Lecture # 18 - Optimization with Equality Constraints

- So far, we have assumed in all (economic) optimization problems we have seen that the variables to be chosen do not face any restriction.
- However, in other occassions such variables are required to satisfy certain constraints. Examples:
  - A consumer chooses how much to buy of each product, such that it satisfies his budget constraint
  - A firm would look to minimize its cost of production, subject to a given output level.
- What do we do? Use the **Lagrange multiplier method**
  - Suppose we want to maximize the function $f(x, y)$ where x and y are restricted to satisfy the equality constraint $g(x, y) = c$

$$\max \; f(x, y) \; \text{subject to} \; g(x, y) = c$$

    * The function $f(x, y)$ is called the objective function
  - Then, we define the **Lagrangian function**, a modified version of the objective function that incorporates the constraint:

$$Z(x, y, \lambda) = f(x, y) + \lambda [c - g(x, y)]$$

    where the term λ is a(n unknown) constant called a **Lagragian multiplier**, associated to the constraint
    * Notice that $Z(\lambda, x, y) = f(x, y)$ when the constraint holds, i.e., when $g(x, y) = c$, regardless of the value of λ

- So $Z(x, y, \lambda)$ is an unconstrained function (in three variables), so we can find its maximum by finding the first order conditions:

$$\begin{aligned}\frac{\partial Z}{\partial \lambda} &= c - g(x, y) = 0 \\ \frac{\partial Z}{\partial x} &= f_x - \lambda g_x = 0 \\ \frac{\partial Z}{\partial y} &= f_y - \lambda g_y = 0\end{aligned}$$

  The first equation automatically ensures that the constraint is satisfied

- So we have a system of 3 equations and 3 unknowns → Find the stationary point

  * So we obtain the stationary points of the constraint function $f(\cdot)$ (with two choice variables), by looking at the stationary points of the unscontrained function $Z(\cdot)$ (three choice variables, one of which is associated with the constraint).

Example 1 *Suppose we want to find the extrema of* $f(x,y) = xy$ *subject to the constraint* $x + y = 6$

The Lagrangian is: $Z(x,y,\lambda) = xy + \lambda[6 - x - y]$, *so the first order conditions are:*

$$\begin{aligned} x + y &= 6 \\ \frac{\partial Z}{\partial x} &= y - \lambda = 0 \\ \frac{\partial Z}{\partial y} &= x - \lambda = 0 \end{aligned}$$

There is then a stationary point at $x^* = y^* = \lambda^* = 3$

Example 2 *Suppose a consumer has utility function* $U(x,y) = Ax^{\alpha}y^{1-\alpha}$ *and faces the budget constraint* $p_x \cdot x + p_y \cdot y = m$

The Lagrangian is: $Z(x,y,\lambda) = Ax^{\alpha}y^{1-\alpha} + \lambda[m - p_x \cdot x - p_y \cdot y]$, *so the first order conditions are:*

$$\begin{aligned} p_x \cdot x + p_y \cdot y &= m \\ \frac{\partial Z}{\partial x} &= \alpha A x^{\alpha-1}y^{1-\alpha} - \lambda p_x = 0 \\ \frac{\partial Z}{\partial y} &= (1-\alpha) A x^{\alpha} y^{-\alpha} - \lambda p_y = 0 \end{aligned}$$

We can express the last two equations as follows:

$$\lambda = \frac{\alpha A x^{\alpha-1}y^{1-\alpha}}{p_x} = \frac{(1-\alpha) A x^{\alpha} y^{-\alpha}}{p_y}$$

Simplifying:

$$\alpha p_y y = (1-\alpha) p_x x$$

Replacing it in the budget constraint, we obtain the demand functions:

$$\begin{aligned} x(p_x, p_y, m) &= \alpha \frac{m}{p_x} \\ y(p_x, p_y, m) &= (1-\alpha) \frac{m}{p_y} \end{aligned}$$

General case:

- I just introduced an example where the objective function has:
  - Two choice variables: $f(x, y)$
  - One constraint: $g(x, y) = c$
- Suppose we have:
  - Four choice variables: $f(x_1, x_2, x_3, x_4)$
  - Two constraints: $g_1(x_1, x_2, x_3, x_4) = c_1 \qquad g_2(x_1, x_2, x_3, x_4) = c_2$
- Then the Lagrangian function is:

$$Z = f(x_1, x_2, x_3, x_4) + \lambda_1 [c_1 - g_1(x_1, x_2, x_3, x_4)] + \lambda_2 [c_2 - g_2(x_1, x_2, x_3, x_4)]$$

- By getting the first order conditions of Z, we get the stationary points of $f(\cdot)$ that satisfy the constraints.

