The Term Structure of Interest Rates

What is it?

The relationship among interest rates over different time-horizons, as viewed from today, $t = 0$.

A concept closely related to this:

The Yield Curve

- Plots the effective annual yield against the number of periods an investment is held (from time t=0).

- Empirical evidence *suggests* the effective annual yield is increasing in *n*, i.e. the number of periods remaining until maturity.

$$y_t^{(n)} > y_t^{(n-1)} > \ldots > y_t^{(2)} > y_t^{(1)},$$

where $y_t^{(n)}$ refers to the yield at time *t* over *n* periods.

We will concern ourselves with possible reasons for this:

- Begin by building simple model that captures essentials. The introduce complexities.
- Assume the future is known with certainty. Then introduce uncertainty

We should note that *time* is an essential element in our analysis. A period is a portion of time that defined over its beginning and end point.

Spot versus Short Rates

Spot rate:

- That rate of effective annual growth that equates the present with the future value.
- Thus, the spot rate is the cost of money over some time-horizon from a certain point in time.
- This is identical with the yield to maturity, or internal rate of return, on a *zero coupon* bond.
- Denote the yield of a bond at time t with n periods to maturity by $y_t^{(n)}$.

Short rate:

- Refers to the interest rate that prevails over a specific time period.
- Only known with certainty *ex-post*.
- The short rate refers to the (annualised) cost of money between any two dates, thus it may provide us with the correct rate of discount to apply over a

certain time period, e.g. the rate that prevailed between year one and year two.

- Denote the short rate applicable between time $t = 1$ & $t = 2$ as r_1.

- We (typically) use a combination (i.e. the product) of short rates to discount over a series of time-periods.

Expectations

If we knew with certainty the *short* interest rates that will hold over the future periods, we could calculate the effective annual yield that applies for a specific time-horizon.

In reality the future sequence of interest rates is unknown.

Similarly, if we know the *spot-rates* (*the yield to maturity* of a *zero coupon bond*) at which money is lent/borrowed over the various time-periods from now (3 month money, six month money, etc.), we have an idea about what the best guess is, as to the likely development of interest rates over the coming periods. [However, these expectations *could* change dramatically in the next instant.]

Another distinction we must draw is between interest rates, short or spot, and the *yield* of an investment.

By taking the interest rates that prevailed over any one period, and forming an average of these (weighted by the amount of time they prevailed for over a given period), we can obtain the effective annual interest rate that prevailed over a specific period, or, equivalently, the *yield* that accrued to our investment.

We can plot these over time to obtain a yield curve. (Strictly speaking the yield is simply the effective annual rate of growth our investment would have to grow by in each period in order for it to grow from the price paid to the value at maturity).

The yields over n-periods are given by the <u>geometric average</u> of the short rates that prevailed in each period, i.e. it is the *single* effective annual yield that would have given our investment the same future value as we obtained from the series of short rates that actually prevailed.

Certainty

If we assume we know the future short rates with certainty, we can calculate the yield of investments locked in at these rates.

E.g. assume r_1 = 8%, r_2 = 10%, r_3 = 10%, r_4 = 11%, where r_1 is the interest rate that applied in the first year. (N.B.: The short rates in consecutive periods are rising!)

Then the yield on a 3-year investment should be:

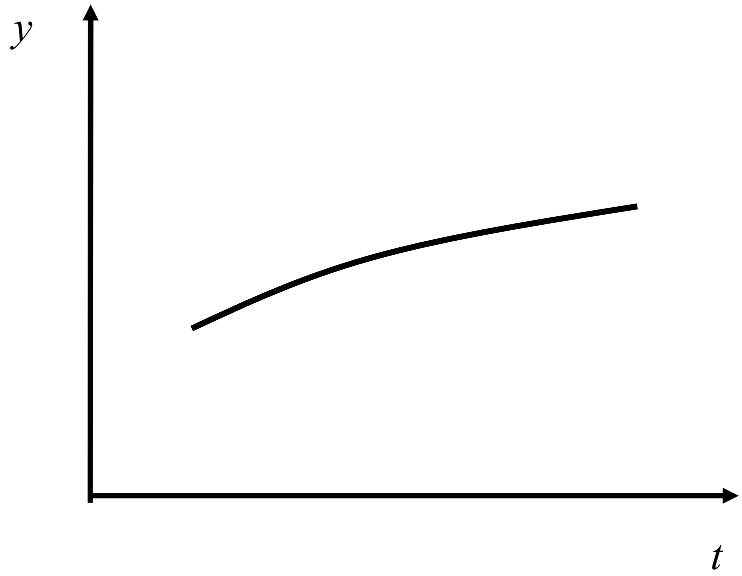
$$(1 + y^{(3)})^3 = (1 + r_1)(1 + r_2)(1 + r_3)$$

