Studies on the Application of the α-stable Distribution in Economics

by

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Declaration

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John C. Frain

Summary

Bubbles, booms and busts in asset prices give rise to a considerable misallocation of resources when they are growing and the subsequent adjustment can be very long and painful. Yet, there is no accepted diagnosis of a bubble. In effect, there is a sense in which a bubble and a bust can not occur in the usual econometric models. These models, almost always, depend on the normal or Gaussian distribution. Yet when one looks at data for asset prices the number and size of extreme losses and gains are orders of magnitude greater than a normal distribution would predict. The very existence of these extreme values must lead one to question the validity of the normality assumption and to look for an alternative.

From time to time several alternatives have been proposed. A common proposal is to use mixtures of normal distributions. The simplest such solution is to have a mixture of two normal distributions — the first, with low volatility, represents the fundamental state with no bubble and the second, with high volatility, the bubble. The price of the asset in question is seen as switching from one state to the other with the switching being determined by some form of deterministic or stochastic process. Other solutions involve what are, in effect, infinite mixtures of normal distributions. Chief amongst these are the various GARCH proceses and the t-distribution. Various other "fat-tailed" distributions have been proposed but these have not received universal acceptance and probably never will. While such distributions often fit the data well, We have not seen any convincing theoretical arguments why they should.

The purpose of this thesis is to examine the use of the α-stable distribution in this context and to determine some of the consequences of its use. The α-stable distribution is a generalisation of the normal distribution. It was first proposed as a distribution for asset returns and commodity prices by Mandelbrot in the early 1960s. It attracted a lot of attention up to the early 1970s and then interest faded. There were two reasons for the waning interest. First the advances made at the time in portfolio and option pricing theory were dependent on the normal distribution. At the time almost all of this work could not have been replicated without the normality assumption. Secondly for actual application the computer power available at the time was simply not sufficient to properly use the α-stable distribution. Thus α-stable analysis was primitive relative to the corresponding normal analysis.

Section 2.1 is a brief history of the application of the α-stable distribution to financial economics. Appendix A contains an account of the theory of such processes. The α-stable distribution allows for the type of extreme and skewed values observed in asset prices. The theoretical arguments that can be used to justify the assump-

tion of a normal distribution can also be used to justify an α-stable distribution. We discuss the relevance of a generalised central limit theorem, domains of attraction and scaling to asset pricing. Statistically, the α-stable distribution is a much better fit to the six total return equity indices that we use to illustrate this study. We then report on three studies that use an assumption of an α-stable distribution.

The first study examines the problem of regression when the disturbances have an α-stable distribution. OLS estimates are not optimum. The maximum likelihood estimator of the regression coefficients is a form of robust estimator that gives less weight to extreme observations. The theory is applied to the estimation of day of week effects in the equity indices. The methodology is feasible and there are sufficient differences in the results to justify the use of the new methodology when sufficient data are available and "fat tails" are suspected. The results support the conclusion that day of week effects no longer exist.

The second study is a simulation exercise to assess the power of normality tests when the alternative is an α-stable distribution. Such tests are sometimes applied to monthly equity returns and when normality can not be rejected it is concluded that the data can not be non-normal α-stable. We show that the power of these test is often so poor that these conclusions can not be sustained.

The third study concerns the use of the α-stable distribution in the measurement of Value at Risk (VaR). We find that a static α-stable distribution gives good measures of VaR at conventional levels for the equity indices examined. The α-stable distribution and a GARCH process with α-stable innovations can give very good measures of VaR.

We may draw two types of conclusion from the studies:

1. The use of the α-stable distribution is feasible in many situations. In the situations examined here it appears to give better results than traditional methods that rely on the normal distribution. It can only be used when there is a large sample of data such as is available in the daily equity return series considered here.

2. From a policy viewpoint there are two consequences of this analysis:

   (a) If economic variables follow an α-stable distribution then we must accept that extremes do occur and must make provision where appropriate.

   (b) It would appear that policy can not reduce the stability parameter. It can change the scale parameter and considerable reductions in the probability of extreme events can be brought about by reductions in the scale parameter. Such policies ought to be designed to be sustainable and effective in the long run.

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At the end of an adventure, and the completion of a thesis such as this is an intellectual journey through some uncharted territory, one must acknowledge the assistance of all who helped in the preparations for the journey and helped chart progress along the way.

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Last but not least I must thank my children John D., Paul, Anne and Diarmaid, my granddaughter Éabha and, in particular, my wife, Helen, for their love, understanding and encouragement. I could not have completed this work without their

help. I must ask their forgiveness for the many times that I was wrestling with some abstruse point in mathematics or computing or economics when I should have been paying attention to other matters.

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CHAPTER 1

Introduction

1.1 Preview

The use of the normal distribution is ubiquitous in statistical analysis in all branches of science. Ever since the days of Bernoulli (1654-1705), De Moivre (1667–1743), Laplace (1749–1827) and Gauss (1777-1855) it has been recognised that, subject to certain fairly unrestrictive conditions, any datum, that is the result of the aggregation of many individual data, has an approximate normal distribution. The return[1] on many assets is the result of agents processing many items of information. It may be argued that the accumulation of such information is the equivalent of many shocks to returns and the result is a normal distribution of returns.

[1]Throughout this thesis the return on an asset is measured as 100 times the log difference of the asset price (including dividends). Thus if P_{t-1} and P_t are the prices of the asset in periods $t-1$ and t, respectively, and D_t the dividend paid in period t the return R_t paid on the asset in period t given by

$$R_t = 100 \log\left(\frac{P_t + D_t}{P_{t-1}}\right) \approx 100\left(\frac{P_t + D_t}{P_{t-1}} - 1\right)$$

Continuously compounded gains (losses), calculated in this way, are numerically less (greater) than standard percentage changes.

When one looks at recent events, in particular, or at the historical performance of equity indices, things are not that simple. On September 15, 2008, Lehman Brothers filed for Chapter 11 bankruptcy protection listing bank debt of \$613 billion, in excess of \$150 billion bond debt, and assets worth \$639.[2] On the same day Merrill Lynch agreed to sell itself to Bank of America for \$50 billion,[3] a third of its 52 week high. The shares of AIG fell from a 52 week high of \$70.13 on 9 October, 2007 to a low of \$1.25 on 16 September, 2008 when the Federal Reserve Board announced a loan of \$85 billion, under terms and conditions[4] that were designed to protect the interests of the U.S. government and taxpayers. The Federal takeover of Fannie Mae and Freddie Mac [5] on 7 September, 2008, could be the most expensive support program undertaken by the federal government. The plan commits the government to provide as much as \$100 billion to each company to backstop any shortfalls in capital. It enables the Treasury to ultimately buy the companies outright at little cost. It also eliminates dividend payments while protecting the principal and interest payments on the debt, now held by foreign central banks, financial institutions, pension funds and others.

These events have been described[6] as once a century events. The use of the term “once a century” probably implies that the user thinks that the events are very rare and that he does not have a good measure of how likely such events are. We would be certain that all of these companies had state of the art risk management systems. It is also likely that the use or interpretation of these systems depended, to some extent, on the normal distribution. With the benefit of hindsight, the problems arising from sub-prime mortgages, the consequent credit shortages and the confidence deficit were the cause of these

[2] http://www.marketwatch.com/news/story/story.aspx?guid={2FE5AC05-597A-4E71-A2D5-9B9FCC MarketWatch, 15 September 2008.

[3] http://www.ft.com/cms/s/2/d285ebc8-82ff-11dd-907e-000077b07658.html, Financial Times, 16 September, 2008.

[4] http://www.federalreserve.gov/newsevents/press/other/20080916a.htm, Federal Reserve press release.

[5] http://topics.nytimes.com/top/news/business/companies/fannie_mae/index.html, New York Times, 16 September, 2008.

[6] Alan Greenspan interviewed on abc NEWS, This Week, 15 September 2008 (http://abcnews.go.com/Video/playerIndex?id=5798760)

problems. It is clear that these events were not foreseen. However a good risk measurement system should be able to give a reasonable estimate of the probability of such unforseen extreme events. The estimates of the probability of such extreme events provided by the normal distribution are wrong by several orders of magnitude. A similar conclusion is reached if we apply the normal distribution to a measure of risk used by LTCM.[7] The resulting probability is so small that the LTCM crash should not have occurred once in the entire life of the universe. The use of the normal distribution in cases such as these is leading to a gross underestimation of the risk of a large loss.

The problems arising from the use of the normal distribution are confirmed when we look at extreme losses on equity indices. A standard measure of process quality control initiated by Motorola is known as six sigma. Basically, the idea is that the standard deviation of the process is controlled so that a defective item occurs when some quality measurement is six sigma (standard deviations) below the average of the measure. In such cases, using a normal distribution suggests that such events have a probability of less than one in a billion of occurring.[8] The six sigma theory allows for a drift in the process and, by convention, calculates the probability as if it were 4.5 standard deviations with a probability of about one in three hundred thousand. If we consider the daily loss on an equity index a six sigma event might occur on average once every 4,000,000 years (or once every 1,200 years if we use the 4.5 rule to determine the probability). These events are much rarer than the "once in a century" events we mentioned earlier. We can apply these concepts to daily returns on the FTSE100 total return price index, which is available since 31 December 1985. The six sigma for this index is 6.2%. On 19 October 1987, 20 October 1987 and 26 October 1987 losses on the index were 11.2%, 12.2% and 6.3%, respectively. Thus there have been three six sigma events since the end of 1985 despite the fact that such events are practically

[7] See Footnote 5 on page 120

[8] Calculations of small normal probabilities such as these are based on the implementation of the normal distribution function in R (R Development Core Team (2008)). This is based on the algorithm given in Wichura (1988). This algorithm gives an estimate of $p = \Phi(z)$, the distribution function of the normal distribution, which is accurate to about 16 figures for $10^{-316} < \min(p, 1-p)$

impossible given a normal distribution. Indeed two of the events are closer to twelve sigma!

The discrepancy remains if we look at smaller but still relatively rare losses. Using a normal distribution we expect a loss of greater than 4 standard deviations to occur once every 126 years. Additional losses on the FTSE100 total return index, greater than 4 standard deviations, were recorded on 11 occasions – 22 October 1987 (5.8%), 30 November 1987 (4.4%), 11 September 2001 (5.9%), 15 July 2002 (5.6%), 19 July 2002 (4.7%), 22 July 2002 (5.1%), 1 August 2002 (4.9%), 30 September 2002 (4.9%), 12 March 2003 (4.6%) and 21 January 2008 (5.6%). We must conclude that we have been very unlucky or that there is a problem with the fit of normal distribution to returns on the FTSE100. We conclude that the problem is the fit of the normal distribution to the data.

This problem is not solely one of recent times. Daily returns on the Dow Jones Industrial Average are available from May 1896. In this period of 112 years we find 29 six sigma events and 103 four sigma events in the daily returns on this index. Six sigma events have occurred in nine of the twelve decades since the index was first calculated. There were four such events in the 1980s and one in each of the 1990s and the first decade of the twenty first century. On Monday 19 October 1987 the index fell by a record 25.6%. Kindleberger (2000) attributes the crash to the excessive growth in prices in the stock market, luxury housing, office building and the dollar exchange rate. Carlson (2007) attributes the deepness of the recession to the impact of margin calls on liquidity, program trading, and uncertainty and herd trading. The fall of 8.3% on the 22 October was a continuation of the same crises. The fall of 7.1% on Friday 8 January 1988 was more than compensated for by the rises earlier that week and the following Monday. The fall of 7.2% on Friday 13 October 1989 was precipitated by a rush of late selling. There was a partial recovery the following Monday when equities were seen as good value.[9] The fall of 7.5% on 27 October 1997 was again recovered over the following week but the index had fallen 6.4% during the month of October. The volatility was attributed to the Asian currency and economic crises. The occurrence of

[9] New York Times BUSINESS DIGEST: 14 October 1989 and following issues.

these six sigma events is evidence of the lack of fit of the normal distribution to the data. There is thus no doubt that the use of the normal distribution leads to very wrong conclusions about the possibility of extreme occurrences in finance. The evidence is so strong that one must conclude that the normal distribution should not be used in evaluating risk. It is not sufficient to say that these events are once off events that could not have been foreseen. The purpose of a risk management system is to get a measure of the possibility of the range of all possible changes including the very unlikely ones that may be a bit more likely than people think.

This failure of the normal distribution has considerable consequences for the conduct of business in the world of finance and in particular for the assessment of risk there. Any methods based on the normal distribution will underestimate risk. Various solutions have been proposed and none appears to have been universally accepted. The solution examined here is the replacement of the normal distribution by the α-stable family of distributions. As we shall show in Chapter 5, this distribution produces good estimates of the probability of extreme events in the equity indices considered. The use of the α-stable distribution demands considerable computational resources but these can be met in the cases considered here. As computer facilities become even more powerful it will be possible to achieve more.

The contents of the remainder of this thesis are as follows. Chapter 2 introduces the α-stable distribution. As a matter of principle we like to use models that can be justified by theory whether that theory is determined by economics, finance or common sense. Various time series models (ARIMA, VAR etc.) can be thought of as reduced forms of structural models. As reduced forms we may be restricted in their use. We base our theoretical arguments for the α-stable distribution on the generalised central limit theorem. The arguments that use the central limit theorem to justify a theory based on the normal distribution can now be used with the generalised central limit theorem to justify an α-stable distribution. The α-stable distribution also has, in common with the normal distribution, attractive scaling properties under time aggregation. The α-stable distribution encompasses the normal distribution and thus one can test the restrictions imposed by the normality assumption. The argument for an α-stable distribution does not rest solely on the statistical fit of the distribution.

An alternative method of modelling “fat tails” uses what is known as extreme value theory. Such procedures use the tails of the empirical distribution to make inferences about extreme values. This provides valuable results in many fields of application including insurance, hydrology, material and life sciences and finance. Here we are more interested in the properties of the entire return series.

We examine the empirical fit of the α-stable distribution to six daily total return indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite (DJAC) and S&P500). We find that the fit is good. We conclude that there are good theoretical and empirical reasons to use α-stable distributions in modelling asset returns.

Our main concern is with the unconditional distribution of returns. Apart from some material on Value at Risk in Chapter 5, we do not examine the conditional distribution of returns. Any statistical analysis of equity returns is a compromise. If we use a long series, we are likely to encounter problems of non-stationarity. If we use a short period, estimates may not be sufficiently precise. In certain circumstances temporal dependencies may reduce the effective size of the sample and bias estimates based on shorter samples. These problems will imply that the fit of the data is not always as good as one might expect. Apart from the DAX30, for which data are available from September 1959, the estimates in Chapter 2 are based on periods from the late 1980s up to September 2005. In Chapter 5 the sample period is extended to January 2008 and includes some of the recent turbulence on the equity markets. The estimated parameters for the extended period are not significantly different from those for the shorter period.

We continue with three studies of the α-stable distribution. These three studies address the implications of the α-stable distribution for three techniques (tests that variables follow a normal distribution, estimating regression coefficients and estimating Value at Risk) that an economist working in a Cental Bank or other financial institution might find useful.

