Topic 6: Differentiation

Jacques Text Book (edition 4):
Chapter 4
1. Rules of Differentiation
2. Applications
Differentiation is all about measuring change!
Measuring change in a linear function:

\[ y = a + bx \]

\[ a = \text{intercept} \]
\[ b = \text{constant slope i.e. the impact of a unit change in } x \text{ on the level of } y \]

\[ b = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]
If the function is non-linear: e.g. if $y = x^2$

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]
gives slope of the line connecting 2 points $(x_1, y_1)$ and $(x_2, y_2)$ on a curve

- $(2,4)$ to $(4,16)$: slope $= \frac{(16-4)}{(4-2)} = 6$
- $(2,4)$ to $(6,36)$: slope $= \frac{(36-4)}{(6-2)} = 8$
The slope of a curve is equal to the slope of the line (or tangent) that touches the curve at that point.

Total Cost Curve

which is different for different values of x
Example: A firm's cost function is

\[ Y = X^2 \]

<table>
<thead>
<tr>
<th>X</th>
<th>( \Delta X )</th>
<th>Y</th>
<th>( \Delta Y )</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td></td>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>+1</td>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>+1</td>
<td>4</td>
<td>+3</td>
</tr>
<tr>
<td>3</td>
<td>+1</td>
<td>9</td>
<td>+5</td>
</tr>
<tr>
<td>4</td>
<td>+1</td>
<td>16</td>
<td>+7</td>
</tr>
</tbody>
</table>

\[ Y = X^2 \]
\[ Y + \Delta Y = (X + \Delta X)^2 \]
\[ Y + \Delta Y = X^2 + 2X \cdot \Delta X + \Delta X^2 \]
\[ \Delta Y = X^2 + 2X \cdot \Delta X + \Delta X^2 - Y \]

since \( Y = X^2 \)  \( \Rightarrow \)  \( \Delta Y = 2X \cdot \Delta X + \Delta X^2 \)

\[ \frac{\Delta Y}{\Delta X} = 2X + \Delta X \]

The slope depends on \( X \) and \( \Delta X \).
The slope of the graph of a function is called the derivative of the function

\[ f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]

- The process of differentiation involves letting the change in x become arbitrarily small, i.e. letting \( \Delta x \to 0 \)
- e.g. if \( f = 2X + \Delta X \) and \( \Delta X \to 0 \)
- \( \Rightarrow = 2X \) in the limit as \( \Delta X \to 0 \)
the slope of the non-linear function
$Y = X^2$ is $2X$

- the slope tells us the change in $y$ that results from a very small change in $X$
- We see the slope varies with $X$
  - e.g. the curve at $X = 2$ has a slope = 4 and the curve at $X = 4$ has a slope = 8
- In this example, the slope is steeper at higher values of $X$
Rules for Differentiation (section 4.3)

1. The Constant Rule

If \( y = c \) where \( c \) is a constant,

\[
\frac{dy}{dx} = 0
\]

e.g. \( y = 10 \) then \( \frac{dy}{dx} = 0 \)
2. The Linear Function Rule
If \( y = a + bx \)

\[
\frac{dy}{dx} = b
\]

e.g. \( y = 10 + 6x \) then \( \frac{dy}{dx} = 6 \)
3. The Power Function Rule

If \( y = ax^n \), where \( a \) and \( n \) are constants

\[
\frac{dy}{dx} = n \cdot a \cdot x^{n-1}
\]

i) \( y = 4x \) \( \Rightarrow \) \( \frac{dy}{dx} = 4 \cdot x^0 = 4 \)

ii) \( y = 4x^2 \) \( \Rightarrow \) \( \frac{dy}{dx} = 8 \cdot x \)

iii) \( y = 4x^{-2} \) \( \Rightarrow \) \( \frac{dy}{dx} = -8 \cdot x^{-3} \)
4. The Sum-Difference Rule