or

$$y^{(3)} = [(1 + r_1)(1 + r_2)(1 + r_3)]^{1/3} - 1.$$

In this case of certainty, we will note how the yield actually increases with the length of time an investment is locked in for. This, however, is <u>only</u> because the *short* rates are rising over time. You can calculate $y^{(i)}$, with i = 1, 2, 3, 4, yourself.

The Yield Curve



Generally,

$$(1+y^{(t)})^t = \prod_{i=1}^{t}(1+r_i)$$

$$\Rightarrow y^{(t)} = [\prod_{i=1}^{t}(1+r_i)]^{1/t} - 1$$

N.B., the holding period return, HPR, on investments of different maturities & with known (future) shorts would have to be identical, even if the yields (over their life-time) differ.

Why?

The HPR of any investment opportunity must be the identical over one period if everything is known (under certainty), since otherwise investments with higher returns would strictly dominate the lower return ones, thus causing the prices to adjust, and hence the rates of return.

Consider a roll-over strategy under certainty:

$$(1+y^{(n)})^n = \left(1+y^{(n-1)}\right)^{n-1}\left(1+r_n\right)$$

$$\left(1+r_n\right) = \frac{(1+y^{(n)})^n}{\left(1+y^{(n-1)}\right)^{n-1}}$$

Forward Rates (under certainty)

A forward rate agreement (FRA) is an agreement at time t to lend money at some future date, say $t+1$, to be repaid with interest at some date thereafter, say $t+2$.

Imagine, the *spot rates* for three month and six month money are given by $r_{0,3}$ and $r_{0,6}$, respectively. What should the forward rate from months four to six, $f_{4,6}$ be?

Clearly, different investment strategies over the same time horizon should have identical returns in a world of certainty. Thus,

$$\left(1+\frac{r_{1,6}}{2}\right)=\left(1+\frac{r_{1,3}}{4}\right)\left(1+\frac{f_{4,6}}{4}\right).$$

What this says is that the future value of a six-month investment should be equal to two successive 3-month investment strategies, when certainty prevails.

More generally (and omitting monthly considerations), the yield on an n-period investment should equal the product of the yield of an (n-1)-period investment and the rate of return of the forward rate for the nth-period, f_n:

$$(1+f_n)(1+y^{(n-1)})^{n-1} = (1+y^{(n)})^n$$

$$\Rightarrow (1+f_n) = \frac{(1+y^{(n)})^n}{(1+y^{(n-1)})^{n-1}}$$

The forward rate is identical to the future spot rate unde certainty.

Thus, we have seen, that under certainty, the only way the yield curve can be rising with an instrument's time-to-maturity is if future short rates rise.

Aside: Synthetic FRA

We can replicate an FRA without explicitly drawing up such an agreement by using spot rates.

Assume:

M = 1000 for all bonds

The price of a one-period zero, $P^{(1)}$, is 925.93.

That on a two period zero, $P^{(2)}$, is 841.68.

By $(P/M)^{1/n} - 1 = y^{(n)}$, we know that $y^{(1)} \approx 8\%$ and $y^{(2)} \approx 9\%$.

Also, by

$$\Rightarrow (1 + f_n) = \frac{(1 + y^{(n)})^n}{(1 + y^{(n-1)})^{n-1}}$$

we know, $f_2 \approx 10.01\%$.

For a sFRA we want to take out a loan at some future date, at a known interest rate, f_n, and repay that loan at some later date again.

Cash-Flow

Zero at t = 0

Positive when we want the loan, at t = 1

Negative when we repay, t = 2.

Strategy

$t=0$

- Buy a single one-period zero at $P^{(1)}$
- Sell $(1+f_2)$ times the two period spot at $P^{(2)}$
- Note, our cash flow is zero at t= 0. [925.93-925.93=0].

$t = 1$

- Receive M for our one-period zero, i.e. 1000.

$t = 2$

- Must pay back the liability incurred by selling f_2 times the two-period spot: $(1+f_2)*M$ = 1.1001*1000 = 1100.1.

It is obvious that the rate of interest between time t=1 and t=2 applied would have been quoted as 10.01%, i.e. f_2.