The first study, in Chapter 3 is the estimation of regression coefficients when the disturbances have a non-normal α-stable distribution. In this case Ordinary Least Squares estimates are consistent but are not efficient.[10] The coefficient t-statistics do not have a t-distribution. The method used is an extension of the maximum likelihood method, for symmetric α-stable distributions, given in McCulloch (1998) to general α-stable distributions. The method is a form of robust estimation of the coefficients, where less weight is given to extreme observations. These weights are determined by the α and β parameters of the α-stable distribution. The methodology is then applied to the estimation of day of week effects in returns on the equity indices listed above and on the Dow Jones Industrial Average for the period covered by Gibbons and Hess (1981), in a classic examination of such effects[11]. The results are compared to those obtained using standard OLS and asymptotic normal theory. We find:

1. Standard errors of coefficients are somewhat smaller using the α-stable methodology.

2. We repeat the analysis of Gibbons and Hess (1981) using returns on the Dow Jones Industrial Average rather than the indices that they use. Our results are similar to theirs, rejecting the hypothesis of no day of the week effects. Our OLS estimates agrees with Gibbons and Hess (1981) in finding that returns on Monday are negative and significantly less than average and that returns are higher than average on Wednesday and Friday. The results of our α-stable analysis are similar except that we do not find higher than average returns on Wednesday.

3. For the ISEQ, CAC40, FTSE100 and DJAC there are no significant day of week effects in either the α-stable or OLS normal analyses. The estimates are based on the data covering the period from the late 1980s to September 2005.

4. There are some indications of a higher return on Mondays and a lower return on Wednesdays in the normal analysis of the S&P500. We do not

[10] They are also unbiased when $\alpha > 1$.

[11] The extent to which conclusions such as these may be attributed to data mining is discussed on page 60

find these effects using the α-stable assumption. Data cover the period from January 1980 to September 2005.

5. Data for the total return index for the DAX30 are available from 1959 as compared to the starting dates of late 1980s for the other series. For the entire period both methods indicate significant day of week effects. The normal distribution indicates significantly higher returns on Wednesdays and Fridays and lower on Mondays. The α-stable results only indicate higher returns on Thursdays. These α-stable results may reflect the timing of Bundesbank/European Central Bank announcements.

6. Conventional wisdom would indicate that a weekend effect (high returns on Fridays and low on Mondays) did exist at some stage but that these effect have now been arbitraged away. To look at this effect the DAX30 data were divided into three periods, September 1959 to January 1975, January 1975 to May 1990 and May 1990 to September 2005. Both methodologies indicate weekend effects in the first two periods (slightly stronger in the first) and no effects in the last period, confirming the conventional wisdom that these effects have been arbitrated away.

There is sufficient evidence here to justify the examination of the robustness of Ordinary Least Squares coefficient estimates when fat-tails are suspected and sufficient data are available

Chapter 4 is a simulation study of the power of tests of normality when the alternative is an α-stable distribution. If daily returns have an α-stable distribution then any time aggregation of these returns (e.g. monthly returns) must have an α-stable distribution. As in Chapter 2, tests of normality reject normality for most daily asset returns. However, when these same returns are aggregated to monthly or quarterly frequencies these tests often do not reject normality for the aggregated data. It is then argued that, as daily and monthly data have different distributions, the distribution of returns can not be α-stable. The results of the completed simulations verify that these tests often have very low power in the sizes of samples available for monthly return series. Thus the acceptance of normality by such tests does not provide a strong argument against the α-stable distribution.

Value at Risk (VaR) is an attempt to give a single number that summarises the risk in an investment, a portfolio or even an entire enterprise. It is one of the most common measures of risk used in financial institutions. Often the models used to measure VaR have an explicit or implicit underlying assumption of normality either in the estimation or scaling of the VaR estimate. Given the heavy tails in returns such an assumption is questionable.

Volatility in financial markets is a matter of considerable concern to financial institutions and their supervisors. Already it is clear that this volatility has had an adverse effect on the real economy. Many measures of risk that are used today do not take full account of the kind of extreme changes in asset prices that have been observed. Chapter 5 finds that the Value at Risk measure of risk can be improved by the use of an α-stable distribution in place of more conventional measures. The chapter describes the use of this measure and implements it for six total return equity portfolios. We find that α-stable based measures can be calculated, in the cases examined, and that, as explained there, they are better measures of risk than conventional measures. They are a useful tool for the risk manager and the financial regulator. If the greater probability of extreme losses as calculated from an α-stable distribution had been recognised, the current market volatility, would not have surprised so many people. The recognition of this greater risk might have prevented some of the riskier ventures that have added to the depth of the current crisis.

Appendix A is a summary or the theory of α-stable distributions. It gathers together and gives a uniform presentation of material that was included in the individual working papers on which this thesis is based.

Appendix B contains two of the programs used in this analysis. The first is an edited version of the output of the MATHEMATICA (Wolfram (2003)) program, used in Chapter 3, to estimate the day of week effects for the ISEQ. The second is a reduced version of the C++ program used to estimate the α-stable GARCH processes in Chapter 5. These are included to demonstrate the kind of facilities available for analysis with the α-stable distribution and to show that such analyses are feasible.

1.2 Postscript

Most of this thesis was researched and written before the current (September/October 2008) period of extreme market volatility. Given our time constraints, it is not feasible to extend our analysis to include this period. However, I feel that I should set this analysis in context with the current situation, even though this involves duplicating some material presented elsewhere.

Our initial intention was to research bubbles and busts in asset markets. Our aim was to concentrate on equity indices where good data are readily available. Very soon we realised that the usual kind of econometric models could not account for the many extreme changes in asset prices that have occurred both over the last century and in more recent times. If the usual normal distribution is used it under-estimates the probability of such changes by many orders of magnitude.

We already had some knowledge of the α-stable distribution and decided to look at it as a probability distribution that might provide a better measure of the probability of these extreme events. Both theory and measurement confirmed that it did. The distribution, to the extent described in Chapter 5, overestimates the number of extreme movements. If the recent turbulence is taken into account the number of extreme events in the sample will increase and the fit to the α-stable distribution should be improved.

If returns follow an α-stable distribution our understanding of the current situation may be clarified. The following points are of particular importance:

- Crashes are much more common than predicted by the usual theories. Regulatory bodies should realise that they do occur and they should make appropriate action plans to meet such contingencies. The prompt proposals made by such bodies in the current situation would make me think that such plans were in existence. In an α-stable world such plans are of prime importance.

- Many of the methods used in measuring risk are based on a the assumption of a normal distribution. This distribution underestimates risk. Thus it is likely that risk is being underestimated and underpriced in financial markets. It is likely that Lehman Brothers had a Value at Risk model that showed that the likelihood of disaster in the sub-prime market was very small. We may never know to what extent this model was based on or interpreted using a normal distribution. However we would assume that the normal distribution played a significant part in their decisions. The implication of the α-stable assumption is that risk is usually underestimated and therefore mispriced.

- Regulators and Financial Institutions relying on the Normal distribution to set prudential ratios may have set these at too low a level. Measures of risk set at a time of low volatility may need to be increased during a period of high volatility. In periods of low volatility these limits are often not binding (see Masschelein (2007)). The implication here is that as these may involve the normal distribution they may be set to low. In a period of high volatility they will again be underestimated but they are more likely to be binding as the institution tries to contract to meet the new increased capital requirement. Such a contraction will tend to amplify any credit cycle. A change to a more realistic long-term measure of Value at Risk based to some extent on the α-stable distribution would be considerably larger than the current measure and might be binding in periods of low and high volatility. At least it would not add to the amplitude of the credit cycle. Some of the more risky investments might also have been avoided.

- If returns follow an α-stable distribution then all risk can not be hedged. Risk that can not be hedged must be priced on the market and its price will depend on the risk appetites of those willing to trade the appropriate insurance. Derivative payoffs may be capped or it may even not be possible to obtain insurance is some cases. To the extent that the Merton Black Scholes theory and its extensions are based on a normal distribution it only provides a benchmark for pricing many derivative products. There is a great need for a reconsideration of these theories and their applications.

CHAPTER 2

The α-stable Distribution and Equity Returns[1]

2.1 Introduction

In this section we give a summary outline of the introduction of stochastic processes as models of asset prices paying particular attention to Brownian motion and α-stable processes. The remainder of the Chapter may be summarised as follows.

Section 2.2 gives a brief introduction to the α-stable distribution and should be read in conjunction with Appendix A

Section 2.3 analyses six daily total return indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite and S&P500). Normal and α-stable distributions are fitted to the daily returns on these indices and the fits are compared. In all cases tests of the fit reject the normal distribution. The normal distribu-

[1] This Chapter is based on a paper presented at:

- TCD Graduate Seminar, January 2006.
- IEA Annual Conference April 2006.
- MACSI seminar, Mathematics and Statistics Department, University of Limerick, March, 2007.

tion can be regarded as a restricted version of the α-Stable distribution and the restrictions can be tested. In all cases the data reject these restrictions. Apart from one case, the fits to α-stable distributions are acceptable. The QQ-plots further show the superior fit of the α-stable distribution.

Section 2.4 summarises the Chapter.

Louis Jean-Baptiste Alphonse Bachelier is often regarded as the father of the modern theory of mathematical finance. His Ph. D. thesis (Bachelier (1900a)):[2]

- Described the institutional details of trading on the French Exchange.
- Defines Brownian motion and argues that stock prices follow a Brownian motion He argues that the increments in stock prices are serially independent, follow a normal distribution and have zero expected value. (The continuity requirement for Brownian motion is implicit).
- Assumes the Markov property i.e. the next price depends only on the current price, regardless of history.
- Provides a method of valuing futures and options on that exchange.

The thesis anticipates much of the developments in stochastic calculus that were refined in the twentieth century and which were used in finance, physics and various other fields. Brownian motion is named after the English botanist Robert Brown whose research dates to the 1820s. It was rediscovered, independently, by Einstein (1905) in a paper that contributed to the acceptance of the atomic theory of matter. It was given a rigorous mathematical foundation by Wiener in the 1920s and is now known as Brownian motion or the Wiener Process. In recognition of Bachelier's contribution Feller (1971, p. 99), refers to Brownian motion as *Brownian motion or Wiener-Bachelier Process.*

[2] It perhaps a little inaccurate to refer here to his Ph.D. thesis as the paper in question was only part of the work for a Ph. D. At the time, a Ph. D. in the Faculty of Sciences at the Academy of Paris required two theses. The first was on a topic chosen by the student and a second on a topic chosen by the faculty. Bachelier's own choice was the "Théory de la spéculation" paper. His second paper was on the topic of fluid mechanics (see Courtault et al. (2000)).

Two biographies of Bachelier, Courtault et al. (2000) and Taqu (2001) were prepared to celebrate the hundredth anniversary of the presentation of his thesis. These give a detailed account of its influence in the development of probability and mathematical finance. Evidence of the importance of the thesis is provided by the fact that the original is still in print as Bachelier (1900b), on the internet[3] and in two English translations (in Cootner (1964b) and in Davis and Etheridge (2006)).

Bachelier's analysis of stock prices is based on the normality of the actual stock prices. He assumes that the change in price is independent of the level Bachelier (1900a, p. 35) and that the price follows a Brownian motion. Today we would assume that the logarithm of the price follows a Brownian motion. He recognises the possible problem and argues that the approximation is justified as the distribution of the price of the stock being examined is close to symmetric and that the probability of price being negative is so small that it is effectively zero. As he is dealing with the distribution of future spot prices and the valuation of close to the money options on short dated low volatility high-liquidity government stock, this approximation would have been satisfactory.

Taqu (2001) relates that Paul Samuelson introduced Bachelier to economists in the 1950s. Around 1955 the statistician, Leonard Jimmie Savage[4] discovered Bachelier (1914) in the Chicago or Yale library. He sent postcards to colleagues, asking "Does anyone know him?". Samuelson was one of the recipients. Samuelson had already heard of Bachelier from two sources. The first was between 1937 and 1940 from Stanislaw Ulam. Ulam was a topologist who was involved with Monte Carlo methods and worked on the atomic bomb at Los Alamos. Samuelson also knew of Bachelier from the classic probability text Feller (1968) the first edition of which appeared in 1950. Prompted by Savage's postcard Samuelson looked for and found the thesis at the MIT Library. Soon afterwards Samuelson, in manuscripts and informal talks, suggested using geometric Brownian motion as a model for stocks.

[3] `http://www.numdam.org/item?id=ASEN_1900_3_17_21_0`

[4] L. J. Savage is best known for his contributions to Bayesian statistics. His most noted work is Savage (1954), in which he put forward a theory of subjective and personal probability which also has applications in game theory.

Kendall (1953), in an examination of the statistical properties of UK price statistics, including equity prices, also examines levels rather than logarithms. He finds very small serial correlation in the first differences of the price levels. It is perhaps somewhat surprising that he and the discussants were somewhat surprised at this result. One discussant (K. S. Rao) demonstrated that it is possible to have zero correlations even when a time-series is completely deterministic. The paper or the discussants did not mention that zero correlation and independence are equivalent only when the distributions are normal. Perhaps there was an implied assumption that the distributions were asymptotically normal or could, for practical purposes, be taken as normal. Apart from this article there appears to have been little attention devoted to the distribution of returns until the 1960s (see, for example, the introduction to Cootner (1964b)).

The purpose of Osborne (1959)[5] is to show that the logarithms of common stock prices follow a Brownian motion. It would appear that Osborne was not familiar with Bachelier's work. Alexander (1961) includes Bachelier (1900a) in his references. He re-analyses the data used in Kendall (1953) and verifies and amends the results found there. His analysis uses the logarithms of the variables rather than their levels.

During the 1960s and the early 1970s the normality assumption underlying various asset returns was questioned by, in particular, Mandelbrot (1962, 1963, 1967, 1997), (see also Mandelbrot and Hudson (2004)) and Fama (1964, 1965a, 1976). The mathematicians had already worked on processes that were a generalisation of Brownian motion, which maintained the assumption of stationary independent increments, dropped the normality assumption, and imposed certain continuity restrictions[6] and are now known as Lévy

[5] M. F. M. Osborne was a physicist working with at the Naval Research Centre of the Defence Department, Washington D. C. He worked on problems related to underwater sound, detection of submarines, underwater explosions and later on the aerodynamics of insect flight and the hydrodynamic performance of migrating salmon. His initial interest in the stock market was as a slow motion source of random noise. In the early 1970s he was a visiting lecturer in finance at the University of California in Berkeley. His views on finance and economics are in Osborne (1977) which is based on the lectures he gave in Berkeley. He is sometimes quoted as the father of the econophysics school.

[6] The paths of the process are almost surely right continuous with left limits.

processes. The class of Lévy processes and the class of infinitely divisible processes are the same. An α-stable process is a Lévy process where the increments follow an α-stable distribution rather than the normal distribution followed by the increments of a Brownian motion.

Mandelbrot examined the variation of prices of cotton (1816-1940), wheat (1883-1936), railroad stock (1857-1936) and interest and exchange rates (similar periods) and found a larger number of extreme values than could be justified by the assumption of a normal distribution. Fama examined the distribution of daily returns for the 30 stocks in the Dow Jones Industrial Average in a period from about the end of 1957 to 26 September 1962. These papers offered support for the hypothesis that returns followed an α-stable[7] rather than a normal process. An α-stable distribution may be thought of as a generalisation of the normal distribution where the generalisation allows greater concentration close to the mean, more extreme values and possible skewness. We will see that the normal distribution is an α-stable with restricted parameter values. This pioneering work of Mandelbrot and Fama and others was extended over the next few years in areas such as:

Fama (1971) CAPM and α-stable processes — see appendix Section A.6 of this thesis.

Blattberg and Sargent (1971) Regression with non-gaussian stable disturbances — see McCulloch (1998) and Chapter 3 of this thesis for a more modern treatment based on Maximum likelihood.

DuMouchel (1971, 1973, 1975) Maximum Likelihood Estimation of the parameters of an α-stable processes.

Chambers et al. (1976) Simulation of α-stable random variables.

Kanter (1976), Logan et al. (1973) Properties of α-stable distributions.