If \( y = f(x) \pm g(x) \)

\[
\frac{dy}{dx} = \frac{d}{dx} \left[ f(x) \right] \pm \frac{d}{dx} \left[ g(x) \right]
\]

If \( y \) is the sum/difference of two or more functions of \( x \):

differentiate the 2 (or more) terms separately, then add/subtract

(i) \( y = 2x^2 + 3x \) then \( \frac{dy}{dx} = 4x + 3 \)

(ii) \( y = 5x + 4 \) then \( \frac{dy}{dx} = 5 \)
5. The Product Rule

If \( y = u \cdot v \) where \( u \) and \( v \) are functions of \( x \), \((u = f(x) \text{ and } v = g(x))\) Then

\[
\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
\]
Examples

If \( y = u \cdot v \)

\[
\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
\]

i) \( y = (x+2)(ax^2+bx) \)

\[
\frac{dy}{dx} = (x + 2)(2ax + b) + (ax^2 + bx)
\]

ii) \( y = (4x^3-3x+2)(2x^2+4x) \)

\[
\frac{dy}{dx} = (4x^3 - 3x + 2)(4x + 4) + (2x^2 + 4x)(12x^2 - 3)
\]
6. The Quotient Rule

• If \( y = \frac{u}{v} \) where \( u \) and \( v \) are functions of \( x \) (\( u = f(x) \) and \( v = g(x) \)) Then

\[
\frac{dy}{dx} = \frac{v \ \frac{du}{dx} - u \ \frac{dv}{dx}}{v^2}
\]
If \( y = \frac{u}{v} \) then 
\[
\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

Example 1

\[
y = \frac{(x + 2)}{(x + 4)}
\]

\[
\frac{dy}{dx} = \frac{(x + 4)(1) - (x + 2)(1)}{(x + 4)^2} = \frac{-2}{(x + 4)^2}
\]
7. The Chain Rule  
(Implicit Function Rule)

- If $y$ is a function of $v$, and $v$ is a function of $x$, then $y$ is a function of $x$ and

\[
\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}
\]
Examples

\[ \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} \]

i) \( y = (ax^2 + bx)^{1/2} \)

let \( v = (ax^2 + bx) \), so \( y = v^{1/2} \)

\[ \frac{dy}{dx} = \frac{1}{2} \left( ax^2 + bx \right)^{-1/2} \cdot (2ax + b) \]

ii) \( y = (4x^3 + 3x - 7)^4 \)

let \( v = (4x^3 + 3x - 7) \), so \( y = v^4 \)

\[ \frac{dy}{dx} = 4 \left( 4x^3 + 3x - 7 \right)^3 \cdot (12x^2 + 3) \]
8. The Inverse Function Rule

If \( x = f(y) \) then \( \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \)

- **Examples**
  
  i) \( x = 3y^2 \) then
  \[
  \frac{dx}{dy} = 6y \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{6y}
  \]

  ii) \( y = 4x^3 \) then
  \[
  \frac{dy}{dx} = 12x^2 \quad \text{so} \quad \frac{dx}{dy} = \frac{1}{12x^2}
  \]
Differentiation in Economics
Application I

- Total Costs = TC = FC + VC
- Total Revenue = TR = P * Q
- $\pi$ = Profit = TR – TC
- Break even: $\pi = 0$, or TR = TC
- Profit Maximisation: MR = MC
Application I: Marginal Functions (Revenue, Costs and Profit)

Calculating Marginal Functions

\[ MR = \frac{d(TR)}{dQ} \]

\[ MC = \frac{d(TC)}{dQ} \]
Example 1

• A firm faces the demand curve $P=17-3Q$

• (i) Find an expression for $TR$ in terms of $Q$

• (ii) Find an expression for $MR$ in terms of $Q$

Solution:

$TR = P.Q = 17Q - 3Q^2$

$MR = \frac{d(TR)}{dQ} = 17 - 6Q$
Example 2

A firm's total cost curve is given by

$$TC = Q^3 - 4Q^2 + 12Q$$

(i) Find an expression for $AC$ in terms of $Q$
(ii) Find an expression for $MC$ in terms of $Q$
(iii) When does $AC = MC$?
(iv) When does the slope of $AC = 0$?
(v) Plot $MC$ and $AC$ curves and comment on the economic significance of their relationship
Solution

(i) \( TC = Q^3 - 4Q^2 + 12Q \)

Then,
\[
AC = \frac{TC}{Q} = Q^2 - 4Q + 12
\]

(ii) \( MC = \frac{d(TC)}{dQ} = 3Q^2 - 8Q + 12 \)

(iii) When does \( AC = MC \)?
\[
Q^2 - 4Q + 12 = 3Q^2 - 8Q + 12
\]

\[\Rightarrow Q = 2\]

Thus, \( AC = MC \) when \( Q = 2 \)
Solution continued….

(iv) When does the slope of AC = 0?
\[ \frac{d(AC)}{dQ} = 2Q - 4 = 0 \]
⇒ \( Q = 2 \) when slope AC = 0

(v) Economic Significance?
MC cuts AC curve at minimum point…
9. Differentiating Exponential Functions

If \( y = \exp(x) = e^x \) where \( e = 2.71828 \ldots \)

then \( \frac{dy}{dx} = e^x \)

More generally,

If \( y = Ae^{rx} \)

then \( \frac{dy}{dx} = rAe^{rx} = ry \)
Examples

1) \( y = e^{2x} \) then \( \frac{dy}{dx} = 2e^{2x} \)

2) \( y = e^{-7x} \) then \( \frac{dy}{dx} = -7e^{-7x} \)
10. Differentiating Natural Logs

*Recall* if \( y = e^x \) then \( x = \ln y \)

• If \( y = e^x \) then \( \frac{dy}{dx} = e^x = y \)

• From The Inverse Function Rule

\[
y = e^x \implies \frac{dx}{dy} = \frac{1}{y}
\]

• Now, if \( y = e^x \) this is equivalent to writing \( x = \ln y \)

• Thus, \( x = \ln y \implies \frac{dx}{dy} = \frac{1}{y} \)
More generally,

if \( y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \)

NOTE: the derivative of a natural log function does not depend on the co-efficient of \( x \)

Thus, if \( y = \ln mx \Rightarrow \frac{dy}{dx} = \frac{1}{x} \)
Proof

• if \( y = \ln mx \quad m>0 \)

• Rules of Logs \( \Rightarrow y = \ln m + \ln x \)

• Differentiating (Sum-Difference rule)

\[
\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}
\]
Examples

1) $y = \ln 5x$ \quad (x>0) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$

2) $y = \ln(x^2+2x+1)$

let $v = (x^2+2x+1)$ \quad so $y = \ln v$

Chain Rule: \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{1}{x^2 + 2x + 1} \cdot (2x + 2)$$

$$\frac{dy}{dx} = \frac{(2x + 2)}{(x^2 + 2x + 1)}$$
3) \( y = x^4 \ln x \)

**Product Rule:** \( \Rightarrow \)

\[
\frac{dy}{dx} = x^4 \cdot \frac{1}{x} + \ln x \cdot 4x^3
\]

\[
= x^3 + 4x^3 \ln x = x^3 \left(1 + 4 \ln x\right)
\]

4) \( y = \ln(x^3(x+2)^4) \)

Simplify first using rules of logs

\( \Rightarrow \) \( y = \ln x^3 + \ln(x+2)^4 \)

\( \Rightarrow \) \( y = 3 \ln x + 4 \ln(x+2) \)

\[
\frac{dy}{dx} = \frac{3}{x} + \frac{4}{x + 2}
\]
Applications II

- how does demand change with a change in price……

- \( e_d = \frac{\text{proportional change in demand}}{\text{proportional change in price}} \)

\[
= \frac{\Delta Q}{Q} / \frac{\Delta P}{P} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}
\]
Point elasticity of demand

\[ e_d = \frac{dQ}{dP} \frac{P}{Q} \]