Uncertainty

In a world of uncertainty, we are unsure of future returns, e.g. interest rates vary, much of which is not entirely predictable. Thus, we tend to consider expected returns as our best guess, as to what future shorts are likely to be.

We will treat the short rate at time t, r_t, as a random variable.

Denote the expectation at time t about the short rate from time $t+i$ to $t+i+1$ as $E_t(r_{t+i})$.

Conditional Expectations:

The expectation, formed at time t, about a random variable, r_t, conditional on a set of information available at time t, Ω_t, is denoted:

$$E_t(r_t \mid \Omega_t).$$

Furthermore, the further away a point in time lies in the future, the less predictable the outcome is from today's data.

We equate uncertainty with risk, which we will measure as the variance of a random variable, *Var*(•).

For (government) bonds we assume default risk is negligible, and hence we do not include a *risk premium* in our considerations for the required rate of return.

However, there is risk associated with the length of time one is locked into an investment, as the return on other instruments in the future change. Thus, bonds *may* have a *term premium*.

<u>Preferences</u>

Investors' preferences may become an important determinant of asset valuation in a world of uncertainty. In particular, agents may be risk averse, which implies that a certain return is preferred to an uncertain (expected) return.

Also, they may have preferences about the length of time they are invested in a project, e.g. short- vs. long-term investors.

If such an investor had to choose between a long-run, but uncertain investment and a short-term but certain investment that offers the same return, he would choose the short-term investment.

In a world of certainty, this does not affect the choice over investments with different horizons. For investments that have a longer horizon than the investor desires, the investor can sell his investment, such that it has the same HPR as would an investment with a shorter life-time.

However, in a world of uncertainty, there is (resale) *price uncertainty* when investors wish to liquidate longer-term investments. On the other hand, if the investor does not wish to liquidate the investment prior to maturity, he faces uncertainty over *future reinvestment rates*.

Thus, the investor could invest in a one-period bond and gain a certain return for that one period. On the other hand, a bond with a longer horizon may only offer an expected return, since its future resale value is uncertain.

If future interest rates increase, the value of the bond will fall.

[Actually, deviations from the expected interest rates (i.e. a shock) will affect the price of longer-term bonds more than that of shorter-term securities. Why? Similar comparing the effect of the shock of you winning the lottery in your twenties (distant from your expiration), rather than winning it in your eighties].

Why should a risk-averse, or short-tem, investor hold a long-term investment, if the short-term investment offers a risk-less HPR that is identical to the (risky) *expected* HPR of a longer-term bond?

Clearly, a risk-averse (or short-term) investor must be compensated for assuming risk, and this may take the form of a *term*, or *liquidity premium*.

Let us assume, there exists a required rate of return, k_t, for any investor to hold an n-period bond between periods $t+i$ and $t+i+1$, that depends on the one period rate of return between those dates, r_{t+i}, (that period's short) and a term premium for an n-period bond, $T^{(n)}_{t+i}$.

$$k_{t+i} \equiv r_{t+i} + T^{(n)}_{t+i} \qquad (i > 1)$$

If this is the case, then the pertinent discount rate when pricing a bond should be given by k_{t+i}, rather than by r_{t+i}.

By the same token, though the forward rate equals the rate of return of that period's short rate under certainty, under uncertainty this is no longer the case.

In fact, for a world dominated by risk-averse agents, the, certain, forward rate should be less than the, uncertain, expected short rate, that prevails at that time.

The liquidity premium would be given by the difference between the forward rate for period n and the expected short rate for period n at time t:

$$premium = f_n - E_t(r_n) \, .$$

Theories of the Term Structure

If we have an n-period coupon bond and market determined spot rates exist for all maturities, then the market price of a bond should be determined by:

$$(1) \quad P_t^{(n)} = \frac{C_{t+1}}{(1+rs_t^{(1)})} + \frac{C_{t+2}}{(1+rs_t^{(2)})} + \ldots + \frac{C_{t+n} + M_{t+n}}{(1+rs_t^{(n)})^n} = \sum_{i=1}^{n} V_i$$

where $rs^{(n)}$ is the n-period spot rate.

If this were not so, then arbitrage would allow for risk-less profits, i.e. we could sell the rights to the payments.

Why?

Now, assume investors require an asset to provide a specific return between time t and $t+1$ in order for them to hold the asset, denoted k_t.

$$E_t\left[\frac{P_{t+1}^{(n-1)} - P_t^{(n)} + C_{t+1}}{P_t^{(n)}}\right] = k_t$$

This can be solved forward to give the current bond price as the discounted present value of future coupon payments discounted at the expected one-period spot returns k_{t+i}.

$$(2) \quad P_t^{(n)} = E_t\left[\sum_{j=1}^{n}\frac{C_{t+j}}{\prod_{i=o}^{j-1}(1+k_{t+i})} + \frac{M}{\prod_{i=0}^{n-1}(1+k_{t+i})}\right]$$

N.B.: The only unknown investors need to form expectations about are the *k*s. We can split these into a risk-less interest rate component and a term premium.

$$k_{t+i} \equiv r_{t+i} + T_{t+i}^{(n)} \; (i > 1).$$

There exist various theories as to the reasons for the yield curve to be rising in *n* under uncertainty. All of the ones we will consider make certain hypothesis about the term premium, $T_t^{(n)}$, whose value may be seen to depend on the time-period we are in, *t*, and the number of periods a security has to maturity, *(n)*. We will use the concept of one-period HPR to illustrate this.

(1) and (2) cannot both determine a bonds price. Underlying the different equations are differing behavioural assumptions:

(1) is determined by arbitrage.

(2) determines a set of expected one-period returns that yield the same bond price as (1).

(1) and (2) will provide the same price on a two period bond if:

$$(1+rs^{(1)}) = (1+k_0)$$
$$(1+rs^{(2)})^2 = (1+k_0)E_t(1+k_1)$$
$$\Rightarrow E_t(1+k_1) = (1+rs^{(2)})^2 / (1+rs^{(1)})$$

The Expectations Hypothesis (EH)

Under the EH the assumption made about the term premium is that it is constant in time and periods-to-maturity.

$$E_t H_{t+1}^{(n)} - r_t = T_t^{(n)} = T \ .$$

A specific version of the EH is the *Pure Expectations Hypothesis* (PEH), which states that the (constant) premium is zero, $T = 0$, for all time-periods and all bonds, regardless of time-to-maturity (simplest assumption).

In turn, this implies *risk-neutral investors* that are only concerned with expected returns, and all bonds' expected one-period holding period return is equal to that of a risk-less one-period bond. In this case all market participants are *plungers*.

Loosely speaking, the forward rate will equal the market consensus expected future short rate, i.e. $f_n = E_t(r_n)$. Thus, agents are risk-neutral, the expected excess HPR is zero, and liquidity premia are zero. (The yields on long-term bonds are given by the current expected future short rates.)

Thus, we can use the yield curve in order to gauge expectations on future interest rates.

Generally, the EH holds that *T* is constant, but not necessarily zero; Thus, the expected excess return is independent of time-period and time-to-maturity. Thus, in order to move from a risk-free to a risky longer term investment, investors require some fixed premium.

The Liquidity Preference Model (PEH)

The term-premium is time-invariant, but does depend on a security's life-time-to-maturity. Hence, the expected excess return is constant for securities of n-period lifespan, but depends on n:

$$T_t^{(n)} = T^{(n)}.$$

$$E_t H_{t+1}^{(n)} - r_t = T^{(n)}$$

Loosely speaking, depending on whether an investor has long- or short-run liquidity preferences, they will require a different term premium: short-run investors would require an expected future short rate in excess of the forward rate, whereas the opposite would hold for long-term investors. Usually, we assume the market is dominated by short-term investors, and

$$R_t^n - E_t(r_t) = T^{(n)}$$

and

$$T^{(n)} > T^{(n-1)} > \ldots > T^{(1)}.$$

Market Segmentation Theory

Here, long- and short-term bonds are perceived to exist in entirely different markets, with their very own and independent equilibria. Excess expected returns are influenced by the outstanding stock of maturities of similar lifetime.

$$E_t H_{t+1}^{(n)} - r_t = T(z_t^{(n)}),$$

where $z_t^{(n)}$ is a measure of the relative weight of securities with lifespan *n* in the portfolio of total assets. Thus, *T* depends, in part, on the outstanding stock of assets of different maturities.

$$R_t^{(n)} - E_t(r_t) = T(z_t^{(n)})$$

<u>*Preferred Habitat Hypothesis*</u>

Bonds with similar *n* have similar *T*, as they can be regarded as substitutes. In this view, investors of one type can be lured into the other market if the premium is large enough.

<u>*Time Varying Risk*</u>

Expected excess returns and the term premium are a function of (vary over) time and *n*.

$$E_t H_{t+1}^{(n)} - r_t = T(n, z_t),$$

where z_t is some set of variables.