[7] There is a certain confusion in the literature about the name to be given to this family of distributions. Mandelbrot used the term L-stable after Lévy. The probability literature uses the term stable which is unfortunate as it implies, to the non-mathematician, properties which are not appropriate. The terms α-stable, stable Paretian, stable Pareto or even Pareto-Levy are also used. Lévy (1954) uses the term “lois quasi-stables”. Here I use the terms α-stable to denote this family of distributions.

After an initial period of interest, research in financial economics regarding α-stable processes waned. There were two likely reasons for the waning interest in α-stable distributions. First the assumption of an underlying normal distribution had contributed, or was about to contribute, to major breakthroughs in empirical and theoretical finance. The success and importance of this work can be gauged by the fact that Nobel prizes have since been awarded to Markovich, Millar, Sharpe, Merton and Scholes for their work on portfolio allocation, Capital Asset Pricing model, Option Pricing and other contributions to the theory of investment. This normal distribution played an important part in these developments. The fear was, quoting Cootner (1964a), page 418.

> *Mandelbrot, like Prime Minister Churchill before him, promises us not utopia but blood, sweat, toil and tears. If he is right, almost all of our statistical tools are obsolete — least squares, spectral analysis, workable maximum-likelihood solutions, all our established sample theory, closed distribution functions. Almost without exception, past econometric work is meaningless....*

For reasons that will become apparent, working with α-stable distributions demands considerable computational resources. Some of us can remember that in the 1970s it could take days to prepare and estimate an ordinary least squares regression on a shared computer system. In many ways these systems were less powerful than many of today's mobile phones or many other electronic gadgets. Even with a technique as elementary as ordinary least squares the modern range of diagnostics were not produced. Even though Mandelbrot worked for IBM, for his early work he had no access to Fortran[8] and his early work was completed in Assembler with the aid of a programmer (Zarnfaller — see Mandelbrot (1997, p. 468)). Even in the late 1960s computer routines for ordinary least squares were not as reliable as might have been expected (see Longley (1967)). Statistical/Econometric programs such as SAS, TSP, TROLL and SPSS were developed in the late 1960s early 1970s and

[8] The high level programming language FORTRAN was invented in IBM about 1957 and is still widely used in scientific computation.

were limited to basic regression and analysis of variance. The user of today's version of any these programs would not recognise the early versions.[9]

The methods used by Mandelbrot, Fama and others in estimating and testing α-stable distributions were ingenious and should be evaluated in the context of the facilities available at the time. The arguments advanced against the suitability of α-stable distributions also need to be re-examined. Given the available technology at the time, both sides of the argument did as much as could have been done at the time. With today's resources much more can be done and we would prefer not to use the empirical work done at that time to argue for or against the validity of the use of the α-stable distribution in finance. It is a pity that a some modern texts (e.g. Taylor (2005)) dismiss the α-stable distribution on the evidence of authors such as Blattberg and Gonedes (1974) [10], Hagerman (1978) and Perry (1983)[11] who did not have the use of modern technology.

The problem with Cootner's view is that he sees models arising from the normal and α-stable distributions as totally contradictory. In the majority of

[9] Renfro (2004) contains a comprehensive account of the development of econometric software.

[10] Blattberg and Gonedes (1974) give empirical arguments for the use of the t-distribution. The methods they use to estimate the parameters of an α-stable distribution need to be updated. While their empirical arguments are strong they do not offer any theoretical reasons why asset returns follow a t-distribution rather than an α-stable.

[11] Perry's argument is that if returns follow an α-stable distribution estimates of the variance should tend to increase with sample size. He finds that there is little evidence to support this fact and uses this finding to argue against returns following an α-stable distribution. If one considers the high peak of an α-stable distribution and the fat tails one would expect recursive estimates to show a jump when a value in the extreme tail is found and to be falling when one encounters a value closer to the centre of the distribution. Simulations confirm this. Figure 2.8 on page 41 shows the result of recursive estimation of the variance of six simulated samples from an α-stable distribution with $\alpha = 1.7$ sample size of 5000 (about 20 years of daily data) The recursive estimates of the variance show jumps coinciding with large returns but otherwise the estimates fall in value. It is difficult to detect an upward trend in the simulated data. Figure2.7 on page 40 shows recursive estimates of variance of the return series under consideration. All six graphs for the data have some similar features and do not give the impression of settling down to a constant value.

Apart from a diminishing trend between extreme tail values, recursive variance estimates do not show any obvious trend Even when the sample size is increased to one million it is difficult sometimes to see a trend for values of $\alpha = 1.7$. If the stability parameter is reduced to 1.2 the increasing trend is more marked but again does not show in all cases.

econometric analyses the sample size is too small to support estimation using an α-stable distribution. With small samples we may observe no extreme values. If an extreme observation does occur we may use a dummy variable to effectively skip it. This will lead to a more robust estimate and one that is likely to be closer to an α-stable based estimate if such were feasible. As the peak of a normal distribution is wider than that of a stable distribution my intuition is that the use of the normal distribution may lead to conservative confidence intervals, at the conventional 10% and 5% confidence levels. Given the dependence of econometrics on asymptotic results that are only approximate in small samples this may not be a disadvantage.

When, as in the analyses here, the data series are long enough they should be analysed using the best tools available. In the analysis here, using the α-stable distribution does give results that correspond more closely with reality. When conventional normal distribution based methods are used the analyst and management should be aware of the possible defects in the model. At least the results should be examined to see how robust they are with respect to the choice of distribution. While α-stable distributions, in general, do provide a much better fit to the returns we examine here, they still give rise to considerable implementation problems both on an empirical and theoretical basis. However, as we show there is much that can be done and as computer power increases and more data become available the α-stable distribution will become easier and cheaper to use and will therefore be used more often. Economists should be aware of these results.

A significant indication of problems with the normal distribution is that extreme events are more frequent than the assumption of a normal distribution would predict. For example[12] there have been 35 falls greater than 6% in the daily Dow Jones Industrial Average since its inception in 1896, about 110 years ago. If the changes in the (logarithm) of the index are normally distributed one would expect that 35 falls of this magnitude would take place about once every 600 million years. The six total return indices considered here are available for much shorter periods but show similar discrepancies in the numbers of large falls in the indices. For example the daily FTSE100 total

[12]For details of calculation of small normal probabilities see Footnote 8 on page 3.

return index which is available from 31 December 1985 shows 7 falls greater than 5% in the period to September 2005. Assuming a normal distribution one would expect 7 such falls to occur every 124,000 years. The daily ISEQ total return index shows 6 such falls in the period from January 1989 to September 2005. The normality assumption would imply an expected period of 12,000 years. The distribution of increases shows similar discrepancies between the empirical distribution and the normal distribution.

These extreme events are the Black Swans of Taleb (2004, 2007). According to Taleb, the *cygnus atratus* is a black swan which is native to Australia. Native swans in Europe are white. Prior to the discovery of the black swan in Australia a European might have assumed that all swans were white and he would have been totally surprised by the finding of a black swan. Taleb attributes the problem to invalid induction. If the distribution or returns is normal the extreme returns on equity returns are black swans. Under an α-stable distribution these black swans become a shade of gray and we should not be taken by surprise if they occur.

With a normal distribution the average loss given that the loss is greater than x% will approach x% as x becomes large. With an α-stable distribution this average will approach $\alpha/x(\alpha-1)$% or about $2.2x$% to $2.7x$% for the range of values of α found in finance. These are basic properties of the normal and α-stable distributions and perhaps one might dwell on them a little longer. Prior to 19 October 1987 the largest[13] loss on the daily close to close Dow Jones Industrial Average was the 14.5% recorded on 28 October 1929. If losses are normally distributed and if that level of losses were to be exceeded then it is probable that they would only be exceeded by an extremely small amount. Thus in a large sample the largest observed loss (14.4%) is close to an effective ceiling on the maximum loss. On 19 October 1987 the loss on the Dow was 25.6%. which is considerably larger than the previous most extreme loss. This is an important example of the problem that over reliance on a false normality assumption can lead to a wrong conclusion.

[13] At the start of World War I the new York Stock Exchange closed on 31 July 1914 and reopened on 14 December 1914. The close on close loss was 23.0% but we do not regard this observation as comparable to the others.

The extreme observations observed are indications that the risk involved in many investments is underestimated by the normality assumption. From a practical viewpoint, this is important to investment companies and to their supervisors. It is of particular importance to those who are measuring risk using a Value at Risk system based on an assumption that returns follow a normal distribution. If the element of risk is underestimated in equity price models, which assume normality, alternative models may provide some explanation of the excess equity premium paradox.

The fat tails of the distribution of returns can be fit by a variety of other distributions in addition to the α-stable. It is often argued that the fat tails can be accommodated by a polynomial decay in the tails of the distribution ie the asymptotic probability density function of the extreme values of the tails is given by

$$f_X(x, \alpha) = cx^{-(1+\alpha)}, \text{ for } x > x_0.$$

When $0 < \alpha \leq 2$ we are in the realm of an α-stable distribution. Extreme value theory often leads to an estimate of α of the order of 4 for the tails of the return distribution. The t and Pareto distributions are examples of such fat tailed distributions. These do not have the scaling properties that we find desirable in return distributions. Also Weron (2001) shows that estimates of α for α-stable distributions with α taking values in the range found here, from extreme value theory, may be biased upward and often give estimates of α greater than 2. Appendix A gives further details.

In this chapter we will concentrate on the application of α-stable distributions. α-stable distributions have been known to mathematicians for a considerable time. According to Gut (2005) the class of α-stable distributions was discovered by Paul Lévy after a lecture in 1919 by Kolmogorov, when someone told him that the normal distribution was the only possible α-stable distribution. He went home and discovered that there was a family of symmetric α-stable distributions the same day. Lévy's early work is summarised in Lévy (1925) and Lévy (1954). The probability books Gnedenko and Kolmogorov (1954) and Feller (1966) were, at the time, the main theoretical resources on the α-stable distribution. Recent mathematical accounts are Zolotarev (1986), Samorodnitsky and Taqqu (1994) and Uchaikin and Zolotarev (1999). Various applications of the α-stable distribution are contained in Adler et al. (1998). Applications to Finance are covered in Mittnik et al. (2000)

2.2 The α-Stable Distribution

This section contains a brief introduction to the α-stable distribution. Appendix A contains a more complete technical description and references. The α-stable distribution is a family of statistical distributions which is indexed by a parameter α which can be any positive number less than or equal to 2. When $\alpha = 2$ the α-stable distribution becomes a normal distribution. When $\alpha = 1$ the distribution becomes a Cauchy distribution. As α is decreased larger extreme values become more likely.

A second parameter, β measures the skewness of the distribution. β can take values from -1 to 1. When $\beta = 0$ the distribution is symmetric. A positive value of β implies that the distribution is skewed to the right (i.e. Large positive values are more likely than large negative values). Larger values of β imply greater positive skewness. Similarly negative values of beta imply that large negative values are more likely than large positive. It is sometimes thought that equity return distributions are negatively skewed. As the normal distribution is symmetric it can not model any skewness in the data.

The α-stable distribution requires two more parameters — a spread parameter, γ, and a location parameter, δ. These are similar in interpretation to the mean, μ and standard deviation, σ, respectively, of the normal distribution.

The α-stable distribution has several features that make it an attractive model of returns:

1. It allows one to take account of the frequency of extreme values outlined in Section 2.1.

2. It allows one to model skewness in the data. Are extreme negative values more likely than extreme positives?

3. The sum of independent observations from an α-stable distribution has, up to a scale and location factor, the same α-stable distribution as the individual observations. The α-stable distribution is the only distribution with this property. If we could account for possible time of day, day of week, other seasonal effects and other non-stationarities that are inherent in the return generating process we might assume that returns aggregated over time have the same distribution, up to a scale and location factor, as the original higher frequency data. Such data then must have an α-stable distribution. The normal distribution is one particular member of the family of α-stable distributions. The general α-stable distribution allows one to retain this property while allowing the data to be modelled in a more flexible manner.

4. The α-stable distribution replaces the normal distribution in what is known as the generalised central limit theorem. A non-normal α-stable distribution may be the limit distribution of sums of random variables that do not satisfy the requirements of the Lindeberg-Lévy-Feller central limit theorem. Thus where an equity or portfolio return is the result of an accumulation of shocks (news) the α-stable distribution may provide a good approximation. This argument is basically the same as that used to justify a normal distribution.

5. In some cases one can model returns as an α-stable distribution with extreme values censored or alleviated by some process. The origin of this suggestion was in physics (Magenta and Stanley (1994)) where there may be physical constraints on a process. There may in certain circumstances be constraints in economic applications. For example the stock exchange may take some action to avoid contagion or there is some other intervention (LTCM) that reduces the measured real effect. Perhaps the measured real effect includes only the private cost of the loss and not the public cost. In such cases the α-stable distribution may be a more accurate picture of returns or losses than the normal as in such cases convergence to the normal may be slow.

Working with the α-stable distribution has several disadvantages:

1. Working with α-stable distributions can be difficult. The normal ($\alpha = 2$), Cauchy ($\alpha = 1$) and Lévy ($\alpha = 1/2$) are the only stable distributions where the probability density function can be written using common mathematical functions. Otherwise the density function must be estimated using numerical methods. Modern computers facilitate this process.

2. Probably the greatest objection to the use of α-stable distributions is that the variance[14] of a non-normal α-stable distribution is infinite. Much of Finance theory taught in courses worldwide is based on what one could call Merton-Black-Scholes normal theory and the assumption of a finite variance (see textbooks such as Elton et al. (2003) or Hull (2006)).

Section A.6 in the Appendix shows how the CAPM analysis can be extended to the case where returns have an α-stable distribution. Almost all the econometrics taught in econometrics programs also makes some assumption about normality or asymptotic normality. We are not suggesting that the entire normality theory be abandoned. The normality assumption is an idealisation and as such facilitates the analysis of what is going on. A point or straight line in geometry is an idealisation and does not exist in reality but is very useful in many applications in science and engineering. Like the line or the point one should not abandon the idea of normality. It is an accurate reflection of reality in some cases. It is important that one should appreciate when a different model is appropriate and know the limitations of normality.

[14] When $\alpha \leq 1$ the mean is also infinite but this case is not important here as in the distribution of asset returns we would expect and find values of α of magnitude of about 1.7

2.3 Comparison of fit of Normal and α-stable Distributions to Returns on Equity Indices

Table 2.1 gives summary statistics for each of the six total return indices. Each return series is given in its domestic currency and no attempt has been made to convert series to a common currency. The most notable features of these statistics are the extreme estimates of the kurtosis of returns. This is an indication of the long tails in the data. The table also estimates three statistics which function as goodness of fit tests[15] to the normal distribution.

Jarque and Bera (JB) test The JB test is a joint test for skewness and excess kurtosis relative to a normal distribution. Given a data sample $\{x_i, \quad i = 1, \ldots, N\}$ with mean $\bar{x}$ the JB test statistic is estimated[16] as follows:

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{N-1}\Sigma_{i=1}^{i=N}(x_i - \bar{x})^2, \\
m_k &= \frac{1}{N}\Sigma_{i=1}^{i=N}(x_i - \bar{x})^k, \quad \text{k=2,3,...} , \\
b_1^{1/2} &= \frac{N^2}{(N-1)(N-2)}\frac{m_3}{\hat{\sigma}^3} \quad \text{(Skewness)}, \\
b_2^{1/2} &= \frac{N^2}{(N-1)(N-2)(N-3)}\frac{(N+1)m_4 - 3(N-1)m_2^2}{\hat{\sigma}^4} \quad \text{(Kurtosis)}, \\
JB &= N\left(\frac{(b_1^{1/2})^2}{24} + \frac{(b_2^{1/2})^2}{6}\right) \quad \text{(Jarque-Bera statistic)}.
\end{aligned}$$

Under the assumption that the x_i are independent identically distributed normal random variables the Jarque-Bera statistic is asymptotically χ^2 with 2 degrees of freedom. In this case the statistics indicate very significant departures from a normal distribution.