\( e_d \) is negative for a downward sloping demand curve
-
Inelastic demand if \( |e_d| < 1 \)
-
Unit elastic demand if \( |e_d| = 1 \)
-
Elastic demand if \( |e_d| > 1 \)
Example 1

Find $e_d$ of the function $Q = aP^{-b}$

$$e_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

$$e_d = -baP^{-b-1} \cdot \frac{P}{aP^{-b}}$$

$$= \frac{-baP^{-b}}{P} \cdot \frac{P}{aP^{-b}} = -b$$

$e_d$ at all price levels is $-b$
Example 2

If the (inverse) Demand equation is
\[ P = 200 - 40\ln(Q+1) \]

Calculate the price elasticity of demand when \( Q = 20 \)

- Price elasticity of demand: \( e_d = \frac{dQ}{dP} \cdot \frac{P}{Q} \quad < 0 \)

- \( P \) is expressed in terms of \( Q \),
  \[ \frac{dP}{dQ} = -\frac{40}{Q + 1} \]

- Inverse rule \( \Rightarrow \) \( \frac{dQ}{dP} = -\frac{Q + 1}{40} \)

- Hence, \( e_d = -\frac{Q + 1}{40} \cdot \frac{P}{Q} \quad < 0 \)

- \( Q \) is \( 20 \) \( \Rightarrow \) \( e_d = -\frac{21}{40} \cdot \frac{78.22}{20} = -2.05 \)

(Where \( P = 200 - 40\ln(20+1) = 78.22 \))
Application III: Differentiation of Natural Logs to find Proportional Changes

The derivative of \( \log(f(x)) = \frac{f'(x)}{f(x)} \), or the proportional change in the variable \( x \)

i.e. \( y = f(x) \), then the proportional \( \Delta x \)

\[
\frac{dy}{dx} \cdot \frac{1}{y} = \frac{d}{dx} (\ln y)
\]

*Take logs and differentiate to find proportional changes in variables*
1) Show that if \( y = x^\alpha \), then \( \frac{dy}{dx} \cdot \frac{1}{y} = \frac{\alpha}{x} \)

and this \( \equiv \) derivative of \( \ln(y) \) with respect to \( x \).

**Solution:**

\[
\frac{dy}{dx} \cdot \frac{1}{y} = \frac{1}{y} \cdot \alpha \cdot x^{\alpha - 1}
\]

\[
= \frac{1}{y} \cdot \alpha \cdot \frac{x^\alpha}{x}
\]

\[
= \frac{\alpha}{y} \cdot \frac{y}{x}
\]

\[
= \frac{\alpha}{x}
\]
Solution Continued…

Now \( \ln y = \ln x^\alpha \)

Re-writing \( \Rightarrow \ln y = \alpha \ln x \)

\[
\Rightarrow \frac{d (\ln y)}{dx} = \alpha \cdot \frac{1}{x} = \frac{\alpha}{x}
\]

Differentiating the \( \ln y \) with respect to \( x \) gives the proportional change in \( x \).
Example 2: If Price level at time $t$ is
$$P(t) = a + bt + ct^2$$
Calculate the rate of inflation.

Solution:

The inflation rate at $t$ is the proportional change in $P$

$$\frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = \frac{b + 2ct}{a + bt + ct^2}$$

Alternatively,

differentiating the log of $P(t)$ wrt $t$ directly

$$\ln P(t) = \ln (a + bt + ct^2)$$

where $v = (a + bt + ct^2)$ so $\ln P = \ln v$

Using chain rule,

$$\frac{d(\ln P(t))}{dt} = \frac{b + 2ct}{a + bt + ct^2}$$