Second Order Conditions

- The second order conditions for a constrained optimization are slightly more complicated than for an unconstraint one. As such, we will only look at the case of two choice variables and one constraint.
- Suppose $f(x, y)$ AND $g(x, y)$ are both twice differentiable in an interval I, and suppose (x^*, y^*) is an interior, stationary point of I, that satisfies the first-order conditions of $Z(\lambda, x, y) = f(x, y) + \lambda [c - g(x, y)]$.
- In this case, the Hessian for the choice variables is:

$$H[Z] = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{xy} & Z_{yy} \end{bmatrix}$$

  where:

  - $Z_{xx} = f_{xx} - \lambda g_{xx}$
  - $Z_{yy} = f_{xx} - \lambda g_{xx}$
  - $Z_{xy} = f_{xy} - \lambda g_{xy}$

- Define the **bordered Hessian** as follows:

$$\overline{H} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{xy} & Z_{yy} \end{bmatrix}$$

- Then:

  - (x^*, y^*) is a maximum point if $\left|\overline{H}\right| > 0$ when evaluated at $x = x^*$, $y = y^*$, $\lambda = \lambda^*$.
  - (x^*, y^*) is a minimum point if $\left|\overline{H}\right| < 0$ when evaluated at $x = x^*$, $y = y^*$, $\lambda = \lambda^*$.

Example 3 *Suppose we want to find the extrema of* $f(x, y) = xy$ *subject to the constraint* $x + y = 6$*. We found there is a stationary point at* $x^* = y^* = \lambda^* = 3$*.*

For the bordered Hessian we need five derivatives:

- $Z_{xx} = f_{xx} - \lambda g_{xx} = 0$
- $Z_{yy} = f_{xx} - \lambda g_{xx} = 0$
- $Z_{xy} = f_{xy} - \lambda g_{xy} = 1$
- $g_x = 1$
- $g_y = 1$

As a result, the bordered Hessian is:

$$\overline{H} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and its determinant is $\left|\overline{H}\right| = 2 > 0$*, so the stationary point is a maximum.*

Example 4 *Suppose a consumer has utility function $U(x,y) = Ax^{\alpha}y^{1-\alpha}$ and faces the budget constraint $p_x \cdot x + p_y \cdot y = m$. We got that there is a stationary point that satisfies the constraint at:*

$$
\begin{aligned}
x(p_x, p_y, m) &= \alpha \frac{m}{p_x} \\
y(p_x, p_y, m) &= (1-\alpha)\frac{m}{p_y}
\end{aligned}
$$

For the bordered Hessian we need five derivatives:

- $Z_{xx} = f_{xx} - \lambda g_{xx} = -\alpha(1-\alpha) Ax^{\alpha-2}y^{1-\alpha} < 0$
- $Z_{yy} = f_{xx} - \lambda g_{xx} = -\alpha(1-\alpha) Ax^{\alpha}y^{-\alpha-1} < 0$
- $Z_{xy} = f_{xy} - \lambda g_{xy} = \alpha(1-\alpha) Ax^{\alpha-1}y^{-\alpha} > 0$
- $g_x = p_x$
- $g_y = p_y$

As a result, the bordered Hessian is:

$$
\overline{H} = \begin{bmatrix} 0 & p_x & p_y \\ p_x & -\alpha(1-\alpha) Ax^{\alpha-2}y^{1-\alpha} & \alpha(1-\alpha) Ax^{\alpha-1}y^{-\alpha} \\ p_y & \alpha(1-\alpha) Ax^{\alpha-1}y^{-\alpha} & -\alpha(1-\alpha) Ax^{\alpha}y^{-\alpha-1} \end{bmatrix}
$$

and its determinant is $\left|\overline{H}\right| = 2\alpha(1-\alpha) Ax^{\alpha-1}y^{-\alpha}p_xp_y + \alpha(1-\alpha) Ax^{\alpha}y^{-\alpha-1}(p_x)^2 + \alpha(1-\alpha) Ax^{\alpha-2}y^{1-\alpha}$ 0 *for any value of* $A > 0$ *and* $\alpha \in [0,1]$. *So the stationary point is a maximum.*