Kolmogorov Smirnov (KS) test The KS test in this case compares the cumulative sample distribution $S(x)$ defined as the proportion of the sample that is less than x with that of the hypothesised distribution $F(x)$ and

[15] A more detailed description of tests of normality is given in Section 4.2.

[16] The definition of the Jarque-Bera statistic given here corresponds to that implemented in the RATS econometric package, see Estima (2004, p. 395). The small sample corrections are not identical to those in Wuertz and Katzgraber (2005) as used in Section 4.2.

is given by

$$KS = Max|F(x) - S(x)|.$$

When the mean and variance of the normal distribution are unknown critical values of the KS test for normality are given in Lilliefors (1967). The 1% critical values in this case are 0.014 for the FTSE100, 0.015 for the ISEQ, CAC40 and DJAC, 0.016 for the S&P500 and 0.009 for the DAX30. The values found indicate very significant departures from normality.

Shapiro Wilk (SW) test The SW test is based on the correlation between $x_{(i)}$ and $F^{-1}(i/N + 1)$ which are the i/n quantiles of the sample and population respectively.[17] Again the normal hypothesis is rejected and the conclusions following the JB and KS statistics are confirmed.

Figures 2.1, 2.2 and 2.3 on pages 34 to 36 are normal QQ-plots of the returns data for each index. A QQ-plot is a diagnostic plot designed to show the closeness of two distributions. Both distributions can be empirical when the aim is to look at the similarity of the two empirical distributions. In this case one distribution is the relevant returns distribution and the other is a normal distribution with the same mean and variance. This normal QQ-plot plots the empirical quantiles of the data against the corresponding quantiles of a normal distribution. These plots also contain distribution free 95% confidence intervals for the empirical quantiles (see Hogg et al. (2005)) and the straight line along which the QQ-plot would lie if the match were perfect. Of particular interest are the regions where the line lies outside the 95% bands. (Note that one might expect of the order of 5% of the points on the QQ curve (231 for ISEQ) to lie outside these bands). All six plots show considerable deviations from the normal distribution. They illustrate the considerable weight in the tails of the distribution relative to the normal as would be expected from the kurtosis statistics in Table 2.1. The graphs also show problems near the centre of the distributions. All show excess concentration of returns in the

[17] More details of the SW test are in Subsection 4.2.6.

Table 2.1: Summary Statistics for Equity Total Return Indices and their Fit to a Normal Distribution

	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
start date	04/01/88	31/12/87	28/09/59	31/12/85	30/09/87	03/01/89
end date	21/09/05	26/09/05	26/09/05	26/09/05	26/09/05	26/09/05
observations	4622	4627	12000	5149	4693	4363
mean	0.052	0.044	0.022	.041	0.038	0.043
St. dev	0.934	1.277	1.148	1.028	1.007	0.980
Skewness	-0.3634	-0.124	-0.282	-0.732	-2.686	-0.198
Kurtosis	5.376	3.002	8.378	9.814	58.1964	4.282
JB test[a]	5690	1749	35254	21123	667907	3362
KS test[b]	0.065	0.054	0.062	0.055	0.074	0.063
SW test[c]	0.941	0.967	NA	NA	0.8689	0.956

[a] The asymptotic distribution of the Jarque-Bera statistic is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.
[b] See text
[c] The 5% critical level for the Shapiro Wilk test is .9992 for a sample of 4500. The smaller values reported here indicate very significant departures from normality.

empirical distribution relative to the normal at the centre of the distribution.

Table 2.2 gives maximum likelihood estimates of the α-stable parameters of all six returns indices along with 95% confidence half-width estimates. These estimates have been derived using John Nolan's program (Nolan (2006)). The estimates of the α-stability parameter found have an average value of 1.688 a minimum of 1.646 for the ISEQ and a maximum of 1.726 for the FTSE. On the basis of the estimated half-width confidence intervals all values are significantly different from 2. The CAC, DAX and FTSE indices show significant negative skewness. The ISEQ, Dow Jones Composite and the S&P 500 also show negative skewness but it is not significant. By applying the restrictions $\alpha = 2$ and $\beta = 0$ and re-estimating one can complete a likelihood ratio test of the restrictions. The restrictions are rejected at very low significance levels. A KS test for goodness of fit to the stable distribution is accepted for all except the S&P500.

Table 2.2: Estimates of Parameters of α-stable distributions of Equity Total Return Indices (complete period)

	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
start date	04/01/88	31/12/87	28/09/59	31/12/85	30/09/87	03/01/89
end date	21/09/05	26/09/05	26/09/05	26/09/05	26/09/05	26/09/05
observations	4622	4627	12000	5149	4693	4364
α^{a}	1.646	1.718	1.687	1.726	1.684	1.668
	(0.045)	(0.043)	(0.027)	(0.041)	(0.044)	(0.046)
β	-0.064	-0.147	-0.076	-0.147	-0.076	-0.105
	(0.111)	(0.128)	(0.075)	(0.125)	(0.119)	(0.118)
γ	0.502	0.746	0.627	0.583	0.529	0.550
	(0.014)	(0.020)	(0.011)	(0.015)	(0.015)	(0.017)
δ	0.054	0.032	0.019	0.036	0.042	0.034
	(0.026)	(0.038)	(0.020)	(0.028)	(0.027)	(0.029)
KS (stable)	0.012	0.014	0.010	0.008	0.018	0.023
p-value[b]	0.518	0.307	0.166	0.892	0.097	0.025
LR[c] test of Normality	838.1	418.6	1945.8	786.7	1236.5	583.0

[a] Figures in brackets below the estimated parameters are the 95% confidence interval half width-estimates

[b] The p-values for the KS statistic are calculated on the assumption that the values of α, β, γ and δ are chosen independently of the sample. As the α-stable parameters are estimates the calculated p-values overestimate true significance level.

[c] Likelihood ratio test of the joint restriction $\alpha = 2$ and $\beta = 0$. The test statistic is asymptotically $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels, respectively.

Figures 2.4, 2.5 and 2.6 on pages 37 to 39 are stable QQ-plots of the return data for each of the six returns indices. The construction of these curves is similar to that of the normal QQ-plots except that the normal distribution is replaced by the α-stable distribution with parameters taken from Table 2.2. The fit for the European indices is good. At the extremes there is a suggestion that the tails of the empirical distribution are a little lighter than the theoretical stable distribution but this is not significant. What is surprising is that the fit in the centre of the distribution is so superior to that of the normal distribution. The fit of the American indices to the stable distribution is again far superior to that of the normal distribution. There are some deviations in the centre of the American distribution which are not obvious in the diagrams. These are very small relative to the deviations from the normal distribution but might be the subject of further work.

If $x_1, x_2, \ldots, x_n$ is a random sample from a non-normal α-stable distribution the estimated sample variance is given by

$$\hat{\sigma_n}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i^2 - \bar{x})^2$$

As the expected variance of a non-normal α-stable process is infinite, recursive estimates of the sample variance $\hat{\sigma_n}^2$ should tend to increase with n. Such an increase is not always observed in equity returns data. Some (e.g. Cochrane (2005)) have argued that an implication of non-increasing recursive variance estimates is that returns do not follow a non-normal α-stable distribution. To examine the validity of this conclusion we completed an examination of recursive estimates of the variance of simulated α-stable processes with similar parameters and sample size to those of the equity returns considered here. Figure 2.8 shows on page 41 shows the results of six such simulations. The existence of an upward trend is not obvious in thee simulations.

For comparison, Figure 2.7 on page 40 shows recursive values of $\hat{\sigma_n}^2$ for each of the six indices considered here. Again upward trends are not obvious in the data. Looking at the samples the estimated recursive variance does show large jumps from time to time but these are often followed by a long period of falling estimates. Looking at both sets of data informally it is difficult to see universal increasing trends in either sets of estimates. Thus, for the sample sizes available, this inference is not valid.

Similar simulations with one million replications do tend to show a greater tendency for variance to increase. Simulations with $\alpha = 1.2$ also tend to show increasing recursive variance estimates. The increases in the recursive estimates of the variance are caused by the large observations that occur in the tail of the distribution. The higher peaks in the centre of the distribution (the majority of observations) tend to cause the recursive variance to fall between extreme peaks. There is some indication of a deficit of extreme observations in the recursive estimates derived from the equity index returns. This point is discussed in Chapter 5

2.4 Summary and Conclusions

In this chapter we have examined the application of the family of α-stable distributions to daily returns on six total return equity indices (ISEQ, FTSE100, DAX40, CAC30, S&P500 and the Dow Jones Composite Total Return (DJAC)).

An α-stable distribution is a generalisation of the normal distribution. It allows data to be more concentrated at the mean (high peaks) and to have a greater number of extreme values (heavy tails) than a normal distribution would predict. While the normal distribution is symmetric an α stable may be skewed. The normal distribution is determined by two parameters, a location parameter μ and a spread parameter σ. An α-stable process depends on four parameters. The α parameter determines both the height of the peak of the distribution and the weight of the tails. A β parameter determines the skewness of the data ($\beta = 0$ implies no skewness). As for the normal distribution the remaining two parameters determine the location and spread of the data. When $\alpha = 2$ and $\beta = 0$ the stable distribution becomes a normal distribution. Thus the normal distribution is a member of the family of α-stable distributions.

α-stable distributions have been well known to mathematicians since at least the 1930s. Some of their properties make them attractive for modelling returns but implementing such procedures requires a lot of computer power. Section 2.2 and Appendix A of the paper outline the basic probability theory underlying these processes and explain their relevance to finance.

Early support for the use of α-stable processes in economics and finance came from the writings of Mandelbrot and Fama during the 1960s and 1970s. They found that many asset returns had features typical of these processes. After an initial period of research, interest in stable processes appeared to decrease. While computation problems were probably the main cause of this decline, a further contributing factor was the major breakthroughs in finance, achieved at that time. These were largely based on the assumption of an underlying normal distribution. The success of this work using the normal distribution can be gauged by the fact that Nobel prizes have since been awarded to Markovich, Millar, Sharpe, Merton and Scholes for their work on portfolio allocation, Capital Asset Pricing model, Option Pricing and other contributions to the theory of investment. These developments have formed the major constituents of the research agenda in quantitative finance ever since. An erroneous opinion circulating at that time was that the acceptance of α-stable distributions would invalidate not only this work but most econometric work completed up to that time. It is only in more recent years that advances in computing facilities have facilitated increased interest in α-stable processes and this trend is likely to continue.

For all six total return indices examined here,

- The normal distribution is a very poor fit to the empirical distribution of the returns on all six indices.

- The α-stable distribution provides a good fit.

The α-stable family of distributions is a valuable resource in Finance particularly when extreme events are being considered. It also is a very good fit to the centre of a distribution. We are not suggesting that the normal distribution should be abandoned. The normal distribution is a mathematical idealisation of certain facets of the functioning of financial markets. It has proven useful and will continue to be used. Users should be aware of its limitations. Where circumstances and data are available the α-stable distribution would be an additional tool that should be used. It is also an idealisation. It encompasses the normal distribution and as such should provide a more realistic picture. Today's computer power and software imply that the distribution can be used.

