regarded by the faculty and students in mathematics at Jena.

• In 1878, he volunteered to take on an extra teaching load when Professor Snell fell ill, until a replacement was hired a year later.

• In 1879, Frege published his *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (“Concept Notation: A formula language of pure thought, modelled upon that of arithmetic.”)

• He was subsequently promoted to Extraordinary Professor.

Frege, around 1879
Begriffsschrift (Conceptual Notation)

• Frege was attempting to reduce arithmetic to logic, but had difficulty using natural language to do so.

“So that something intuitive could not force itself in unnoticed here, it was most important to keep the chain of reasoning free of gaps. As I endeavoured to fulfil this requirement to the strongest degree, I found an obstacle in the inadequacy of the language; despite all the unwieldiness of the expressions, the more complex the relations became, the less precision—which my purpose required—could be obtained. From this deficiency arose the idea of the “conceptual notation”.” (From Preface of the Begriffsschrift, in Bynum (ed.) 1972)

• Frege’s conceptual notation is an attempt to develop a more exact language than our natural languages.

• This kind of philosophical project has a history that goes back at least to Leibniz (1646–1716).
Leibniz’ Characteristica

• Leibniz imagined a universal language that he called in Latin, *characteristica universalis* (‘universal characteristic’).

• This language would be an exact means of expressing all mathematical, scientific, and philosophical concepts, by way of complex pictograms.

• This could be used together with what he called a *calculus ratiocinator*, a system of reasoning that is able to show the connections between statements, and calculate the consequences of any statement of the universal language.

• Leibniz thought that all he needed in order to construct such a language was a good team of assistants and five years of hard work.

• However, by the end of his life, Leibniz had realised that this was highly over ambitious, and was struggling to work out even a small part of such a language.
Conceptual notation

- Frege thought at the time that his conceptual notation was a step towards Leibniz’ universal language, but it turned out to be a major breakthrough in logic as well. The *Begriffsschrift* was the first system of quantificational predicate logic.

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Function and argument (Concept and object)

• Frege developed a new analysis of the unity of a proposition, i.e., the content of a sentence, in terms of concept and object (function and argument).

• A numerical function in mathematics ‘$f(x)$’, is a piece of machinery that takes in an argument, an object, the number $x$, and does something to it. For example, the square function ‘$(x)^2 = x \times x$’; ‘$(2)^2 = 4$’.

• Frege generalised this notion of function to propositional functions, which can be about anything, not just numbers.

• A propositional function is an incomplete sign, e.g., ‘H( )’, something that has a ‘gap’, an argument place, which can be filled with an object of the correct kind to form a true proposition. Like a glove to a hand; the glove here is analogous to the function representing the concept ‘hand’.

• If the object ‘$o$’, is of the correct kind, e.g., a hand, the resulting proposition, e.g., ‘H(o)’, is true, otherwise it is false.
Quantifiers

• Quantifiers: Words like ‘All’ and ‘Some’.

• ‘All men are mortal’ : \( \forall x (\text{Man}(x) \to \text{Mortal}(x)) \)

• Frege’s notation is able to express statements of **Multiple Generality** (multiple quantification), such as ‘Everyone loves someone.’, which is ambiguous between two readings that can be distinguished in quantificational logic:

1. ‘There is someone whom everybody loves.’: \( \exists y \forall x (L(x,y)) \) [where \( x \) and \( y \) are people].

2. ‘Everyone has someone whom they love.’ : \( \forall x \exists y (L(x,y)) \)

• The first entails the second, but the second does not entail the first.

• Certain mathematical statements can only be formalised by a logic that can express multiple generality. So, Frege’s **quantifiers are a major breakthrough** in this regard.
After the *Begriffsschrift* (1879)

- In 1884, Frege published *Die Grundlagen der Arithmetik: eine logisch mathematische Untersuchung über den Begriff der Zahl* (“The Foundations of Arithmetic: a logico-mathematical inquiry into the concept of number.”) A more philosophical work in which he explains his project.

- 1887: Marries Margarete Lieseberg. They have two children, both of whom died young.

- 1891: ‘Function and Concept’

- 1892: ‘On Sense and Reference’ and ‘On Concept and Object’

- 1893: First volume of the major technical work *Grundgesetze der Arithmetik* (“Basic laws of Arithmetic”).

- 1896: Promoted to Ordinary Honorary Professor at Jena.
Defining the Cardinal Numbers

• Frege’s basic insight is the following: Whenever we say, for example, “I have two hands.” or “I have 8 fingers”, **we are saying something about a concept.**

• The concept ‘hands that I have’ has two objects that fall under it, my left hand and my right hand.

• So, the cardinal number of the concept ‘hands that I have’ is two. There are two such things.

• This is a higher order concept ‘2’ predicated of (said of) a lower order one (‘hands that I have’).

• The following is a **condition for the cardinal number of a concept F being exactly two**: ‘There are objects x and y, which fall under concept F, and x and y are distinct objects. If any other object z falls under F, then it is the same as x or it is the same as y.’

