Lecture # 12 - Derivatives of Functions of Two or More Variables (cont.)

Some Definitions: Matrices of Derivatives

- **Jacobian matrix**
  - Associated to a system of equations
  - Suppose we have the system of 2 equations, and 2 exogenous variables:

$$\begin{aligned} y_1 &= f^1(x_1, x_2) \\ y_2 &= f^2(x_1, x_2) \end{aligned}$$

    * Each equation has two first-order partial derivatives, so there are 2x2=4 first-order partial derivatives
  - Jacobian matrix: array of 2x2 first-order partial derivatives, ordered as follows

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

  - Jacobian determinant: determinant of Jacobian matrix

Example 1 *Suppose* $y_1 = x_1 x_2$, *and* $y_2 = x_1 + x_2$. *Then the Jacobian matrix is*

$$J = \begin{bmatrix} x_2 & x_1 \\ 1 & 1 \end{bmatrix}$$

and the Jacobian determinant is $|J| = x_2 - x_1$

  - Caveat: Mathematicians (and economists) call 'the Jacobian' to both the matrix and the determinant

– Generalization to system of n equations with n exogenous variables:

$$\begin{aligned} y_1 &= f^1(x_1, x_2) \\ y_2 &= f^2(x_1, x_2) \\ &\vdots \\ y_2 &= f^2(x_1, x_2) \end{aligned}$$

Then, the Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

- **Hessian matrix:**
  - Associated to a single equation
  - Suppose $y = f(x_1, x_2)$
    - There are 2 first-order partial derivatives: $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$
    - There are 2x2 second-order partial derivatives: $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$
  - <u>Hessian matrix</u>: array of 2x2 second-order partial derivatives, ordered as follows:

$$H[f(x_1, x_2)] = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_2 \partial x_1} \\ \\ \frac{\partial y_2}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix}$$

Example 2 *Example* $y = x_1^4 + x_2^2 x_1^2 + x_2^3$. *Then the Hessian matrix is*

$$H[f(x_1, x_2)] = \begin{bmatrix} 12x_1^2 + 2x_2^2 & 4x_1 x_2 \\ \\ 4x_1 x_2 & 2x_1^2 + 6x_2 \end{bmatrix}$$

  - **Young's Theorem**: The order of differentiation does not matter, so that if $z = h(x, y)$:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{dy}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{d^2 z}{\partial y \partial x} = \frac{d^2 z}{\partial x \partial y}$$

- Generalization: Suppose $y = f(x_1, x_2, x_3, ..., x_n)$
  * There are n first-order partial derivatives
  * There are nxn second-order partial derivatives
- Hessian matrix: nxn matrix of second-order partial derivatives, ordered as follows

$$H[f(x_1, x_2, ..., x_n)] = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 y}{\partial x_n \partial x_1} \\ \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 y}{\partial x_1 \partial x_n} & \frac{\partial^2 y}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 y}{\partial x_n^2} \end{bmatrix}$$

Chain Rules for Many Variables

- Suppose $y = f(x, w)$, while in turn $x = g(t)$ and $w = h(t)$. How does y change when t changes?

$$\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt}$$

- Suppose $y = f(x, w)$, while in turn $x = g(t, s)$ and $w = h(t, s)$. How does y change when t changes? When s changes?

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial t}\\ \frac{\partial y}{\partial s} &= \frac{\partial y}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial s}\end{aligned}$$

- Notice that the first point is called the **total derivative**, while the second is the **'partial total' derivative**

Example 3 *Suppose* $y = 4x - 3w$, *where* $x = 2t$ *and* $w = t^2$

$\Longrightarrow$ *the total derivative* $\frac{dy}{dt}$ *is* $\frac{dy}{dt} = (4)(2) + (-3)(2t) = 8 - 6t$

Example 4 *Suppose* $z = 4x^2y$, *where* $y = e^x$

$\Longrightarrow$ *the total derivative* $\frac{dz}{dx}$ *is* $\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = (8xy) + \left(4x^2\right)(e^x) = 8xy + 4x^2y = 4xy(2 + x)$

Example 5 *Suppose* $z = x^2 + \frac{1}{2}y^2$ *where* $x = st$ *and* $y = t - s^2$

$\Longrightarrow \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x)(s) + \frac{1}{2}(2)(y)(1) = 2xs + y = 2s^2t + t - s^2$

$\Longrightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x)(t) + \frac{1}{2}(2)(y)(2s) = 2xt + 2sy = 2st^2 + 2st - 2s^3$

Derivatives of implicit functions

- So far, we have had functions like $y = f(x)$ or $z = g(x, w)$, where a (endogenous) variable is expressed as a function of other (exogenous) variables $\Longrightarrow$ **explicit functions.** Examples: $y = 4x^2$, or $z = 3xw + \ln w$
- Suppose we instead have a equation $y^2 - 2xy - x^2 = 0$. We can write $F(y, x) = 0$, but we cannot express y explicitly as a function of x. However, it is possible to define a set of conditions so that an **implicit function** $y = f(x)$ exists:
  1. The function $F(y, x)$ has continuous partial derivatives F_y, F_x
  2. $F_y \neq 0$
- Derivative of an implicit function. Suppose we have a function $F(y, x) = 0$, and we know an implicit function $y = f(x)$ exists. How do we find how much y changes when x changes? (i.e., we want $f_x = \frac{dy}{dx}$)
  - Find total differential for $F(y, x) = 0 \Longrightarrow F_y \cdot dy + F_x \cdot dx = d0 = 0$
  - Find total differential for $y = f(x) \Longrightarrow dy = f_x \cdot dx$
  - Replace $dy = f_x \cdot dx$ into $F_y \cdot dy + F_x \cdot dx = 0$:

$$\begin{aligned} F_y \cdot dy + F_x \cdot dx &= 0 \\ F_y \cdot (f_x \cdot dx) + F_x \cdot dx &= 0 \\ [F_y \cdot f_x + F_x]\, dx &= 0 \end{aligned}$$

  - Since $dx \neq 0$, then the term in brackets has to be zero:

$$F_y \cdot f_x + F_x = 0 \Longrightarrow f_x = -\frac{F_x}{F_y}$$

  - Alternative notation:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Example 6 $F(y, x) = y^2 - 2xy - x^2 = 0$. *Then* $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{-2y-2x}{2y-2x} = \frac{y+x}{y-x}$

Example 7 $F(y, x) = y^x + 1 = 0$. *Then* $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{y^x \ln y}{xy^{x-1}} = -\frac{y}{x} \ln y$

- **Generalization: One Implicit Equation**

  - Suppose $F(y, x_1, x_2) = 0$. Then

$$\frac{dy}{dx_1} = -\frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial y}}$$
$$\frac{dy}{dx_2} = -\frac{\frac{\partial F}{\partial x_2}}{\frac{\partial F}{\partial y}}$$

Example 8 *Suppose* $y^3 x + 2yw + xw^2 = 0$. *Then*

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{y^3 + w^2}{3y^2 x + 2w}$$
$$\frac{dy}{dw} = -\frac{\frac{\partial F}{\partial w}}{\frac{\partial F}{\partial y}} = -\frac{2y + 2xw}{3y^2 x + 2w}$$

  - Suppose $F(y, x_1, x_2, x_3, ..., x_n) = 0$. Then

$$\frac{dy}{dx_i} = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}}, \text{ for any } i = 1, 2, 3, ..., n$$