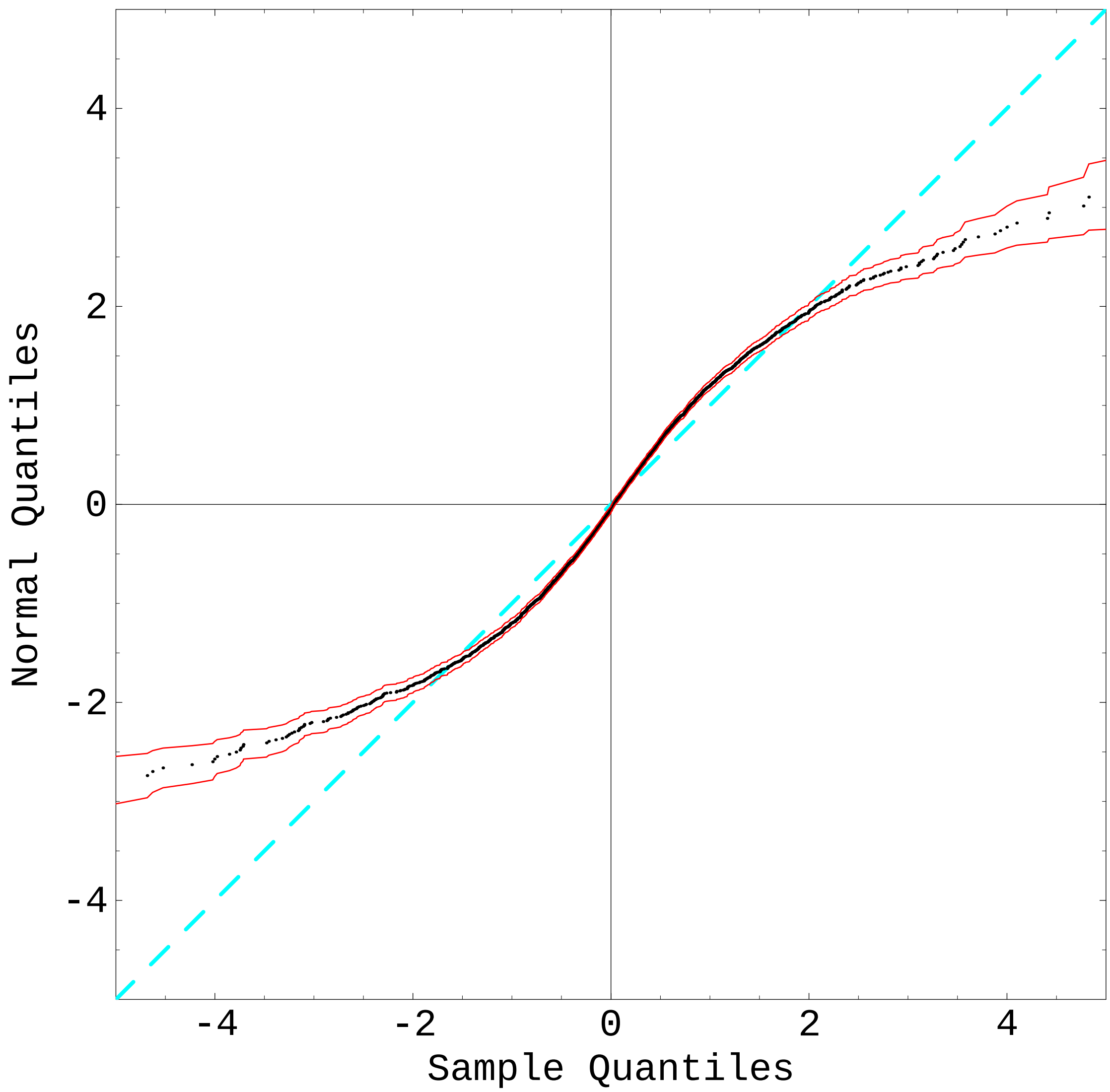


Figure 2.1: Normal QQ Plot (ISEQ returns) with 95% Limits

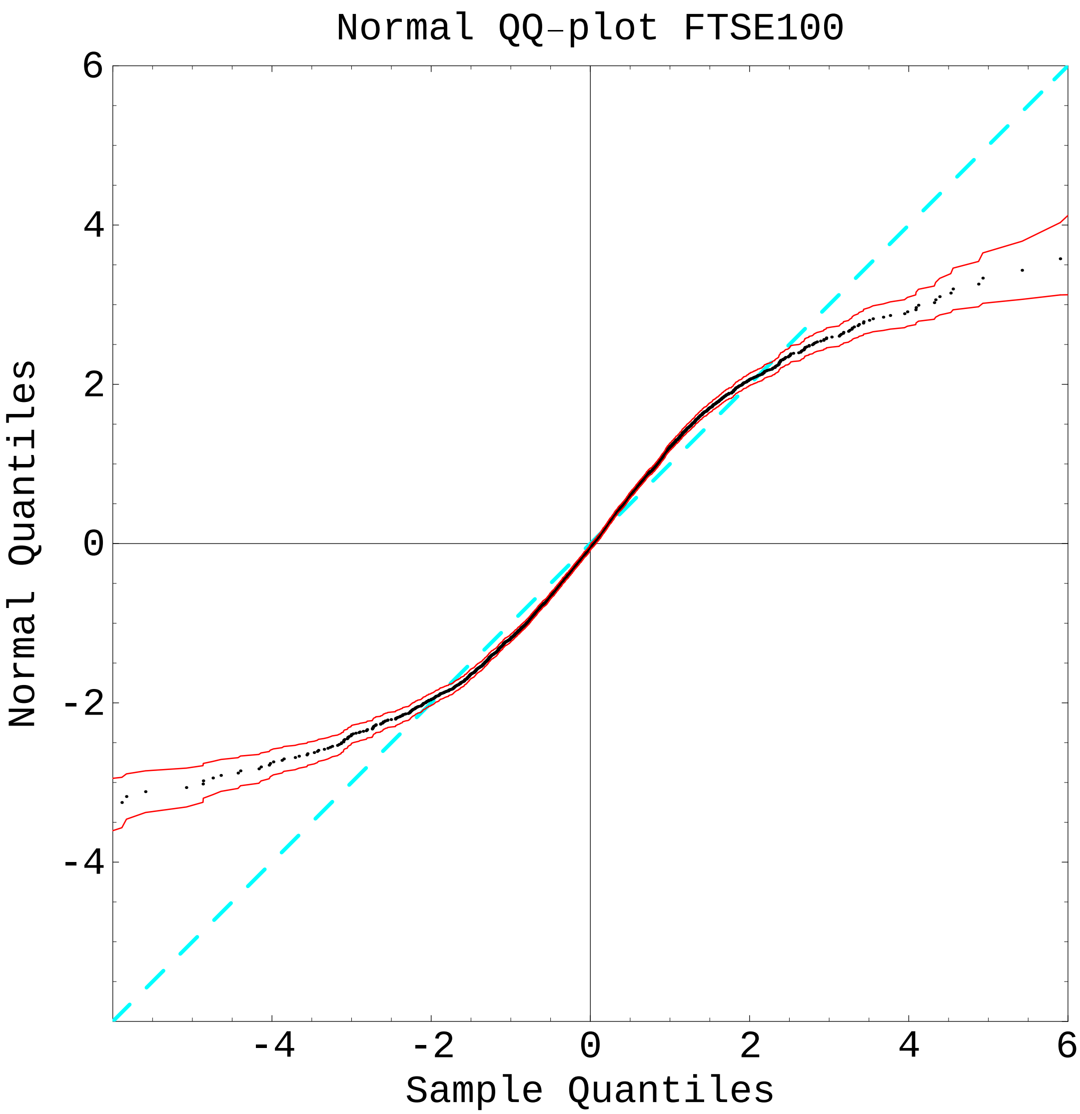


Figure 2.2: Normal QQ Plot (FTSE100 returns) with 95% Limits

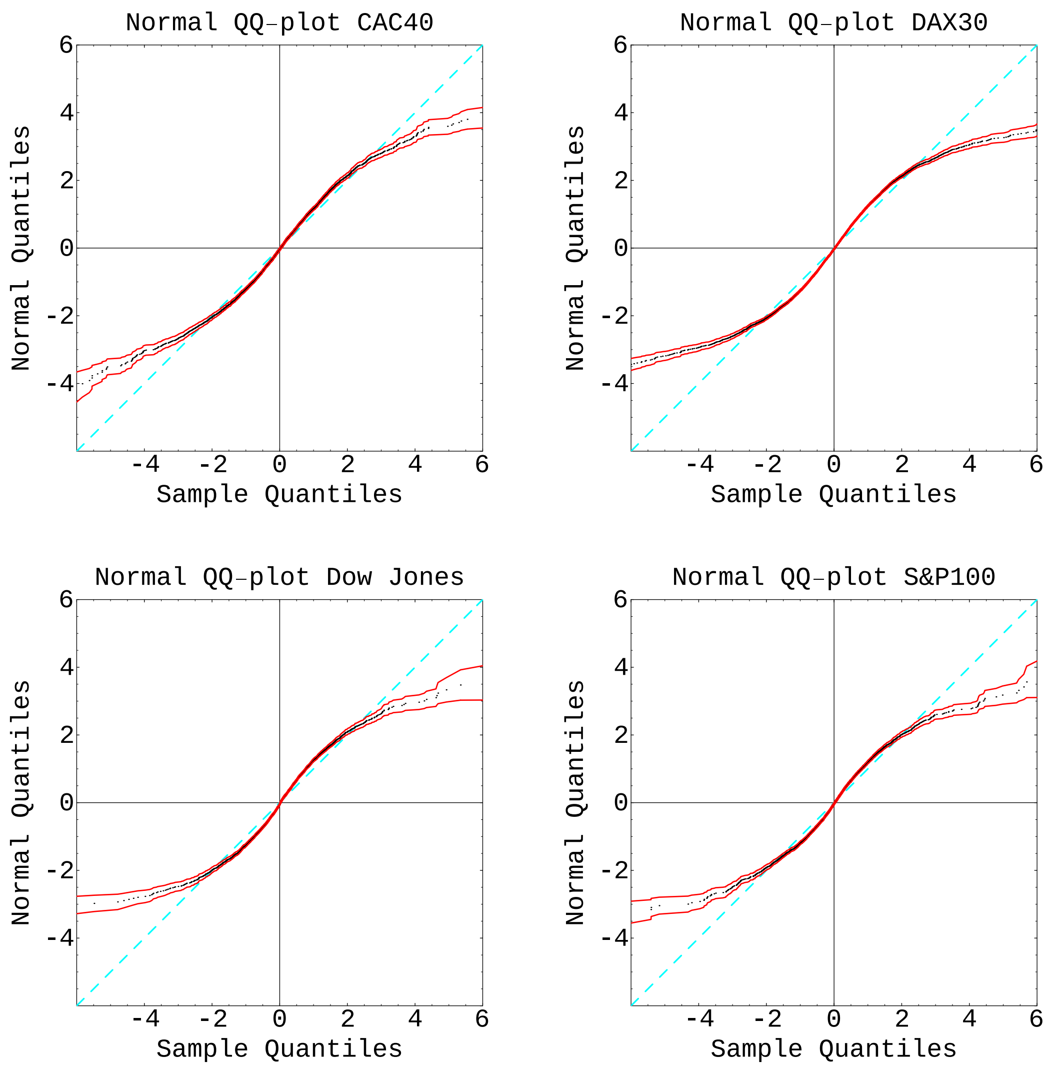


Figure 2.3: Normal QQ Plot (CAC40, DAX30, Dow Composite and S&P100 returns) with 95% Limits

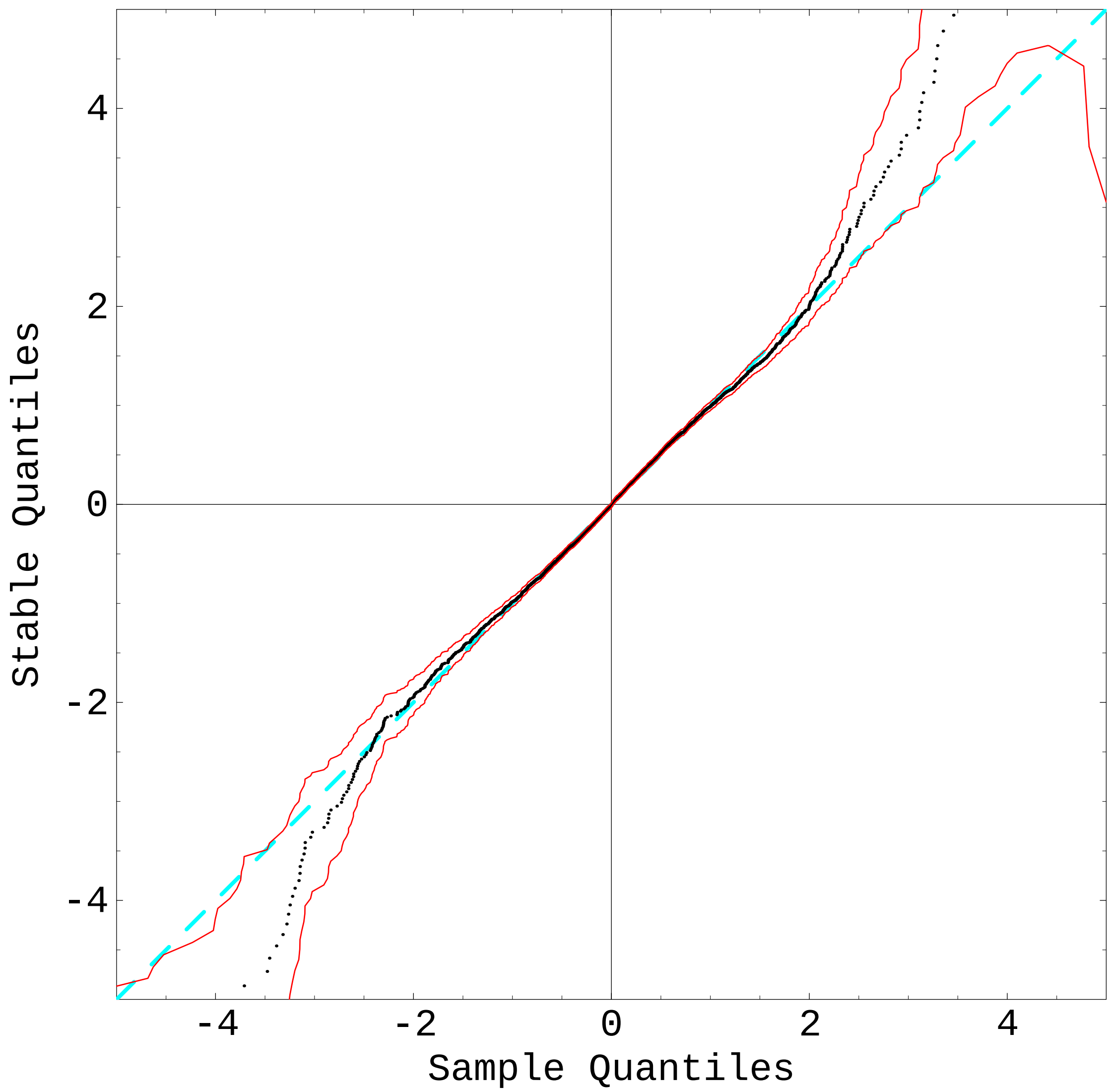


Figure 2.4: Stable QQ Plot (ISEQ returns) with 95% Limits

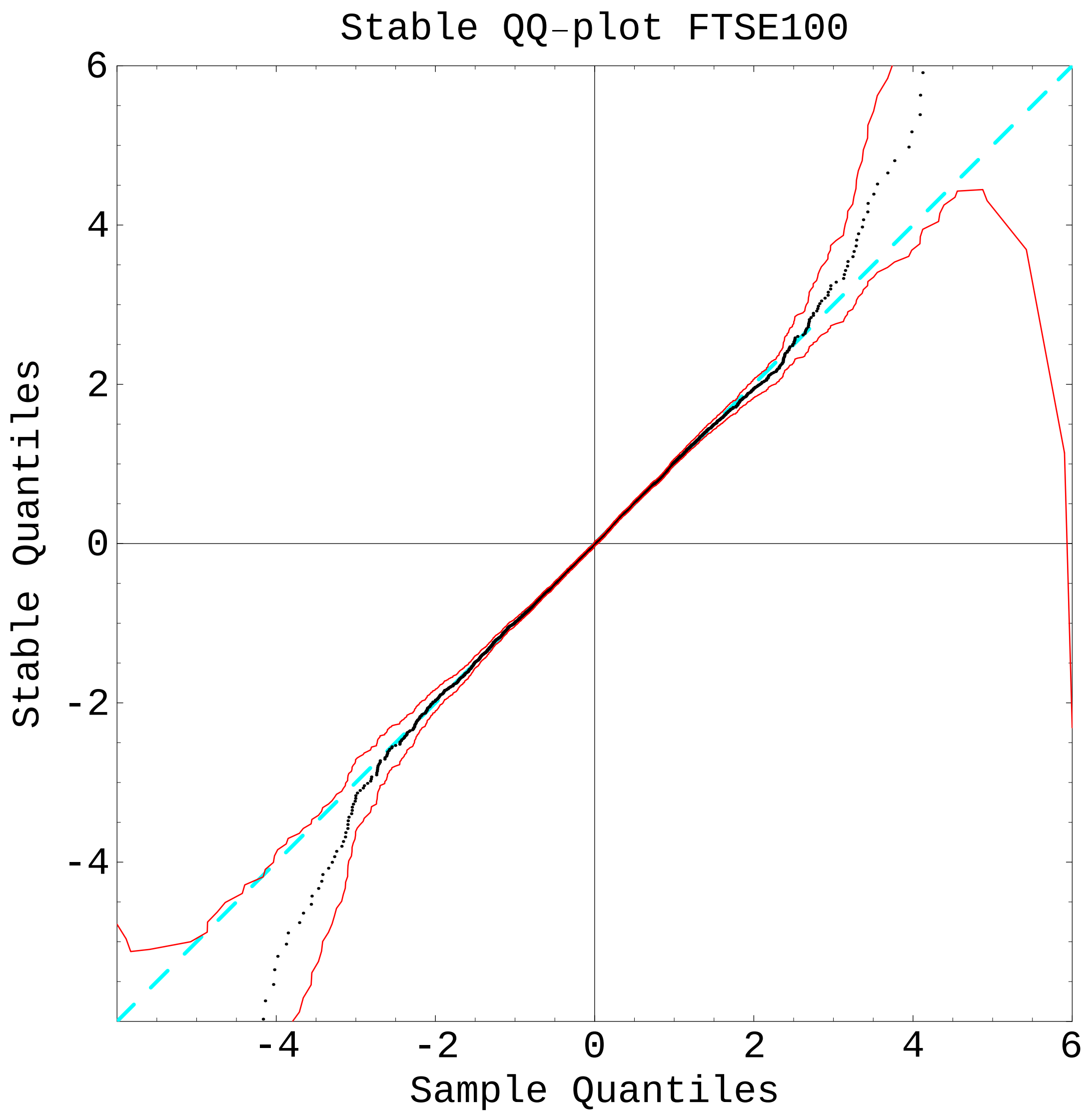


Figure 2.5: Stable QQ Plot (FTSE100 returns) with 95% Limits

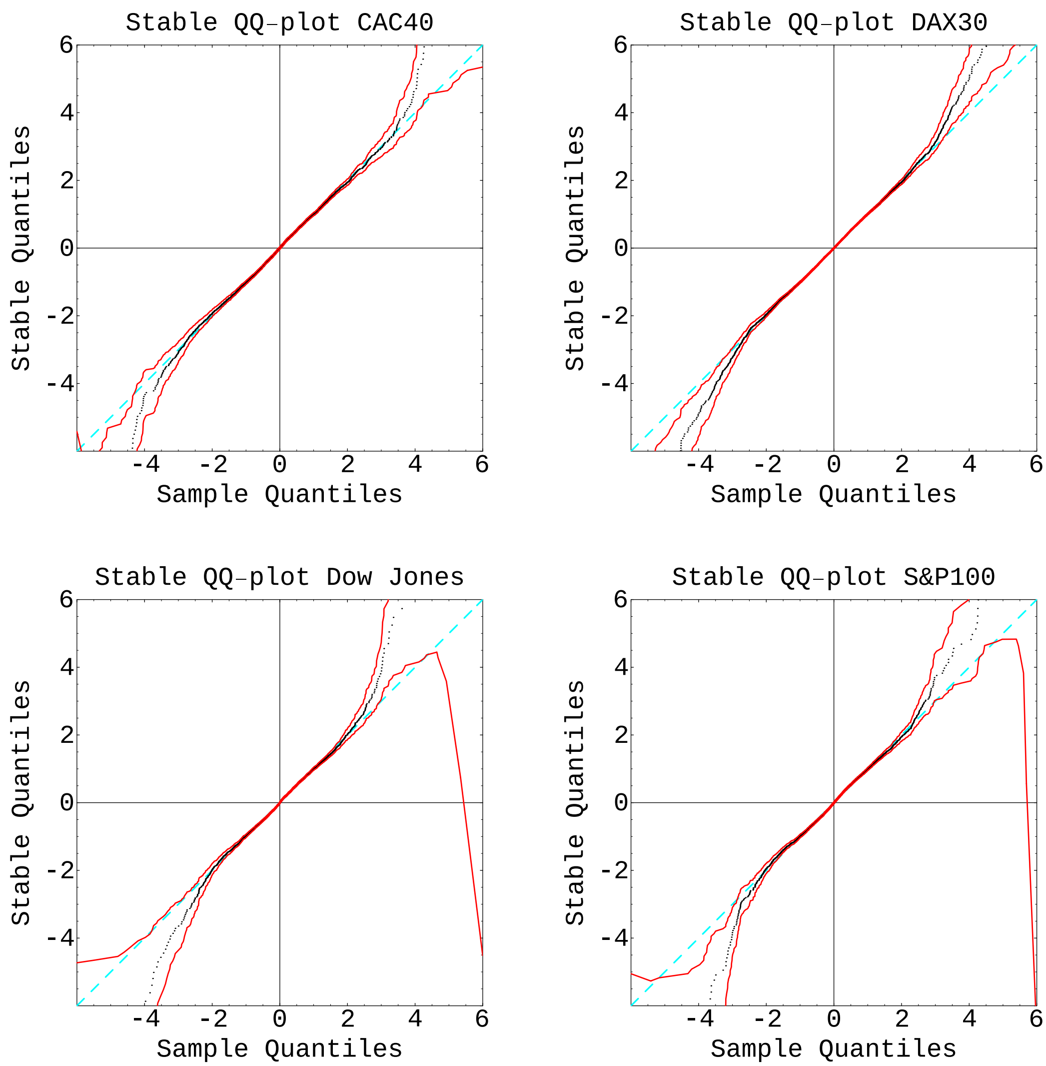


Figure 2.6: Stable QQ Plot (CAC40, DAX30, Dow Composite and S&P100 returns) with 95% Limits

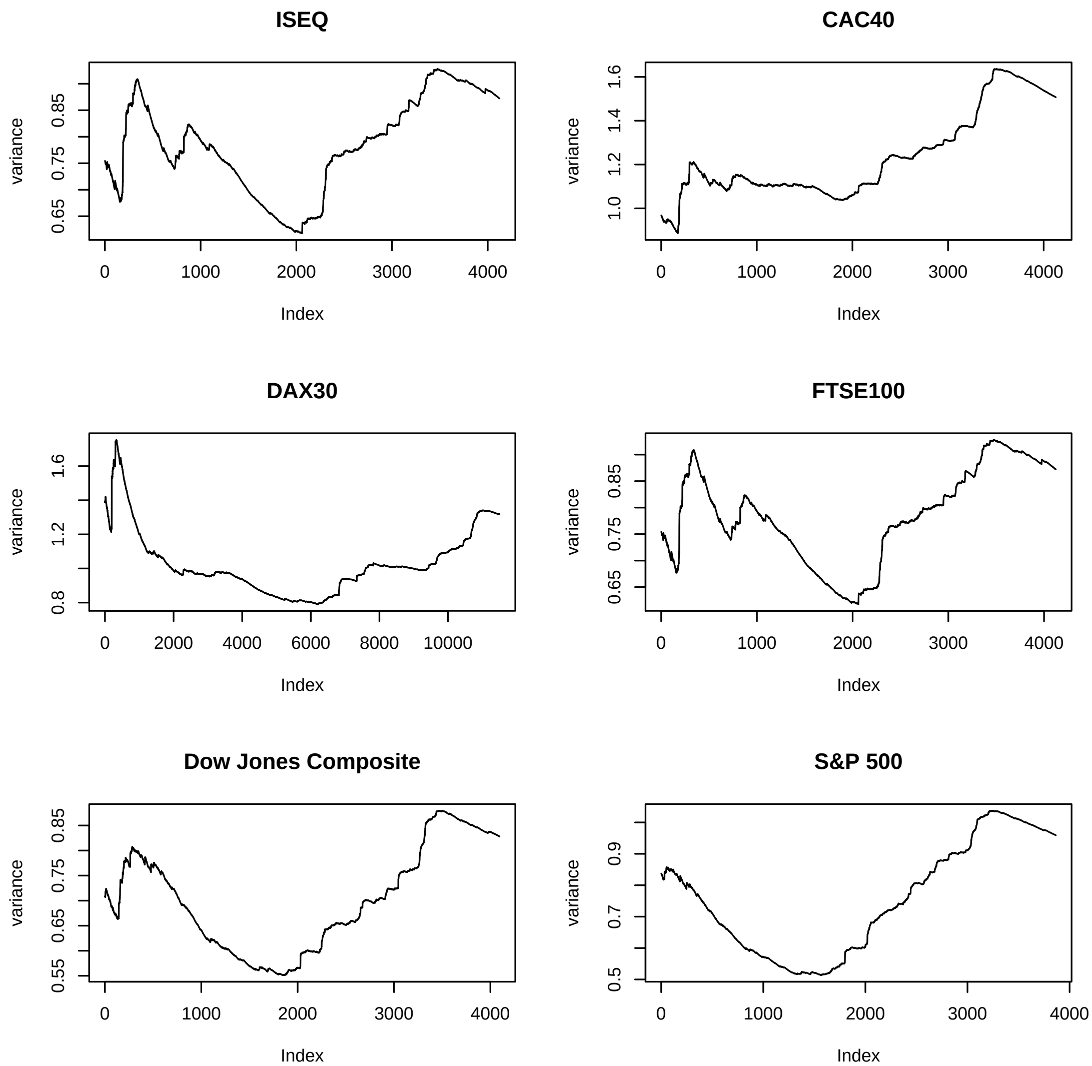


Figure 2.7: Recursive Estimates of the Variance of Returns on Total Return Equity Indices

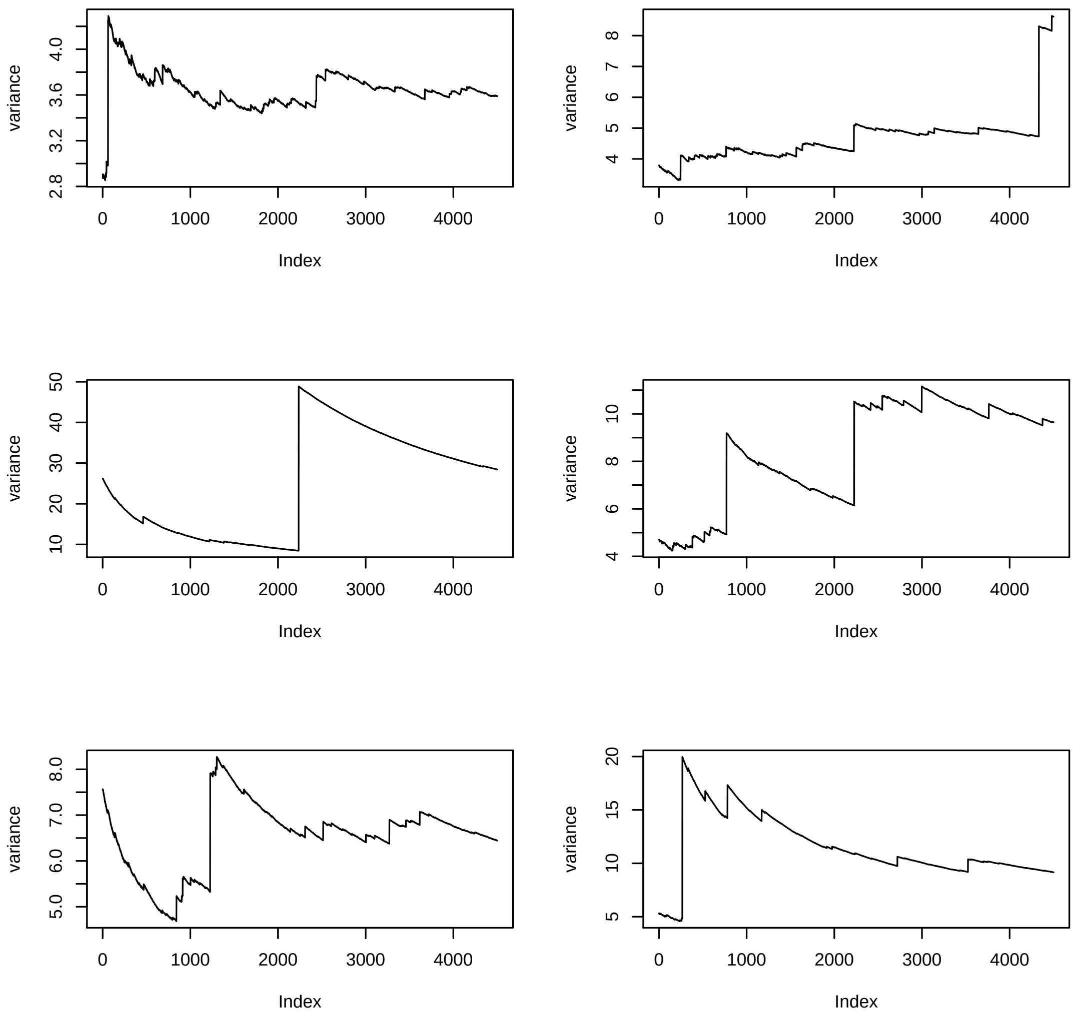


Figure 2.8: Six Simulations of the Recursive Estimation of the Variance of an α-stable Process with $\alpha = 1.7$

CHAPTER 3

Maximum Likelihood Estimates of Regression Coefficients with α-stable Disturbances and Day of Week effects in Total Returns on Equity Indices[1]

3.1 Introduction

Returns on many assets are known to have fat tails and are often skewed. The almost universally used normal or Gaussian distribution can model neither fat tails nor skewness. The α-stable distribution can model these features. The use of this distribution in finance was originally proposed by Mandelbrot (see Mandelbrot (1962, 1963, 1967) or Mandelbrot and Hudson (2004)) to model various goods and asset prices. It became popular in the sixties and seventies but interest waned thereafter. This decline in interest was due not only to its mathematical complexity and the considerable computation resources required but to the considerable success of the Merton Black Scholes Gaussian approach to finance theory which was developed at the same time.

[1]This Chapter is based on a paper (Frain (2008a)) presented at the INFINITY Conference in June 2006.

Recently there has been some renewed interest in the distribution. It has found applications in radiophysics, astrophysics, cosmology, biology, genetics, physiology, ecology and geology (see Uchaikin and Zolotarev (1999)). Recent mathematical accounts are given in Zolotarev (1986), Samorodnitsky and Taqqu (1994), Weron (1998) and Uchaikin and Zolotarev (1999). Rachev and Mittnik (2000) surveys the use of α-stable models in finance.

The availability of cheap powerful computer hardware has made advanced computation resources available to scientists in many fields. The resulting increased demand for good software has provided the incentive to produce and distribute widely software packages such as MATHEMATICA (Wolfram (2003)) and R (R Development Core Team (2008)) which have facilitated the calculations in this Chapter. Programs to compute α-stable distribution and density functions are available in both of these packages — MATHEMATICA (Rimmer (2005)), Rmetrics for R (Wuertz (2007)) — or as the stand-alone program STABLE (Nolan (2006)). These resources allow one to examine the consequences of replacing the normal assumption with the more general α-stable. Further advances in theory and computation facilities will facilitate this process in the coming years and the use of the α-stable distribution will become more common.

In particular, this chapter examines the consequences of α-stable disturbances in OLS estimation. In Section 3.2 the following results are set out:

- Standard OLS estimates are consistent if $\alpha > 1$ but inefficient.
- Coefficient estimates have an α-stable distribution and standard t-statistics do not have the usual distribution.
- Maximum likelihood estimators have the usual asymptotic properties. Confidence intervals and inference may be based on the usual maximum likelihood theory.
- The maximum likelihood estimator with α-stable disturbances is a form of robust estimator which gives less weight to extreme observations.

In Section 3.3 this theory is applied to estimating and testing calendar effects in daily returns on equity indices. These day of week effects are often estimated by the coefficients in an OLS regression of daily returns on five dummy variables — one for each day of the week. We compare the results of estimating such regressions using standard OLS and α-stable maximum likelihood. Estimates are made for six total return indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite (DJAC) and S&P500) and the DJIA[2] for the period used in the often quoted study of these effects in Gibbons and Hess (1981). My results can be summarised as follows -

- The α-stable maximum likelihood and OLS estimates for the DJIA for the Gibbons and Hess (1981) period are almost identical to theirs.
- Data for the total return indices are only available from the late 1980s (apart from the DAX30) and there are no significant day of week effects in the total return indices in that period
- When the data for the CAC40 are split into three equal periods there are indications of day of week effects in the two early periods but they are absent in the last period.

These results are a demonstration of the shifting Monday effect reported in the literature (see Pettengill (2003) and the references there and Hansen et al. (2005)). Such results are, therefore, not sensitive to the use of the “robust” α-stable Maximum Likelihood Estimator.

An examination of the significance of the results for individual coefficients shows that some α-stable coefficients are significant where the corresponding OLS estimates are not. Sullivan et al. (2001) sets out the danger of data mining in cases such as this. I would not draw any conclusions about day of week effects from these discrepancies. They do, however, draw attention to the possible different results that may arise from α-stable maximum likelihood estimation.

[2] Dow Jones Industrial Average

3.2 Regression with Non-normal α-stable Errors

Consider the standard regression model

$$y_i = \sum_{j=1}^{k} x_{ij}\beta_j + \varepsilon_i, \quad i = 1, \dots, N, \tag{3.1}$$

where y_i is an observed dependent variable, the x_{ij} are observed independent variables, β_j are unknown coefficients to be estimated and ε_i are identically and independently distributed. Equation (3.1) may be written in matrix form as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{3.2}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \dots & x_{Nk} \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}, \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix}. \tag{3.3}$$

The standard OLS estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}}_{OLS} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}. \tag{3.4}$$

Thus

$$\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\varepsilon}. \tag{3.5}$$

Thus in the simplest case, where $\boldsymbol{X}$ is predetermined, $\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta}$ is a linear sum of the elements of $\boldsymbol{\varepsilon}$. If the elements of $\boldsymbol{\varepsilon}$ are independent identically distributed non-normal α-stable variables, then $\hat{\boldsymbol{\beta}}_{OLS}$ has a non-normal α-stable distribution. The variance of ε_i does not even exist. Thus standard OLS inferences are not valid. Logan et al. (1973) prove the following properties of the asymptotic t-statistic

1. The tails of the distribution function are normal-like as $t \to \pm\infty$.
2. The density has infinite singularities $|1 \mp x|^{-\alpha}$ at ± 1 for $0 < \alpha < 1$ and $\beta \neq \pm 1$. When $1 < \alpha < 2$ the distribution has peaks at ± 1.
3. As $\alpha \to 2$ the density tends to normal and the peaks vanish.

When $1 < \alpha < 2$ the OLS estimates are consistent but converge at a rate of $n^{\frac{1}{\alpha}-1}$ rather than $n^{-\frac{1}{2}}$ in the normal case.

DuMouchel (1971, 1973, 1975) shows that, subject to certain conditions ($\exists \varepsilon > 0$, $\alpha > \varepsilon$, $|\beta| < 1$) the maximum likelihood estimates of the parameters of an α-stable distribution have the usual asymptotic properties of a maximum likelihood estimator. They are asymptotically normal, asymptotically unbiased and have an asymptotic covariance matrix $n^{-1}\mathcal{I}(\alpha, \beta, \gamma, \delta)^{-1}$ where $\mathcal{I}(\alpha, \beta, \gamma, \delta)$ is Fisher's Information. McCulloch (1998) examines linear regression in the context of α-stable distributions paying particular attention to the symmetric case. Here the symmetry constraint is not imposed. Assume that $\varepsilon_i = y_i - \sum_{j=1}^{k} x_{ij}\beta_j$ is α-stable with parameters $\{\alpha, \beta, \gamma, 0\}$. If we denote the α-stable density function by $s(x, \alpha, \beta, \gamma, \delta)$ then we may write the density function of ε_i as

$$s(\varepsilon_i, \alpha, \beta, \gamma, \delta) = \frac{1}{\gamma}\, s\left(\frac{y_i - \sum_{j=1}^{k} x_{ij}\beta_j}{\gamma}, \beta, 1, 0\right), \tag{3.6}$$

the likelihood as

$$L(\boldsymbol{\varepsilon}, \alpha, \beta, \gamma, \beta_1, \beta_2, \dots) = \left(\frac{1}{\gamma}\right)^n \prod_{i=1}^{n} s\left(\frac{y_i - \sum_{j=1}^{k} x_{ij}\beta_j}{\gamma}, \beta, 1, 0\right), \tag{3.7}$$

and the loglikelihood as

$$\begin{aligned} l(\boldsymbol{\varepsilon}, \alpha, \beta, \gamma, \beta_1, \beta_2, \dots) &= \sum_{i=1}^{n} \left(-\log(\gamma) + \log\left(s\left(\frac{y_i - \sum_{j=1}^{k} x_{ij}\beta_j}{\gamma}, \beta, 1, 0\right)\right)\right) \\ &= \sum_{i=1}^{n} \phi(\varepsilon_i). \end{aligned} \tag{3.8}$$

The maximum likelihood estimators are the solutions of the equations

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_m} = \sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i)x_{im} &= 0, \quad m = 1, 2, \ldots, k \\
\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i}\hat{\varepsilon}_i x_{im} &= 0, \quad m = 1, 2, \ldots, k \\
\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i}(y_i - \sum_{j=1}^{k} x_{ij}\beta_j)x_{im} &= 0, \quad m = 1, 2, \ldots, k \\
\sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i} y_i x_{im} &= \sum_{i=1}^{n} -\frac{\phi'(\hat{\varepsilon}_i)}{\hat{\varepsilon}_i}\sum_{j=1}^{k} x_{ij}\beta_j x_{im}
\end{aligned}
\tag{3.9}
$$

If $\boldsymbol{W}$ is the diagonal matrix

$$
\boldsymbol{W} = \begin{pmatrix}
-\frac{\phi'(\hat{\varepsilon}_1)}{\hat{\varepsilon}_1} & 0 & \ldots & 0 \\
0 & -\frac{\phi'(\hat{\varepsilon}_2)}{\hat{\varepsilon}_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{-\phi'(\hat{\varepsilon}_n)}{\hat{\varepsilon}_n}
\end{pmatrix}, \tag{3.10}
$$

Using the notation in Equation (3.3) we may write Equation (3.9) in matrix format as

$$
\boldsymbol{X}'\boldsymbol{W}\boldsymbol{y} = (\boldsymbol{X}'\boldsymbol{W}\boldsymbol{X})\hat{\boldsymbol{\beta}}, \tag{3.11}
$$

or if $\boldsymbol{X}'\boldsymbol{W}\boldsymbol{X}$ is not singular

$$
\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{W}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{W}\boldsymbol{y}. \tag{3.12}
$$

Thus the maximum likelihood regression estimator has the format of a Generalised Least Squares estimator in the presence of heteroscedasticity where the variance of the error term ε_i is proportional to $\frac{\phi'(\varepsilon_i)}{\varepsilon_i}$.[3] The effect of the "Generalised Least Squares" adjustment is to give less weight to

[3] This is only an analogy. The variance of the error term does not exist. The diagonal elements of $\boldsymbol{W}$ are also random

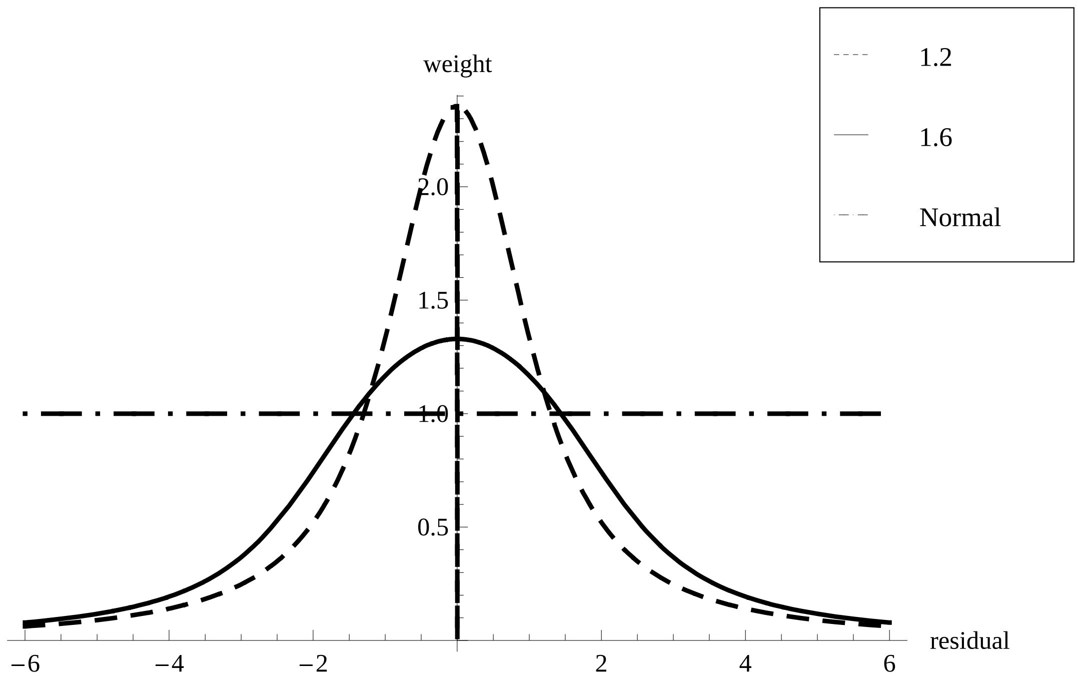


Figure 3.1: Comparison of Implied Weights in GLS Equivalent of Maximum Likelihood Estimates of Regression Coefficient when Disturbances are Distributed as Symmetric α stable Variates

larger observations. Figure 3.1 compares the weighting pattern derived from Equation (3.10) for α-stable processes with $\alpha = 1.2$ and 1.6 with those of a standard normal distribution. For compatibility purposes the α-stable curves are drawn with $\gamma = 1/\sqrt{2}$. As expected the normal distribution gives equal weights to all observations. The estimator for α-stable processes gives higher weights to the centre of the distribution and extremely small weights to extreme values. This effect increases as α is reduced.

This result explains the results obtained by Fama and Roll (1968) who completed a Monte Carlo study of the use of truncated means as measures of location in α-stable distributions. They found:

> *When $\alpha = 1.1$ the .25 truncated*[4] *means are still dominant for all n. For $\alpha = 1.3$ and $\alpha = 1.5$ the .50 truncated means are generally best, and when $\alpha = 1.9$ the distributions of the .75 truncated means are uniformly less disperse than those of other estimators. Finally, when the generating process is Gaussian ($\alpha = 2$) the mean is the "best" estimator. Of course it is also minimum-variance, unbiased in this case.*

The shape of the weight curves in the skewed case is shown in Figure 3.2. The weights are based on the same α-stable distributions as those in Figure 3.1 except that the skew parameter, β, is now -0.1. The most surprising aspect of the weighting systems is the negative weights given to small positive observations. Again the effects are more pronounced as α is reduced.

[4] A g truncated mean retains $100g\%$ of the data. Thus a .25 truncated mean is an average of the central 25% of the data

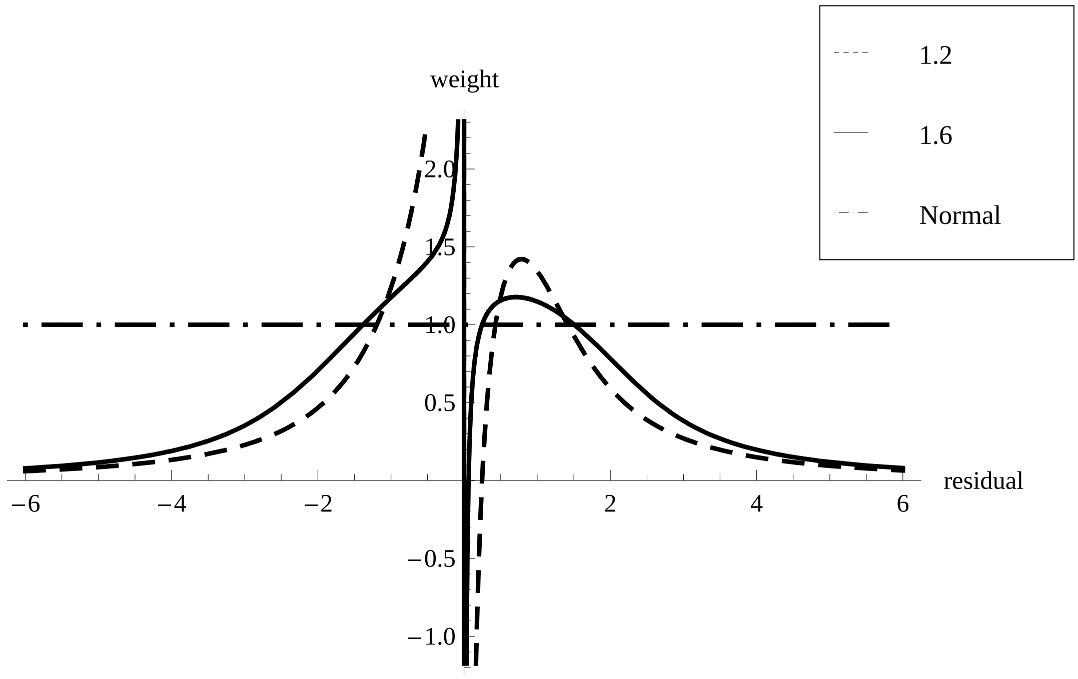


Figure 3.2: Comparison of Implied Weights in GLS Equivalent of Maximum Likelihood Estimates of Regression Coefficient when Residuals are Distributed as Skewed α Stable variables with $\beta = -0.1$

3.3 Maximum Likelihood Estimates of Day of Week Effects with α-stable Errors

Empirical analysis suggests that there is a recurrent low or negative return on equities from Friday to Monday. This effect is known as the weekend effect. The existence of this effect would allow one to design a strategy to make excess profits and would have implications for the Efficient Markets Hypothesis. It is likely that if disturbances are α-stable, then the usual Ordinary Least Squares inferences may lead to spurious results. The use of α-stable disturbances and maximum likelihood will lead to more robust results.

The analysis is based on daily data for six equity indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite(DJAC) and S&P500) which have been adjusted to include dividends. Thus if P_t and D_t are the price and dividend of the index in period t the return on the index in period t is given by

$$R_t = 100 \log\left(\frac{P_t + D_t}{P_{t-1}}\right) \approx 100\left(\frac{P_t + D_t}{P_{t-1}} - 1\right). \tag{3.13}$$

We have also used returns based on the historic values of the Dow Jones Industrial Average equity price index covering the period 3 July 1962 to 28 December 1978, the period analysed in Gibbons and Hess (1981). These have not been adjusted for dividends.

Descriptive statistics and details of goodness of fit of the return series to Normal and α-stable distributions are given in Table 3.1.[5] The goodness of fit normality tests indicate considerable problems with the fit of a Normal distribution. The α-stable distribution provides a better fit.

To estimate and test for day of week effects returns were regressed on five dummy variables, one for each day of the week. The presence of a day of week effect is indicated by the rejection of the hypothesis that all five regression coefficients are equal.

Table 3.2 gives OLS estimates for the longest sample available for each total return index and for the DJIA for the period 3 July 1962 to 28 December

[5] The data in this table are taken from Tables 2.1 and 2.2. Chapter 2 contains a description of the underlying data and the results in this table.

Table 3.1: Summary Statistics Equity Total Return Indices and their Fit to Normal and α-stable Distributions

	ISEQ	CAC40	DAX 30	FTSE100	DJAC	S&P500
start date	04/01/88	31/12/87	28/09/59	31/12/85	30/09/87	03/01/89
end date	21/09/05	26/09/05	26/09/05	26/09/05	26/09/05	26/09/05
observations	4622	4627	12000	5149	4693	4363
mean	0.052	0.044	0.022	.041	0.038	0.043
St. dev	0.934	1.277	1.148	1.028	1.007	0.980
Skewness	-0.3634	-0.124	-0.282	-0.732	-2.686	-0.198
Kurtosis	5.376	3.002	8.378	9.814	58.1964	4.282
Goodness of Fit Tests for Normal Distribution						
JB test[a]	5690	1749	35254	21123	667907	3362
KS test[b]	0.065	0.054	0.062	0.055	0.074	0.063
SW test[c]	0.941	0.967	NA	NA	0.8689	0.956
Maximum Likelihood Estimates of Parameters of α-stable distribution						
α[d]	1.646	1.718	1.687	1.726	1.684	1.668
	(0.045)	(0.043)	(0.027)	(0.041)	(0.044)	(0.046)
β	-0.064	-0.147	-0.076	-0.147	-0.076	-0.105
	(0.111)	(0.128)	(0.075)	(0.125)	(0.119)	(0.118)
γ	0.502	0.746	0.627	0.583	0.529	0.550
	(0.014)	(0.020)	(0.011)	(0.015)	(0.015)	(0.017)
δ	0.054	0.032	0.019	0.036	0.042	0.034
Goodness of Fit Tests for α-Stable Distribution						
KS (stable)	0.012	0.014	0.010	0.008	0.018	0.023
p-value	0.518	0.307	0.166	0.892	0.097	0.025
LR[e] test of Normality	838.1	418.6	1945.8	786.7	1236.5	583.0

[a] The asymptotic distribution of the Jarque-Bera statistic is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels, respectively.

[b] For the sample sizes here the 1% critical value for the Kolmogorov-Smirnov statistic is less that .02. See Marsaglia et al. (2003)

[c] The 5% critical level for the Shapiro-Wilk test is .9992 for a sample of 4500. The smaller values reported here indicate very significant departures from normality.

[d] Figures in brackets under each coefficient estimate are 95% confidence interval half-width estimates

[e] Likelihood ratio test of the joint restriction $\alpha = 2$ and $\beta = 0$. The test statistic is asymptotically $\chi^2(2)$ with critical values given in footnote a to this table.