• Frege then defines the cardinal number ‘2’ as the class of all those concepts that are **equinumerous** with a concept having two objects falling under it.
Hume’s Principle

• A way to contextually define ‘cardinal number’ simpliciter (#): Hume’s Principle: $\forall F \forall G (\#F = \#G \equiv F \approx G)$

• For any concepts F and G ($\forall F \forall G$), the number of Fs ($\#F$) is equal to (=) the number of Gs ($\#G$) if and only if ($\equiv$) there is a one-to-one correspondence ($\approx$) between the Fs and the Gs, i.e., the Fs and the Gs are equinumerous.

• Call it instead ‘The Waiter’s Principle’ or, perhaps, ‘Hume’s Knife and Fork’:

• There are a certain number of places ($\#F$) at a table. In order to put the same number of knifes and forks ($\#G$) on the table as there are places, the waiter does not necessarily need to count the number of places ($\#F$) and then count out the same number of knifes and of forks ($\#G$). All that they need do is put a knife and fork at every place ($F \approx G$). That is, the waiter has no need for the concept of number (#), because one-to-one correspondence ($\approx$) does just as well, this is because it will always lead to the same result ($\equiv$).

• Make sure to test this at your next meal, and to complain if the principle fails!
Defining the Natural Numbers

• Predecessors: \((0, 1), (1, 2), (2, 3), (3, 4), \ldots\)

• The number of Fs \((\#F)\) precedes the number of Gs \((\#G)\) if and only if there is an object \(x\) that is G, and the number of Fs is equal to the number of Gs excluding \(x\).

• We are at a restaurant, and I say that I (‘\(x\)’) have to leave unexpectedly, but the waiter has already laid out knives and forks for each person at the table \((\#G)\). The manager of the restaurant asks the waiter to reduce the number of knives and forks at the table \((\#G)\) to the predecessor of the number of people at the table \((\#F)\). The waiter has been reading Frege, and knows that I (‘\(x\)’) am leaving, so takes away my knife and fork (or lays the table again after I have left, using Hume’s Principle).

• Using what Frege calls the \textit{ancestral} of the predecessor relation he defines the relation \(x < y\). The predecessors are like links in a chain that form a \textit{series of predecessors}: \(x < y\) if and only if \(x\) is the predecessor of \(y\), or the predecessor of the predecessor of \(y\), etc. (i.e., \(x\) is in the series of predecessors of \(y\)).

• He then uses this to define the natural numbers: \(x\) is a \textit{natural} number if and only if \(x = 0\) or \(0 < x\).
On Sense and Reference (1892)

1. ‘The Morning star = The Morning star’ : True, but obvious; ‘a = a’.
2. ‘The Morning star = The Evening star’ : True, and not obvious; ‘a = b’.
   - ‘The Morning star’ and ‘The Evening star’ refer to the same object, i.e., the planet Venus. That is, they have the same reference, Venus.
   - On the view that meaning is reference, this implies that the two terms have the same meaning (reference).
   - However, 1 and 2 would appear to have different meanings. 1 is obvious. 2 is not obvious. Someone could know that 1 is true without knowing that 2 is.
   - So, there must be some other element to their meanings, which are distinct.
   - Frege’s answer is that the terms of 2 do not have the same sense. That is, the mode of presentation of the reference is different in each case.
   - ‘The Morning star’ picks out the bright star that appears in the morning. ‘The Evening star’ picks out the bright star that appears in the evening. These are two modes of presenting, or ways of finding, the same object, Venus.
Russell’s Paradox (1902)

- 1902: In June, Bertrand Russell rereads Frege’s work and realises that his paradox reveals a flaw in it.

- He communicates this to Frege, who was about to publish the second volume of his Grundgesetze, and has to withdraw it from the publisher and add an appendix.


- Russell’s paradox showed that one of Frege’s basic laws, ‘Basic Law V’, leads to a contradiction. This is also the law that Frege used to derive Hume’s principle, which, as we saw, he then uses to define number.

- These days, some philosophers of mathematics, called neo-logicists, point out that Hume’s Principle together with a logic like the one used by Frege, is sufficient for arithmetic. This is now known as ‘Frege’s Theorem’.

- The problem for them is how to justify Hume’s principle.
After the *Grundgesetze*

- Frege’s logicist project never recovered from Russell’s paradox, though he continued to publish and attempted to address the paradox for a number of years.

- His wife died in 1904.

- From 1910–1913, the philosopher Rudolf Carnap was a student of Frege.

- In 1911, Wittgenstein writes to Frege about the former’s solution to Russell’s paradox. The two meet.

- Frege retired in 1918, and moved to a town called Bad Kleinen, on the shore of a lake just south of Wismar where he grew up.

- Frege seems to have been quite xenophobic and anti-Semitic, at least towards the end of his life.

- He died on 26th of July 1925, aged 76.
Thank you for listening!

References and Further reading:


