Maximum Likelihood Estimates of Regression Coefficients with α-stable residuals and Day of Week effects in Total Returns on Equity Indices

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Abstract

This Paper summarizes the theory of Maximum Likelihood Estimation of regressions with α-stable residuals. Day of week effects in returns on equity indices, adjusted for dividends (total returns) are estimated and tested using this and traditional OLS methodology. I find that the α-stable methodology is feasible. There are some differences in the results from the two methodologies. The conclusion remains that if individual coefficients are of interest and the residuals have fat tails and a possible α-stable distribution, the results should be checked for robustness using methods such as those employed here.

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*Comments are welcome. My email address is `frainj@tcd.ie`. This document is work in progress. Please consult me before quoting. Thanks are due to Prof. Antoin Murphy and to Michael Harrison for help and suggestions and to participants at a seminar in TCD for comments reseived. Any remaining errors in the paper are my responsibility. I would also like to thank my wife, Helen, for her great support and encouragement.

1 Introduction

Returns on many assets are known to have fat tails and are often skewed. The almost universally used Normal or Gaussian distribution can model neither fat tails nor skewness. The α stable distribution can model these features. The use of this distribution in Finance was originally proposed by Mandelbrot (see Mandelbrot (1962, 1964, 1967) or Mandelbrot and Hudson (2004)) to model vaious goods and asset prices. It became popular in the sixties and seventies but interest waned thereafter. This decline in interest was due not only to its mathematical complexity and the considerable computation resources required but to the considerable success of the Merton Black Scholes Gaussian approach to Finance theory which was developed at the same time.

Recently there has been some renewed interest in the distribution. Recent Mathematical accounts are given in Zolotarev (1986), Samorodnitsky and Taqqu (1994), Weron (1998) and Uchaikin and Zolotarev (1999). Rachev and Mittnik (2000) survey the use of α-stable models in finance.

The availability of cheap powerful computer hardware has made advanced computation resources available to scientists in many fields. The resulting increased demand for good software has provided the incentive to produce and distribute widely software packages such as Mathematica (Wolfram (2003)) and R (R Development Core Team (2006)) which have facilitated the calculations in this paper. Programs in to compute α-stable distribution and density functions are available in both of these packages (Mathematica (Rimmer (2005)), Rmetrics for R (Wuertz (2005)) or as the stand-alone program STABLE (Nolan (2005))). These resources allow one to examine the consequences of replacing the Normal assumption with the more general α-stable. Further advances in theory and computation facilities will facilitate this process in the coming years and the use of the α-stable distribution will become more common.

In particular, this paper examines the consequences of α-stable residuals in OLS estimation. In section (2) the following results are set out:

- Standard OLS Estimates are consistent but inefficient.
- Coefficient Estimates have an α-stable distribution and standard t-statistics do not have the usual distribution
- Maximum Likelihood estimation have the usual asymptotic properties of Maximum Likelihood estimators and confidence intervals and inference may be based on the usual maximum likelihood theory.
- Maximum Likelihood estimation with α-stable residuals is a form of robust estimator which gives less weight to extreme observations

In section (3) this theory is applied to estimating and testing calendar effects in daily returns on equity indices. These day-of-week effects are often estimated by the coefficients in an OLS regression of daily returns on five dummy variables - one for each day of the week. I compare the results estimating of estimating such regressions using standard OLS and and α-stable maximum likelihood. Estimates are made for six total returns indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite (DJC) and S&P500) and the DJIA for the period used in the often quoted study of these effects in Gibbons and Hess (1981). My results can be summarized as follows -

- The α-stable maximum likelihood and OLS estimates for the DJIA for the Gibbons and Hess (1981) are almost identical.
- Data for the total returns indices are only available from the late 1980's (apart from the DAX30) and there are no significant week-day effects in the total returns indices in that period
- When the data for the CAC40 are split into three equal periods there are indications of weekday effects in the two early periods but they are absent in the late period.

These results are a demonstration of the shifting Monday effect reported in the literature (see Pettengill (2003) and the references there and Hansen et al. (2005)). Such results are, therefore, not sensitive to the use of the "robust" α-stable Maximum Likelihood Estimator.

An examination of the significance of the results for individual coefficients shows that some α-stable coefficients are significant where the corresponding OLS are not. Sullivan et al. (2001) sets out the danger of data mining in cases such as this. I would not draw any conclusions about weekday effects from these discrepancies. They do, however, draw attention to the possible different results that may arise from α-stable maximum likelihood estimation.

2 Regression with non-normal α-Stable Errors

Consider the standard regression model

$$y_i = \sum_{j=1}^{k} x_{ij}\beta_j + \varepsilon_i, \quad i = 1, \dots, N \tag{1}$$

where y_i is an observed dependent variable, the x_{ij} are observed independent variables, β_j are unknown coefficients to be estimated and ε_i are identically and independently distributed. Equation 1 may be written in matrix form as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{2}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \dots & x_{Nk} \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}, \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix} \tag{3}$$

The standard OLS estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}}_{OLS} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'y} \tag{4}$$

Thus

$$\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'\varepsilon} \tag{5}$$

Thus in the simplest case where $\boldsymbol{X}$ is predetermined $\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta}$ is a linear sum of the elements of $\boldsymbol{\varepsilon}$. If the elements of $\boldsymbol{\varepsilon}$ are independent identically distributed non-normal α-stable variables then $\hat{\boldsymbol{\beta}}_{OLS}$ has an α-stable distribution. The variance of ε_i does not even exist. Thus standard OLS inferences are not valid. (Logan et al. (1973)) prove the following properties of the asymptotic t-statistic

1. The tails of the distribution function are normal-like at $\pm\infty$
2. The density has infinite singularities $|1 \mp x|^{-\alpha}$ at ± 1 for $0 < \alpha < 1$ and $\beta \neq \pm 1$. When $1 < \alpha < 2$ the distribution has peaks at ± 1.
3. As $\alpha \to 2$ the density tends to normal and the peaks vanish

When $1 < \alpha < 2$ the OLS estimates are consistent but converge ar a rate of $n^{\frac{1}{\alpha}-1}$ rather than $n^{-\frac{1}{2}}$ in the normal case.

DuMouchel (1971, 1973, 1975) shows that, subject to certain conditions, the maximum likelihood estimates of the parameters of an α-stable distribution have the usual asymptotic properties of a Maximum Likelihood estimator. They are asymptotically normal, asymptotically unbiased and have an asymptotic covariance matrix $n^{-1}I(\alpha, \beta, \gamma, \delta)^{-1}$

where $I(\alpha,\beta,\gamma,\delta)$ is Fisher's Information. McCulloch (1998) examines linear regression in the context of α-stable distributions paying particular attention to the symmetric case. Here the symmetry constraint is not imposed. Assume that $\varepsilon_i = y_i - \sum_{j=1}^{k} x_{ij}\beta_j$ is α-stable with parameters $\{\alpha,\beta,\gamma,0\}$. If we denote the α-stable density function by $s(x,\alpha,\beta,\gamma,\delta)$ then we may write the density function of ε_i as

$$s(\varepsilon_i,\alpha,\beta,\gamma,\delta) = \frac{1}{\gamma}\ s\left(\frac{y_i - \sum_{j=1}^{k} x_{ij}\beta_j}{\gamma}, \beta, 1, 0\right), \tag{6}$$

the Likelihood as

$$L(\boldsymbol{\varepsilon},\alpha,\beta,\gamma,\beta_1,\beta_2,\dots) = \left(\frac{1}{\gamma}\right)^n \prod_{i=1}^{n} s\left(\frac{y_i - \sum_{j=1}^{k} x_{ij}\beta_j}{\gamma}, \beta, 1, 0\right), \tag{7}$$

and the Loglikelihood as

$$\begin{aligned} l(\boldsymbol{\varepsilon},\alpha,\beta,\gamma,\beta_1,\beta_2,\dots) &= \sum_{i=1}^{n}\left(-n\log(\gamma) + \log\left(s\left(\frac{y_i - \sum_{j=1}^{k} x_{ij}\beta_j}{\gamma}, \beta, 1, 0\right)\right)\right) \\ &= \sum_{i=1}^{n} \phi(\hat{\varepsilon}_i). \end{aligned} \tag{8}$$

The maximum likelihood estimators are the solutions of the equations

$$\begin{aligned} \frac{\partial l}{\partial \beta_m} &= \sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) x_{im} = 0, \quad m = 1,2,\dots,k \\ &\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i}\hat{\varepsilon}_i x_{im} = 0, \quad m = 1,2,\dots,k \\ &\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i}(y_1 - \sum_{j=1}^{k} x_{ij}\beta_j) x_{im} = 0, \quad m = 1,2,\dots,k \\ &\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i}(y_i - \sum_{j=1}^{k} x_{ij}\beta_j) x_{im} = 0, \quad m = 1,2,\dots,k \\ &\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i} y_i x_{im} = \sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i} \sum_{j=1}^{k} x_{ij}\beta_j \end{aligned} \tag{9}$$

If $\boldsymbol{W}$ is the diagonal matrix

$$\boldsymbol{W} = \begin{pmatrix} -\frac{\phi'(\hat{\varepsilon}_1)}{\hat{\varepsilon}_1} & 0 & \dots & 0 \\ 0 & -\frac{\phi'(\hat{\varepsilon}_2)}{\hat{\varepsilon}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{-\phi'(\hat{\varepsilon}_n)}{\hat{\varepsilon}_n} \end{pmatrix}, \tag{10}$$

Using the notation in equation (3)we may write equation (9) in matrix format.

$$\boldsymbol{X}'\boldsymbol{W}\boldsymbol{y} = (\boldsymbol{X}'\boldsymbol{W}\boldsymbol{X})\hat{\boldsymbol{\beta}} \tag{11}$$

or if $\boldsymbol{X'WX}$ is not singular

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X'WX})^{-1}\boldsymbol{X'Wy} \tag{12}$$

Thus the maximum likelihood regression estimator has the format of a Generalized Least Squares estimator in the presence of heteroscedasticity where the variance[1] of the error term ε_i is proportional to $\frac{\phi'(\varepsilon_i)}{\varepsilon_i}$. The effect of the "Generalized Least Squares" adjustment is to give less weight to larger observations. Figure 1 compares the weighting pattern derived from equation (10) for α-stable processes with $\alpha = 1.2$ and 1.6 with those of a standard normal distribution. For compatibility purposes the α-stable curves are drawn with $\gamma = 1/\sqrt{2}$. As expected the normal distribution gives equal weights to all observations. The estimator for α-stable processes gives higher weights to the center of the distribution and extremely small weights to extreme values. This effect increases as α is reduced.

This result explains the results obtained by Fama and Roll (1968) who completed a Monte Carlo study of the use of truncated means as measures of location in α-stable distributions. They found

> *When* $\alpha = 1.1$ *the* .25 *truncated*[2] *means are still dominant for all n. For* $\alpha =$ 1.3 *and* $\alpha = 1.5$ *the* .50 *truncated means are generally best, and when* $\alpha = 1.9$ *the distributions of the* .75 *truncated means are uniformly less disperse than those of other estimators. Finally, when the generating process is Gaussian* ($\alpha = 2$) *the mean is the "best" estimator. Of course it is also minimum-variance, unbiased in this case.*

The shape of the weight curves in the skewed case is shown in figure (2). The weights are based on the same α-stable distributions as those in figure 1 except that β is now -0.1. The most surprising aspect of the weighting systems is the negative weights given to small positive observations. Again the effects are more pronounces as α is reduced.

[1] This is only an analogy. The vatiance of the error term does not exist

[2] A g truncated mean retains $100g\%$ of the data. Thus a .25 truncated mean is an average of the central 25% of the data

3 Maximum Likelihood Estimates of Day of Week Effects with α-Stable errors

Empirical analysis suggests that there is a recurrent low or negative return on equities from Friday to Monday. This effect is known as the weekend effect. The existence of this effect would allow one to design a strategy to make excess profits and would have implications for the Efficient Markets Hypothesis. It is likely that, if residuals are *alpha*-stable ,then the usual Ordinary Least Squares inferences may lead to spurious results. The use of α-stable residuals and maximum likelihood will lead to a more robust result.

The analysis is based on daily data for six equity indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite(DJC) and S&P500) which have been adjusted to include dividends. Thus if P_t and D_t are the price and dividend of the index in period t the return on the index in period t is given by

$$R_t = 100 \log \left(\frac{P_t + D_t}{P_{t-1}} \right) \approx 100 \left(\frac{P_t + D_t}{P_{t-1}} - 1 \right). \tag{13}$$

I have also used returns based on the historic values of the Dow Jones Industrial Average equity price index covering the period July 3, 1962 to December 28, 1978, the period analyzed in Gibbons and Hess (1981). These have not been adjusted for dividends.

Descriptive statistics and details of goodness of fit of the return series to Normal and α-stable distributions are given in table (1). The goodness of fit normality tests indicate considerable problems with the fit of a Normal distribution. The α-stable distribution provides a better fit.

To estimate and test for weekday effects returns were regressed on five dummy variables, one for each day of the week. The presence of a weekday effect is indicated by the rejection of the hypothesis that all five regression coefficients are equal.

Table (2) gives OLS estimates for the longest sample available for each total returns index and for the DJIA for the period July 3, 1962 to December 28, 1978 as used in Gibbons and Hess (1981). Table (3) gives corresponding results for estimation using α-stable maximum likelihood methods

Maximum likelihood estimation is carried out by numerically maximizing the log of the likelihood function in equation (8). In the present case Ordinary Least Squares is used to derive initial values for the regression parameters. An α-stable distribution was fitted to the residuals of this regression using the Mathematica (Wolfram (2003)) α-stable density functions in Rimmer (2005). The resulting estimates values of α, β and γ were used as initial values for these parameters in the likelihood estimation. Standard errors of the estimates were estimated by the square root of the diagonal elements of the inverse of the hessian the loglikelihood function. While these estimates of the variance of the estimates appear to be numerically stable corresponding estimates of the covariances were not, in some cases. Thus joint hypotheses on the coefficients are Likelihood Ratio tests.

Table 1: Summary Statistics Equity Total Returns and their fit to Normal and α-Stable Distributions

	ISEQ	CAC40	DAX 30	FTSE100	DJC	S&P500
start date	04/01/88	31/12/87	28/09/59	31/12/85	30/09/87	03/01/89
end date	21/09/05	26/09/05	26/09/05	26/09/05	26/09/05	26/09/05
observations	4622	4627	12000	5149	4693	4363
mean	0.052	0.044	0.022	.041	0.038	0.043
St. dev	0.934	1.277	1.148	1.028	1.007	0.980
Skewness	-0.3634	-0.124	-0.282	-0.732	-2.686	-0.198
Kurtosis	5.376	3.002	8.378	9.814	58.1964	4.282
Goodness of Fit Tests for Normal Distribution						
JB test[a]	5690	1749	35254	21123	667907	3362
KS test[b]	0.065	0.054	0.062	0.055	0.074	0.063
SW test[c]	0.941	0.967	NA	NA	0.8689	0.956
Maximum Likelihood Estimates of Parameters of α-stable distribution						
α[d]	1.646	1.718	1.687	1.726	1.684	1.668
	(0.045)	(0.043)	(0.027)	(0.041)	(0.044)	(0.046)
β	-0.064	-0.147	-0.076	-0.147	-0.076	-0.105
	(0.111)	(0.128)	(0.075)	(0.125)	(0.119)	(0.118)
γ	0.502	0.746	0.627	0.583	0.529	0.550
	(0.014)	(0.020)	(0.011)	(0.015)	(0.015)	(0.017)
δ	0.054	0.032	0.019	0.036	0.042	0.034
Goodness of Fit Tests for α-Stable Distribution						
KS (stable)	0.012	0.014	0.010	0.008	0.018	0.023
p-value	0.518	0.307	0.166	0.892	0.097	0.025
LR[e]test of Normality	838.1	418.6	1945.8	786.7	1236.5	583.0

[a] The asymptotic distribution of the Jarque-Bera statistiv is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

[b] For the sample sizes here the 1% critical value for the Kolmogorov-Smirnov statistic is less that .02. See Marsaglia et al. (2003)

[c] The 5% critical level for the Shapiro Wilk test is .9992 for a sample of 4500. The smaller values reported here indicate very significant departures from normality.

[d] Figures in brackets under each coefficient estimate are the 95% confidence interval half width estimates

[e] Likelihood ratio test of the joint restriction $\alpha = 2$ and $\beta = 0$. The test statistic is asymptotically $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

Table 2: OLS Estimates of Day-of-Week Effects in Returns Indices

	ISEQ	CAC40	DAX 30	FTSE100	DJC	S&P500	Gibbons Hess(1981)
OLS coefficient estimates and Standard Errors							
monday[a]	0.046	-0.032	-0.086	0.026	0.062	0.084	-0.128
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
tuesday	0.047	0.063	0.005	0.047	0.062	0.040	-0.007
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
wednesday	0.041	0.027	0.060	0.025	0.061	0.056	0.080
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
thursday	0.062	0.088	0.043	0.036	-0.003	0.023	0.032
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
friday	0.057	0.072	0.087	0.071	0.017	0.011	0.065
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
Test of equality of weekday coefficients							
F-test	0.061	1.297	8.239	0.349	0.786	0.7361	9.684
significance	(0.993)	(0.269)	(0.00)	(0.845)	(.534)	(0.5672)	(0.000)
Goodness of Fit Tests of Residuals to Normal Distribution							
JB test[b]	5934	1728	34481	20843	665075	3380	1437
Estimates of α-stable Parameters of OLS residuals							
α	1.646	1.727	1.688	1.733	1.683	1.661	1.738
	(0.025)	(0.023)	(0.015)	(0.021)	(0.025)	(0.027)	(0.026)
β	-0.053	-0.136	-0.062	-0.145	-0.077	-0.103	-0.005
	(0.052)	(0.064)	(0.037)	(0.062)	(0.057)	(0.056)	(0.070)
γ	0.500	0.749	0.627	0.584	0.528	0.549	0.467
	(0.008)	(0.011)	(0.006)	(0.008)	(0.008)	(0.009)	(0.007)
δ	0.003	-0.009	- 0.001	-0.004	0.004	-0.009	-0.007
	(0.016)	(0.021)	(0.011)	(0.015)	(0.015)	(0.017)	(0.013)

[a] Figures in brackets under each coefficient or parameter estimate are the standard errors of the estimate

[b] The asymptotic distribution of the Jarque-Bera statistiv is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

Table 3: α-stable Estimates of Day-of-Week Effects in Returns Indices

	ISEQ	CAC40	DAX 30	FTSE100	DJC	S&P500	Gibbons Hess(1981)
α-stable coefficient estimates and Standard Errors							
monday[a]	0.044	0.035	0.003	0.043	0.035	0.009	-0.136
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
tuesday	0.049	0.014	0.048	0.011	0.030	0.028	-0.005
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
wednesday	0.069	0.073	0.047	0.025	-0.008	0.004	0.071
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
thursday	0.052	0.048	0.081	0.063	-0.005	0.030	0.010
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
friday	0.059	0.003	-0.007	0.044	0.011	0.010	0.066
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
Test of equality of weekday coefficients							
LM-test	0.62	2.42	39.00	2.10	5.9	7.3	47.04
significance	(0.961)	(0.659)	(0.000)	(0.717)	(0.207)	(0.121)	(0.000)
Goodness of Fit Test of Residuals to α-stable distribution							
KS (stable)[b]	0.0151	0.0185	0.0082	0.0100	0.0199	0.0239	0.0166
	(0.243)	(.085)	(0.391)	(0.687)	(0.048)	(0.014)	(0.186)
Maximum Likelihood Estimates of Parameters of α-stable distribution							
α	1.632	1.725	1.688	1.733	1.683	1.662	1.738
	(0.024)	(0.023)	(0.015)	(0.021)	(0.025)	(0.027)	(0.025)
β	-0.054	-0.136	-0.062	-0.145	-0.079	-0.105	-0.000
	(0.052)	(0.064)	(0.037)	(0.062)	(0.057)	(.056)	(.069)
γ	0.500	0.749	0.627	0.584	0.528	0.549	0.467
	(0.007)	(0.011)	(0.006)	(0.008)	(0.008)	(.009)	(0.007)

[a] Figures in brackets under each coefficient or parameter estimate are the standard errors of the estimate

[b] Kolmogorov-Smirnov test with a null of an α-stable distribution. Significance Levels are approximate

Table 4: Summary Statistics DAX30 Total Returns and fit to Normal and α-Stable Distributions for three sub periods

start date end date	29/09/59 27/01/75	28/01/75 28/05/90	29/05/90 26/09/05
observations	4000	4000	4000
mean	0.004	0.036	0.025
St. dev	0.989	0.978	1.424
Skewness	0.518	-1.061	-0.276
Kurtosis	8.633	17.940	4.110
Goodness of Fit Tests for Normal Distribution			
JB test[a]	1287	54389	2874.23
KS test[b]	0.044	0.058	0.072
SW test[c]	0.952	0.907	0.949
Estimates of of α-stable Parameters of Return Distribution			
α[d]	1.820	1.777	1.636
	(0.022)	(0.023)	(0.027)
β	0.059	-0.066	-0.168
	(0.095)	(0.082)	(0.056)
γ	0.601	0.553	0.774
	(0.009)	(0.008)	(0.013)
δ	0.001	0.040	0.008
Goodness of Fit Tests for α-Stable Distribution			
KS (stable)	0.012	0.012	0.023
p-value	0.619	0.652	0.034

[a] The asymptotic distribution of the Jarque-Bera statistiv is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

[b] For the sample sizes here the 1% critical value for the Kolmogorov-Smirnov statistic is less that .02. See Marsaglia et al. (2003)

[c] The 5% critical level for the Shapiro Wilk test is .9992 for a sample of 4500. The smaller values reported here indicate very significant departures from normality.

[d] Figures in brackets under each coefficient estimate are asymptotic standard errors

Table 5: OLS Estimates of Day-of-Week Effects in Returns on DAX30 Index in three periods

	DAX30		
start date end date	29/09/59 27/01/75	28/01/75 28/05/90	29/05/90 26/09/05
observations	4000	4000	4000
OLS coefficient estimates and Standard Errors			
monday[a]	-0.192	-0.123	0.057
	(.035)	(0.034)	(0.050)
tuesday	-0.043	0.034	0.023
	(.035)	(0.034)	(0.050)
wednesday	0.102	0.071	0.007
	(.035)	(0.034)	(0.050)
thursday	0.058	0.064	0.007
	(.035)	(0.034)	(0.050)
friday	0.094	0.135	0.032
	(.035)	(0.034)	(0.050)
Test of equality of weekday coefficients			
F-test significance	12.70 (0.000)	7.800 (0.000)	0.171 (0.953)
Goodness of Fit Tests of Residuals to Normal Distribution			
JB test[b]	12837	51386	2874
Estimates of α-stable Parameters of OLS Residuals			
α	1.818	1.778	1.635
	(0.023)	(0.023)	(0.027)
β	0.089	-0.044	-0.169
	(0.095)	(0.082)	(0.056)
γ	0.596	0.552	0.774
	(0.009)	(0.008)	(0.013)
δ	0.002	-0.005	- 0.018
	(0.016)	(0.016)	(0.026)

[a] Figures in brackets under each coefficient or parameter estimate are the standard errors of the estimate

[b] The asymptotic distribution of the Jarque-Bera statistiv is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

Table 6: Maximum Likelihood α-stable Estimates of Day-of-Week Effects in DAX30 Total Return Index

	DAX30		
start date end date	29/09/59 27/01/75	28/01/75 28/05/90	29/05/90 26/09/05
observations	4000	4000	4000
α-stable coefficient estimates and Standard Errors			
monday[a]	-0.195	-0.069	0.058
	(.033)	(0.030)	(0.046)
tuesday	-0.041	-0.029	0.011
	(.032)	(0.030)	(0.045)
wednesday	0.084	0.066	-0.029
	(.032)	(0.030)	(0.045)
thursday	0.070	0.059	-0.011
	(.032)	(0.030)	(0.045)
friday	0.090	0.120	-0.009
	(.032)	(0.030)	(0.045)
Test of equality of weekday coefficients			
LM-test	60.50	23.08	2.34
significance	(0.000)	(0.000)	(0.673)
Goodness of Fit Test of Residuals to α-stable Distribution			
KS (stable)[b]	0.0098	0.0098	0.0279
	(0.837)	(.0836)	(0.004)
Maximum Likelihood Estimates of Parameters of α-stable Residuals			
α	1.818	1.777	1.634
	(0.023)	(0.024)	(0.027)
β	0.090	-0.048	-0.170
	(0.094)	(0.082)	(0.056)
γ	0.596	0.551	0.774
	(0.009)	(0.008)	(0.013)

[a] Figures in brackets under each coefficient or parameter estimate are the standard errors of the estimate

[b] Kolmogorov-Smirnov test with a null of an α-stable distribution. Significance Levels are approximate

The final column in each table gives results corresponding to those in Gibbons and Hess (1981). Although I use a different equity index my results are very similar. The Monday effect is negative and very significant in both cases and the OLS and α-stable estimates are very similar. Thus the Gibbons and Hess (1981) results are robust with respect to the specification of residals.

This “Monday effect” effect found in the Gibbons and Hess (1981) sample period is not significant in the five total returns indices (ISEQ, CAC40, FTSE100, DJC and S&P500). This corresponds to recent analysis which has found that the “Monday effect” has been becoming smaller and even positive in recent times (see for example Hansen et al. (2005)). It should be noted that although there are some differences in the return patterns when comparing the OLS and Maximum Likelihood α-stable estimates it is not obvious how any statistical significance could be attached to this result. Specific weekday effects such as the “Monday effect” have often been justified only after a significant weekday effect has been found. In an experimental science this could be verified by further independent experimental studies. In economics we rarely have this facility. Sullivan et al. (2001) lists a large number of possible seasonal effects. Some of these are likely to occur by chance and will then be found by a specification search. The danger of data-mining is very real.

The longer return series for the DAX30 shows a significant day of the week effect in both OLS and α-stable cases. The OLS analysis points to significantly low returns on Monday and high returns on Friday as the cause of the problem. The α-stable results point to significantly higher returns on Thursday. The contradiction in this result highlights the danger of data mining in this case. The point has been discussed at length in Sullivan et al. (2001).

Table 4 gives summary statistics of returns on the DAX30 for three periods, 29 September 1959 to 27 January 1975, 28 January 1975 to 28 May 1990 and 29 May 1990 to 27 September 2005. The normal distribution is a poor fit to the data. The α-stable distribution provides a good fit for the first two periods. The goodness of fit test for an α-stable distribution fails for the third period The p-values in are based on the assumption of known parameters and thus possible underestimate the fit.

Tables (5) and (3) set out OLS and α-stable maximum likelihood estimates of the day of week effects in each or these subperiods. The results are very similar in both sets of tests. In the first two periods the hypothesis of no weekday effect is rejected and in the third period the hypothesis can not be rejected in both the OLS and α-stable analysis. In the α-stable analysis for the two early periods the Thursday return is significantly higher than the average. This is not so in the OLS analysis.

It should be noted that the stability parameter in the fit of all residuals to an α-stable distribution is significantly less than 2 indicating deficiencies in the assumption of a normal distribution.

4 Summary and Conclusions

This paper sets out the theory of maximum likelihood estimation and of a linear regression when residuals follow an α-stable distribution. This theory is then applied to the estimation and testing for a “weekday effects” in returns on equity indices. I have found that maximum likelihood estimation of a linear regression with α-stable residuals is feasible.

Traditional OLS estimation and testing is carried out in parallel and the results are compared. Of the ten regressions completed significant “weekday effects” effects were found in the same four in both the α-stable and OLS systems. However the alternative methodologies attributed significance to different daily effects. The α-stable distribution appeared to be a better fit to the residuals in both OLS and α-stable estimates and on the basis of specification the α-stable estimation is to be preferred.

“weekday effects” such as the “Monday effect”, found in some of the regressions here, are often justified by theories relating to institutional arrangements. A “Monday effect” has been explained by delays in trading and settlement caused by the weekend. Such explanations are often given after the significant result has been found leading to accusations of data mining. As I did not have a prior theory explaining the extra effects found in the α-stable estimates any conclusions that I might draw would, justifiably, be subject to the same criticisms. The conclusion remains that if individual coefficients are of interest, the residuals have fat tails and a possible α-stable distribution the results should be checked for robustness using methods such as those employed here.

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A An Introduction to α-Stable Processes

Some limit properties of normal random variables

This section outlines the properties of the α-stable family of distributions and compares those with the standard normal distribution. Proofs are not given. These and further details may be found in Feller (1966), Janicki and Weron (1994), Rachev and Mittnik (2000), Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999) or Zolotarev (1986).

An assumption of a normal distribution has formed part of almost all developments in theoretical and empirical finance in the last half century. In the introduction we have already referred to the Capital Asset Pricing Model, optimal portfolio allocation and option pricing as depending on a normality assumption. Some form of central[3] limit theorem has been implicit in all these developments. The theorem as quoted in many econometric tests may be weakened. A version in Gut (2005) is as follows

Lindeberg-Levy-Feller Theorem: Let $X_1, X_2, \ldots$ be independent random variables with finite variances, and set, for $k \geq 1$, $E\,X_k = \mu_k$, $Var\,X_k = \sigma_k^2$, and, for $n \geq 1$, $S_n = \sum_{k=1}^n X_k$ and $s_n^2 = \sum_{k=1}^n \sigma_k^2$. The Lindeberg conditions are

$$L_1(n) = \max_{1 \leq k \leq n} \frac{\sigma_k^2}{s_n^2} \rightarrow 0 \text{ as } n \rightarrow \infty \tag{14}$$

$$L_2(n) = \frac{1}{s_n^2} \sum_{k=1}^{n} E|X_k - \mu_k|^2 I\{|X_k - \mu_k| > \varepsilon s_n\} \rightarrow 0 \text{ as } n \rightarrow \infty \tag{15}$$

If equation 15 is satisfied then so is equation 14 and[4]

$$\frac{1}{s_n^2} \sum_{k=1}^{n} (X_k - \mu_k) \xrightarrow{d} N(0,1) \text{ as } n \rightarrow \infty \tag{16}$$

Thus the sum of independent random variables is normal subject to some fairly unrestrictive conditions on the tails of the distribution. There is even a form of inverse central limit theorem. If Y has a normal distribution and $Y = X_1 + X_2$ and X_1 and X_2 are not degenerate then X_1 and X_2 have a normal distribution.

To each random variable X we can assign a *type* $\{aX + b : a \in \boldsymbol{R}^+, b \in \boldsymbol{R}\}$. As all normal random variables can be transformed to an $N(0,1)$ they are of one type. As the distribution of any sum of random variables, with finite variance, tends to a normal, the normal *type* is regarded as a domain of attraction for such random variables. We shall be ask if there are there other domains of attraction for random variables and what random variables are "attracted" to these domains of attraction?

[3] The name "Central Limit Theorem" is attributed to Pólya. In the German the adjective central modifies the word theorem and not the word limit. The theorem is central to probability and statistics.

[4] The notation $\xrightarrow{d}$ implies a limit in distribution. The notation $\stackrel{d}{=}$ implies that the variables on either side of the sign have the same distribution.

Definition of α-stable random variable

Let X, X_1, X_2, ... be independent identically distributed normal random variables and let

$$(X_1 + X_2 + \cdots + X_n) \stackrel{d}{=} B_n X + A_n. \tag{17}$$

where

- $B_n > 0$ and A_n are real constants. A_n is a centralizing parameter and B_n is a normalizing factor
- $\lim_{n\to\infty} \max\{P(|X_j B_n^{-1}| > \varepsilon) : j = 1, 2, \ldots, n\} = 0$. A sufficient condition for this to hold is that $B_n \to \infty$ as $n \to \infty$

Then X is an α-stable random variable. The term stable refers to the property that the sum of identically distributed independent random variables having the same distribution as the original up to a scale (B_n) and location factor (B_n). If the X's are normal with mean μ and variance σ^2 we may put

$$B_n = \sqrt{n} = n^{\frac{1}{2}}$$
$$A_n = (1 - n^{\frac{1}{2}})\mu$$

to show that the normal distribution satisfies these conditions and is α-stable.

It can also be shown that B_n can only take the value

$$B_n = n^{\frac{1}{\alpha}}, \quad 0 < \alpha \leq 2. \tag{18}$$

with, as shown, the value $\alpha = 2$ corresponding to a normal distribution. This explains the use of α in the term α-stable.

The characteristic function of a Stable distribution S is given by

$$\int e^{itx} dS(x) = \begin{cases} \exp(-\gamma^\alpha |t|^\alpha [1 - i\beta(\tan \frac{\pi\alpha}{2}) \text{ sign } t] + i\delta t), & \text{if, } \alpha \neq 1 \text{ ;} \\ \exp(-\gamma\, |t|\, [1 + i\beta \frac{2}{\pi} (\text{ sign } t) \log(|t|)] + i\delta t), & \text{if, } \alpha = 1. \end{cases} \tag{19}$$

(see Zolotarev (1986) or Samorodnitsky and Taqqu (1994)). The sign t function is defined as

$$\text{sign } t = \begin{cases} -1, & u < 0; \\ 0, & u = 0; \\ 1, & u > 0. \end{cases} \tag{20}$$

The distribution depends on four parameters α, β, γ and δ. These parameters[5] can be interpreted as follows

[5]Note that different notation is adopted by various authorities. The principal differences include

- reversal of the sign of β
- Substitution of $c = \gamma^\alpha$

- α is the basic stability parameter. It determines the weight in the tails.
- β is a skewness parameter and $-1 \leq \beta \leq 1$. A zero beta implies that the distribution is symmetric. Negative or positive β imply that the distribution is skewed to the left or right respectively
- The parameter γ is positive and measures dispersion.
- The parameter δ is a real number and may be thought of as a location measure

Figures 3 to 6 illustrate various properties of α-stable distributions. Figure 3 shows the density functions for symmetric ($\beta = 0$) α stable distributions with $\alpha = 2$ (normal), $\alpha = 1.5$ and $\alpha = 1.0$ (Cauchy). As α is reduced note that the peak gets higher and the tails get heavier. This process continues as α is reduced. Figure 4 is an enlarged version of the left tail of the distribution and shows clearly the heavier tails.

Figure 5 shows the effect of varying the symmetry parameter β for fixed α. With $\alpha = 1.5$ As β falls from 0 to -1 the left tail becomes heavier relative to the right tail and the mode of the distribution shifts to the left of the mean. Similar transformations occur in the opposite direction when β moves from 0 to 1. The skewness caused by a particular value of β increases as α is reduced.

Figure 6 shows the left tail of the empirical distribution of the ISEQ return data, the normal distribution with parameters from table 2, and an α-stable distribution with parameters from table A. The departures from the normal distribution are very clear as is the fit of the normal distribution.

The density function of the stable distribution may be shown to be differentiable (and continuous) on the real line. Except in three special cases the density function of the Stable distribution can not be expressed in terms of elementary functions. The special cases are:

- Substitution of $\sqrt{2}\sigma$ for γ
- The characteristic function in equation 19 is not continuous at $\alpha = 1$. This may lead to problems in certain circumstances. If one makes the substitution

$$\delta_0 = \begin{cases} \delta + \beta\gamma \tan \frac{\pi\alpha}{2} & \alpha \neq 1 \\ \delta + \beta \frac{2}{\pi} \gamma \log \gamma & \alpha = 1 \end{cases}$$

  Following the notation of Nolan (2006) we may write the characteristic function of an α-stable function as

$$\int e^{itx} dS(x) = \begin{cases} \exp\left(-\gamma^\alpha |t|^\alpha [1 + i\beta(\tan \frac{\pi\alpha}{2})(\text{sign } u)(|\gamma t|^{1-\alpha} - 1)] + i\delta_0 t]\right) & \alpha \neq 1 \\ \exp\left(-\gamma |t| [1 + i\beta \frac{2}{\pi} (\text{sign } u) \log(\gamma |t|)] + i\delta_0 t\right) & \alpha = 1 \end{cases}$$

Because of the better behavior of this parametrization at $\alpha = 1$ it is the form most often used in numerical calculations. Nolan refers to this as an $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 0)$ distribution. The parametrization in equation 19 is referred to as an $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 1)$ distribution and is the form most often used here. In the $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 1)$ note that when $0 < \alpha \leq 1$ $EX = \mu$. In the $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 0)$ this does not hold, in general. Note than if $\beta = 0$ or $\alpha = 2$ the two parameterizations coincide. Here we shall use the $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 0)$ parametrization and the density and distribution functions will be denoted by $s(x, \alpha, \beta, \gamma, \delta)$ and $S(x, \alpha, \beta, \gamma, \delta)$ respectively. If the variables are standardized ($\gamma = 1$ and $\delta = 0$ we may use the symbols $s(x, \alpha, \beta)$ and $S(x, \alpha, \beta)$ for the density and distribution

Normal Description If $\alpha = 2$ the characteristic function in equation (19) reduces to

$$\phi(it) = \int e^{itx} dH(x) = \exp(i\delta t + \gamma^2 t^2) \tag{21}$$

Which is the characteristic function of a normal distribution

$$\frac{1}{\gamma\sqrt{\pi}} \exp \frac{(x-\delta)^2}{\gamma^2}, \quad -\infty < x < \infty$$

with mean δ and variance $2\gamma^2$. Note that the symmetry parameter does not appear in the characteristic function in this case.

Cauchy Distribution When $\alpha = 1$ and $\beta = 0$ the characteristic function reduces to

$$\exp(-\gamma|t| + i\delta t)$$

which is the characteristic function of the Cauchy Distribution

$$\frac{1}{\pi(\gamma^2 + (x-\delta)^2)}, \quad -\infty < x < \infty$$

Levy Distribution When $\alpha = 1/2$ and $\beta = -1$ the distribution becomes a Levy distribution

$$\left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left(-\frac{\sigma}{2(x-\mu)}\right), \quad \mu < x < \infty$$

Generalized Central Limit Theorem – Domains of attraction

Consider a random variable X with density function such that

$$F(x) \sim \begin{cases} B_-|x|^{-(1+a)} & \text{as} \quad x \to -\infty \\ B_+|x|^{-(1+a)} & \text{as} \quad x \to \infty \end{cases} \tag{22}$$

where $0 < a < 2$. Thus the tails of the distribution have an asymptotic Pareto[6] distribution. Put

$$b = \frac{B_+ - B_-}{B_+ + B_-} \tag{23}$$

[6] The Pareto distribution was used by Pareto almost on hundred years ago to model the distribution of incomes above a certain threshold. A random variable has a Pareto distribution if its density function is of the form:

$$f_X(x; a, b) = ab^a x^{-(1+a)} \quad x > b,\ a > 0,\ b > 0$$

This distribution has a remarkable property known as scaling. If we increase the threshold the shape of the distribution remains the same apart from a scaling factor. For example, by integration, $P[X \geq cb] = c^{-a}$. Then the distribution of X given that $X \geq cb$, where $c > 1$ is given by

$$f_X(x; a, cb) = a(cb)^a x^{-(1+a)} \quad x > cb,\ a > 0,\ b > 0,\ c > 1$$

Thus $P[X \geq c^2 b | X \geq 2b] = c^{-a}$. To illustrate let the distribution of the wealth of persons with wealth greater than say $e1,000,000$ be Pareto with parameter $a = 1.5$ Then the probability that a person in this group will have wealth of twice the threshold is about 0.35. Now let the threshold be $e2,000,000$ then the probability that a person above that threshold will have a wealth of twice that threshold ($e4,000,000$) is again 0.35. This is in complete contrast to the normal or lognormal distribution. Note that the mean of this distribution exists if $a > 1$ and the variance if $a > 2$.

Then if $X_1,\ X_3,\ \ldots\ X_n$ are independent, identically distributed random variables with this asymptotic distribution then the random variable

$$S = \frac{1}{n^{\frac{1}{a}}} \sum_{i=1}^{n} X_i \tag{24}$$

has a limit in distribution which is α-stable with parameters $\alpha = a$ and $\beta = b$

Thus each member of the family of α-stable distributions possesses a domain of attraction. This domain includes all distributions with the Pareto tails described in equation 22.

Some properties of α-stable distributions

Some of the more important properties of α-stable distributions ar given below

- The only α-stable distribution for which moments of all orders exist is the normal distribution. When $1 < \alpha < 2$ the variance is not defined (infinite) and only the mean exists. In our notation the mean is given by $EX = \delta$. Apart from the lack of a simple form for the density function of an α-stable density function the non-existence of a variance is the greatest barrier to their use. Put simply measures of the variance of an α-stable process will increase with sample size and will not converge.

  If $0 < \alpha \leq 1$ the mean does not exist. If $\alpha < 1$ the mean is even more dispersed than the individual measurements. In applications of α-stable distributions to finance values of α are usually of the order of 1.5 to 1.8 are usually appropriate. The values estimated in section 3 vary from 1.63 to 1.73.

- The α-stable density is symmetric with respect to simultaneous changes of the sign of x and β, that is

  $$s(x, \alpha, \beta, \gamma, \delta) = s(-x, -\beta, \gamma, \delta) \tag{25}$$

- If a and $b > 0$ are real constants then the density of $\frac{x-a}{b}$ is given by

  $$\frac{1}{b} s(\frac{x-a}{b}, \alpha, \beta, \frac{\gamma}{b}, \frac{\delta - a}{b}) \tag{26}$$

  or in particular that of $\frac{x-\delta}{\gamma}$ by

  $$\frac{1}{\gamma} s(\frac{x-\delta}{\gamma}, \alpha, \beta, 1, 0) \tag{27}$$

  where $s(x, \alpha, \beta, \gamma, \delta)$ is the density of x. δ and γ are described as location and scale parameters respectively.

- Let X_1 and X_2 be α-stable random variables with densities $s(x, \alpha, \beta_i, \gamma_i, \delta_i)$, $i = 1, 2$. Then $X_1 + X_2$ is α-stable with

  $$\beta = \frac{\beta_1 \gamma_1^\alpha + \beta_2 \gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, \quad \gamma = (\gamma_1^\alpha + \gamma_2^\alpha)^{\frac{1}{\alpha}}, \quad \delta = \delta_1 + \delta_2 \tag{28}$$

In general use, the density functions of an α-stable process may be estimated by an inverse numerical transform of the characteristic function. For some purposes the numerical integration routines in Mathematica (Wolfram (2003)) may be sufficient. To provide greater accuracy in the tails of the distribution some form of series or integral expansion of the characteristic function is often used. Programs to compute α-stable density and distribution functions are available in Mathematica (Rimmer (2005)), R (Wuertz (2005)) or as the stand-alone program STABLE (Nolan (2005)). The calculations in this paper make considerable use of these packages.

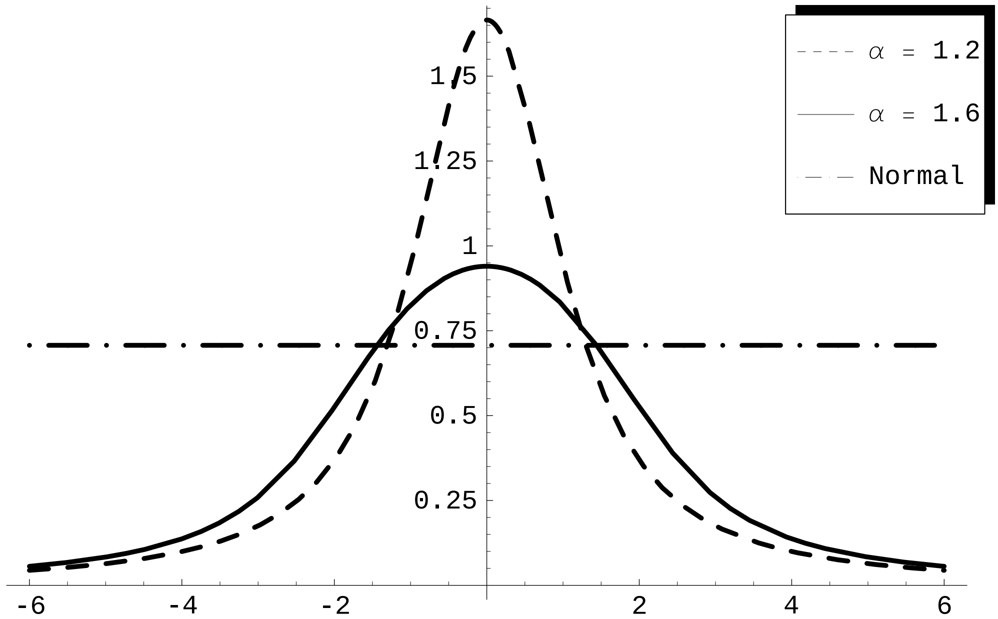

Figure 1: Comparison of implied weights in GLS equivalent of Maximum Likelihood estimates of regression coefficient when residuals are distributed as symmetric α stable variates

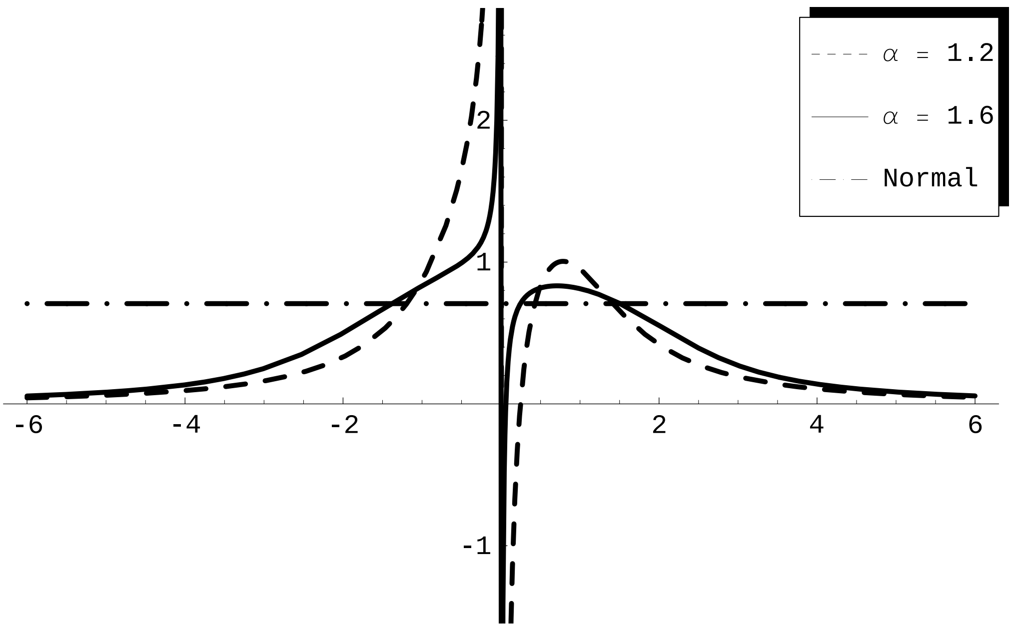

Figure 2: Comparison of implied weights in GLS equivalent of Maximum Likelihood estimates of regression coefficient when residuals are distributed as skewed α stable variables with $\beta = -0.1$

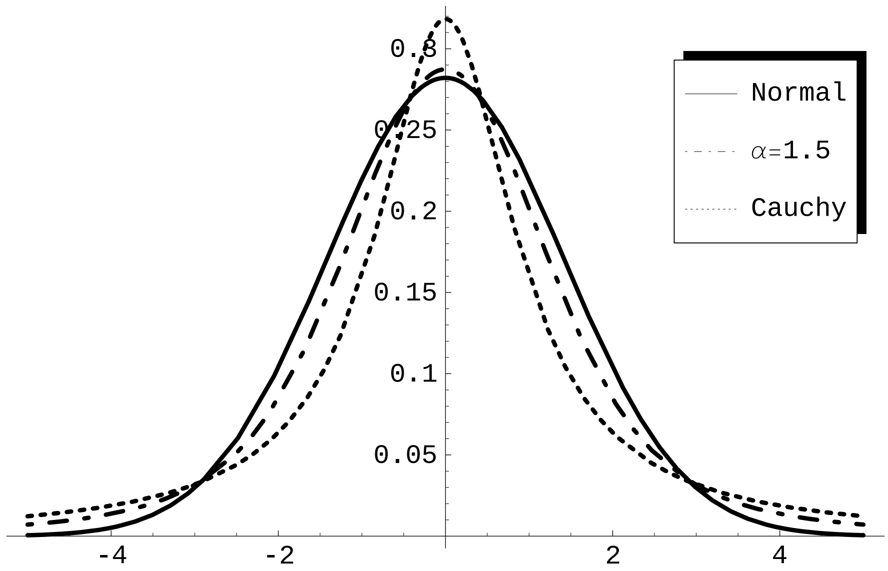

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Figure 3: Normal, α-Stable and Cauchy Distributions

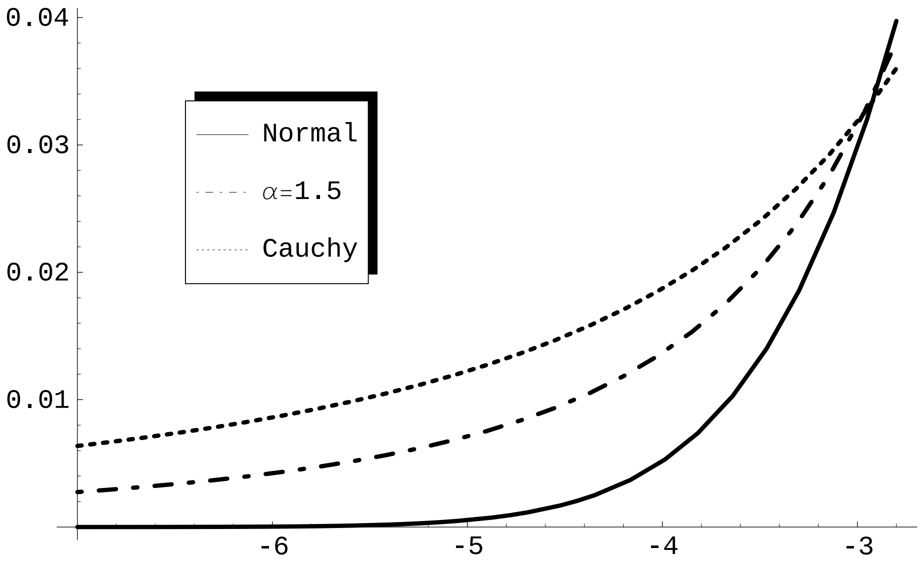

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Figure 4: Tails of Normal, α-Stable and Cauchy Distributions

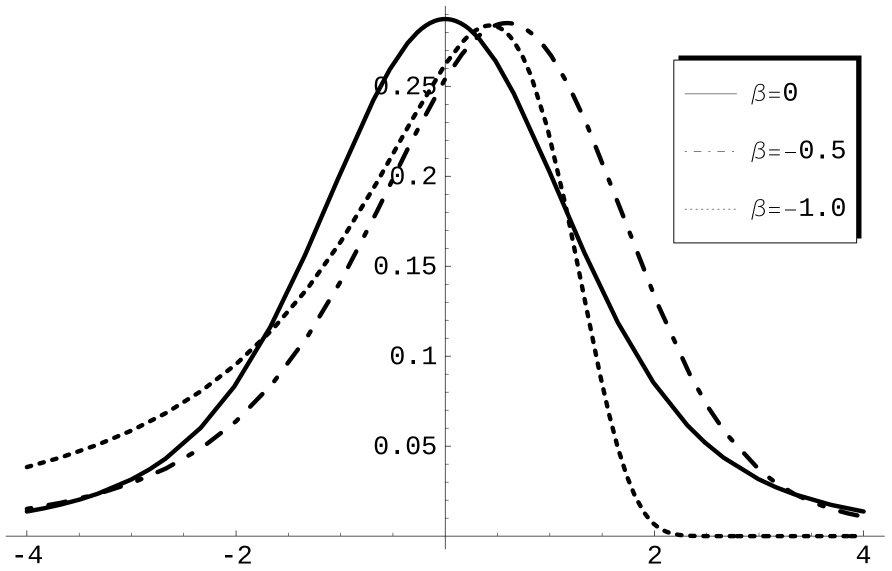

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Figure 5: α-Stable Distribution, $\alpha = 1.5$, β various

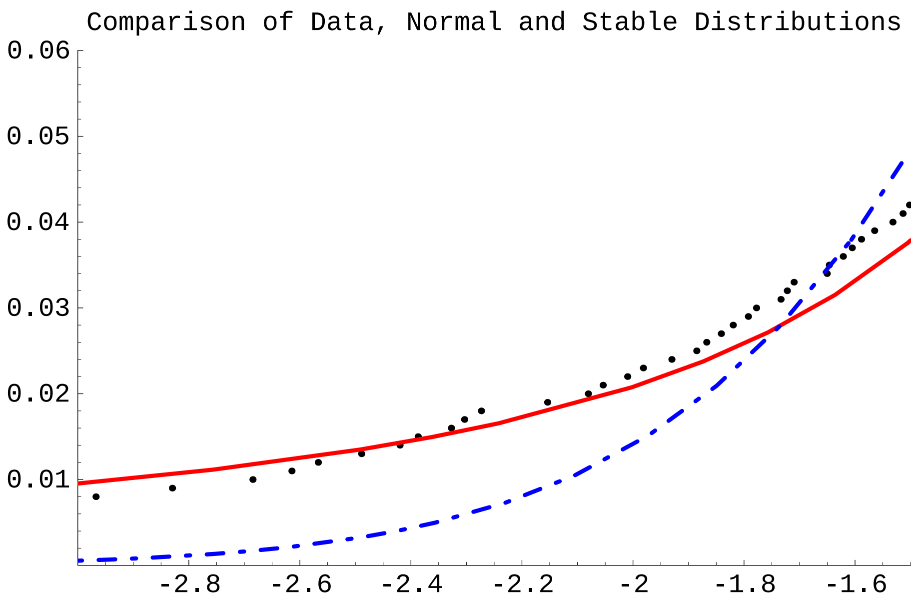

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Figure 6: Comparison of Data, Stable and Normal Distributions