Table 3.2: OLS Estimates of Day of Week Effects in Returns on Equity Indices

	ISEQ	CAC40	DAX 30	FTSE100	DJAC	S&P500	Gibbons Hess(1981)
OLS coefficient estimates and Standard Errors							
Monday[a]	0.046	-0.032	**-0.086**	0.026	0.062	0.084	**-0.128**
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
Tuesday	0.047	0.063	0.005	0.047	0.062	0.040	-0.007
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
Wednesday	0.041	0.027	0.060	0.025	0.061	0.056	**0.080**
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
Thursday	0.062	0.088	0.043	0.036	-0.003	0.023	0.032
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
Friday	0.057	0.072	**0.087**	0.071	0.017	0.011	**0.065**
	(.031)	(0.042)	(0.023)	(0.032)	(0.033)	(0.033)	(0.027)
Test of equality of day of week coefficients							
F-test	0.061	1.297	8.239	0.349	0.786	0.7361	9.684
significance	(0.993)	(0.269)	(0.00)	(0.845)	(.534)	(0.5672)	(0.000)
Goodness of Fit Tests of Residuals to Normal Distribution							
JB test[b]	5934	1728	34481	20843	665075	3380	1437
Estimates of α-stable Parameters of OLS residuals							
α	1.646	1.727	1.688	1.733	1.683	1.661	1.738
	(0.025)	(0.023)	(0.015)	(0.021)	(0.025)	(0.027)	(0.026)
β	-0.053	-0.136	-0.062	-0.145	-0.077	-0.103	-0.005
	(0.052)	(0.064)	(0.037)	(0.062)	(0.057)	(0.056)	(0.070)
γ	0.500	0.749	0.627	0.584	0.528	0.549	0.467
	(0.008)	(0.011)	(0.006)	(0.008)	(0.008)	(0.009)	(0.007)
δ	0.003	-0.009	- 0.001	-0.004	0.004	-0.009	-0.007
	(0.016)	(0.021)	(0.011)	(0.015)	(0.015)	(0.017)	(0.013)

[a] Figures in brackets under each coefficient or parameter estimate are standard errors of the estimate. Coefficients in boldface are significantly different (5% level) from the average, indicating a day of week effect.

[b] The asymptotic distribution of the Jarque-Bera statistic is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels, respectively.

Table 3.3: α-stable Estimates of Day of Week Effects in Returns on Equity Indices

	ISEQ	CAC40	DAX 30	FTSE100	DJAC	S&P500	Gibbons Hess(1981)
α-stable coefficient estimates and Standard Errors							
Monday[a]	0.044	0.035	0.003	0.043	0.035	0.009	**-0.136**
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
Tuesday	0.049	0.014	0.048	0.011	0.030	0.028	-0.005
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
Wednesday	0.069	0.073	0.047	0.025	-0.008	0.004	**0.071**
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
Thursday	0.052	0.048	**0.081**	0.063	-0.005	0.030	0.010
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
Friday	0.059	0.003	-0.007	0.044	0.011	0.010	**0.066**
	(.027)	(0.039)	(0.021)	(0.029)	(0.028)	(0.030)	(0.025)
Test of equality of day of week coefficients							
LR-test	0.62	2.42	39.00	2.10	5.9	7.3	47.04
significance	(0.961)	(0.659)	(0.000)	(0.717)	(0.207)	(0.121)	(0.000)
Goodness of Fit Test of Residuals to α-stable distribution							
KS (stable)[b]	0.0151	0.0185	0.0082	0.0100	0.0199	0.0239	0.0166
	(0.243)	(.085)	(0.391)	(0.687)	(0.048)	(0.014)	(0.186)
Maximum Likelihood Estimates of Parameters of α-stable distribution							
α	1.632	1.725	1.688	1.733	1.683	1.662	1.738
	(0.024)	(0.023)	(0.015)	(0.021)	(0.025)	(0.027)	(0.025)
β	-0.054	-0.136	-0.062	-0.145	-0.079	-0.105	-0.000
	(0.052)	(0.064)	(0.037)	(0.062)	(0.057)	(.056)	(.069)
γ	0.500	0.749	0.627	0.584	0.528	0.549	0.467
	(0.007)	(0.011)	(0.006)	(0.008)	(0.008)	(.009)	(0.007)

[a] Figures in brackets under each coefficient or parameter estimate are standard errors of the estimate. Coefficients in boldface are significantly different (5% level) from the average, indicating a day of week effect.

[b] Kolmogorov-Smirnov test with a null of an α-stable distribution. Significance Levels are approximate.

Table 3.4: Summary Statistics Returns on DAX30 and their Fit to Normal and α-stable Distributions for Three Sub-periods

start date end date	29/09/59 27/01/75	28/01/75 28/05/90	29/05/90 26/09/05
observations	4000	4000	4000
mean	0.004	0.036	0.025
St. dev	0.989	0.978	1.424
Skewness	0.518	-1.061	-0.276
Kurtosis	8.633	17.940	4.110
Goodness of Fit Tests for Normal Distribution			
JB test[a]	1287	54389	2874.23
KS test[b]	0.044	0.058	0.072
SW test[c]	0.952	0.907	0.949
Estimates of α-stable Parameters of Return Distribution			
α[d]	1.820	1.777	1.636
	(0.022)	(0.023)	(0.027)
β	0.059	-0.066	-0.168
	(0.095)	(0.082)	(0.056)
γ	0.601	0.553	0.774
	(0.009)	(0.008)	(0.013)
δ	0.001	0.040	0.008
Goodness of Fit Tests for α-Stable Distribution			
KS (stable)	0.012	0.012	0.023
p-value	0.619	0.652	0.034

[a] The asymptotic distribution of the Jarque-Bera statistic is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels, respectively.

[b] For the sample sizes here the 1% critical value for the Kolmogorov-Smirnov statistic is less that .02. See Marsaglia et al. (2003).

[c] The 5% critical level for the Shapiro Wilk test is .9992 for a sample of 4500. The smaller values reported here indicate very significant departures from normality.

[d] Figures in brackets under each coefficient estimate are asymptotic standard errors.

Table 3.5: OLS Estimates of Day of Week Effects in Returns on DAX30 Index in Three Sub-periods

	DAX30		
start date end date	29/09/59 27/01/75	28/01/75 28/05/90	29/05/90 26/09/05
observations	4000	4000	4000
OLS coefficient estimates and Standard Errors			
Monday[a]	**-0.192**	**-0.123**	0.057
	(.035)	(0.034)	(0.050)
Tuesday	-0.043	0.034	0.023
	(.035)	(0.034)	(0.050)
Wednesday	**0.102**	0.071	0.007
	(.035)	(0.034)	(0.050)
Thursday	0.058	0.064	0.007
	(.035)	(0.034)	(0.050)
Friday	**0.094**	**0.135**	0.032
	(.035)	(0.034)	(0.050)
Test of equality of day of week coefficients			
F-test significance	12.70 (0.000)	7.800 (0.000)	0.171 (0.953)
Goodness of Fit Tests of Residuals to Normal Distribution			
JB test[b]	12837	51386	2874
Estimates of α-stable Parameters of OLS Residuals			
α	1.818	1.778	1.635
	(0.023)	(0.023)	(0.027)
β	0.089	-0.044	-0.169
	(0.095)	(0.082)	(0.056)
γ	0.596	0.552	0.774
	(0.009)	(0.008)	(0.013)
δ	0.002	-0.005	- 0.018
	(0.016)	(0.016)	(0.026)

[a] Figures in brackets under each coefficient or parameter estimate are standard errors of the estimate. Coefficients in boldface are significantly different (5% level) from the average, indicating a day of week effect.

[b] The asymptotic distribution of the Jarque-Bera statistic is $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels, respectively.

Table 3.6: Maximum Likelihood α-stable Estimates of Day of Week Effects in Returns on DAX30 in Three Sub-periods.

	DAX30		
start date end date	29/09/59 27/01/75	28/01/75 28/05/90	29/05/90 26/09/05
observations	4000	4000	4000
α-stable coefficient estimates and Standard Errors			
monday[a]	**-0.195**	**-0.069**	0.058
	(.033)	(0.030)	(0.046)
Tuesday	-0.041	-0.029	0.011
	(.032)	(0.030)	(0.045)
Wednesday	**0.084**	0.066	-0.029
	(.032)	(0.030)	(0.045)
Thursday	**0.070**	0.059	-0.011
	(.032)	(0.030)	(0.045)
Friday	**0.090**	**0.120**	-0.009
	(.032)	(0.030)	(0.045)
Test of equality of day of week coefficients			
LR-test significance	60.50 (0.000)	23.08 (0.000)	2.34 (0.673)
Goodness of Fit Test of Residuals to α-stable Distribution			
KS (stable)[b]	0.0098 (0.837)	0.0098 (.0836)	0.0279 (0.004)
Maximum Likelihood Estimates of Parameters of α-stable Residuals			
α	1.818	1.777	1.634
	(0.023)	(0.024)	(0.027)
β	0.090	-0.048	-0.170
	(0.094)	(0.082)	(0.056)
γ	0.596	0.551	0.774
	(0.009)	(0.008)	(0.013)

[a] Figures in brackets under each coefficient or parameter estimate are standard errors of the estimate. Coefficients in boldface are significantly different (5% level) from the average, indicating a day of week effect.

[b] Kolmogorov-Smirnov test with a null of an α-stable distribution. Significance Levels are approximate.

1978 as used in Gibbons and Hess (1981). Table 3.3 gives corresponding results assuming non-normal α-stable disturbances and using the methods set out in Section 3.2.

Maximum likelihood estimation is carried out by numerically maximising the log of the likelihood function in Equation (3.8). In the present case Ordinary Least Squares is used to derive initial values for the regression parameters. An α-stable distribution was fitted to the residuals of this regression using the MATHEMATICA (Wolfram (2003)) α-stable density functions in Rimmer (2005).[6] The resulting estimates values of α, β and γ were used as initial values for these parameters in the likelihood estimation. Standard errors of the estimates were estimated by the square root of the diagonal elements of the inverse of the Hessian of the loglikelihood function. While these estimates of the variance of the estimates appear to be numerically stable corresponding estimates of the covariances were not, in some cases. For the α-stable disturbances the tests of equality of the day of week coefficients are likelihood ratio tests.

The final column in each table gives results corresponding to those in Gibbons and Hess (1981). Although We use a different equity index our results are very similar. The Monday effect is negative and very significant in both cases and the OLS and α-stable estimates are very similar. Thus the Gibbons and Hess (1981) results are robust with respect to the specification of disturbances.

This “Monday effect” effect found in the Gibbons and Hess (1981) sample period is not significant in the five total return indices (ISEQ, CAC40, FTSE100, DJAC and S&P500). This corresponds to recent analysis which has found that the “Monday effect” has been becoming smaller and even positive in recent times (see for example Hansen et al. (2005)). It should be noted that although there are some differences in the returns patterns when comparing the OLS and maximum likelihood α-stable estimates it is not obvious how any statis-

[6] The Rimmer routines have since been revised to run in MATHEMATICA Version 6. It appears that the versions used in this Chapter are no longer available. For current versions see Rimmer (2007). I have re-run the regressions with the new versions and they produce the same results. The output of a sample run of the program (for MATHEMATICA 6) is given in Appendix B.

tical significance could be attached to this result. Specific day of week effects such as the "Monday effect" have often been justified only after a significant day of week effect has been found. In an experimental science this could be verified by further independent experimental studies. In economics we rarely have this facility. Sullivan et al. (2001) lists a large number of possible seasonal effects. Some of these are likely to occur by chance and will then be found by a specification search. The danger of data-mining is very real.

The longer return series for the DAX30 shows a significant day of the week effect in both OLS and α-stable cases. The OLS analysis points to significantly low returns on Monday and high returns on Friday as the cause of the problem. The α-stable results point to significantly higher returns on Thursday. One might be tempted to attribute this result to the timing of Bundesbank/European Central Bank monetary policy announcements (Thursdays) but we would be concerned about the data-mining aspect of such a conclusion[7] Data mining and seasonal effects in return data have been discussed at length in Sullivan et al. (2001).

Table 3.4 gives summary statistics of returns on the DAX30 for three periods, 29 September 1959 to 27 January 1975, 28 January 1975 to 28 May 1990 and 29 May 1990 to 27 September 2005. The normal distribution is a poor fit to the data. The α-stable distribution provides a good fit for the first two periods. The goodness of fit test for an α-stable distribution fails for the third period.

Tables 3.5 and 3.6 set out OLS and α-stable maximum likelihood estimates of the day of week effects in each or these subperiods. The results are very similar in both sets of tests. In the first two periods the hypothesis of no day of week effect is rejected and in the third period the hypothesis can not be rejected in both the OLS and α-stable analysis. In the α-stable analysis for the two early periods the Thursday return is significantly higher than the average. This is not so in the OLS analysis.

It should be noted that the stability parameter in the fit of all residuals to

[7] If we run a five orthogonal variable regression and the true values of the coefficients are zero then there is almost a 23% chance that one or more of the five coefficients will be significantly different from zero at the 5% level.

an α-stable distribution is significantly less than 2 indicating deficiencies in the assumption of a normal distribution.

3.4 Summary and Conclusions

This chapter sets out the theory of maximum likelihood estimation and of a linear regression when disturbances follow an α-stable distribution. This theory is then applied to the estimation and testing for a day of week effects in returns on equity indices. We have found that maximum likelihood estimation of a linear regression with α-stable disturbances is feasible.

Traditional OLS estimation and testing is carried out in parallel and the results are compared. Of the ten regressions completed, significant day of week effects effects were found in the same four regressions in both the α-stable and OLS systems. However the alternative methodologies attributed significance to different day of week effects. The α-stable distribution appeared to be a better fit to the residuals in both OLS and α-stable estimates. On the basis of specification tests α-stable estimation is to be preferred.

Day of week effects such as the “Monday effect”, found in some of the regressions here, are often justified by theories relating to institutional arrangements. A “Monday effect” has been explained by delays in trading and settlement caused by the weekend. Such explanations are often given after the significant result has been found leading to accusations of data mining. As I did not have a prior theory explaining the extra effects found in the α-stable estimates any conclusions that I might draw would, justifiably, be subject to the same criticisms. The conclusion remains that if individual coefficients are of interest, the disturbances have fat tails and a possible α-stable distribution and there is sufficient data then the results should be checked for robustness using methods such as those employed here.

CHAPTER 4

Small Sample Power of Tests of Normality when the Alternative is an α-stable Distribution[1]

4.1 Introduction

In this chapter I give an account of a series of simulations to measure the power of various tests of the null hypothesis of normality when the alternative is an α-stable distribution. Large samples of high frequency financial data generally reject this null (see, for example, Rachev and Mittnik (2000) and Chapter 2 of this thesis). These tests applied to the smaller samples of monthly data, aggregated from the same daily data, do not always reject normality. For example, when the six normality tests examined here are applied to one hundred months of daily observations of returns on six total return equity indices the normality hypothesis is overwhelmingly rejected by tests for all six indices. When the six tests are applied to monthly aggregates derived

[1]This Chapter is based on a paper (Frain (2007)) presented at:

- TCD Graduate Seminar, December 2006.
- IEA Annual Conference April 2007.

from the six indices (covering a period of 100 months) the null of normality is accepted in fifteen of the thirty six cases.

A property of the α-stable distribution is that aggregated monthly data, derived from α-stable distributed daily data, have an α-stable distribution with the same α parameter. The apparent failure of monthly data to reject the normality hypothesis has been taken as an indication that the daily data can not have an α-stable distribution. The tests examined here are shown to be of low power when applied to the short samples of monthly data typically available from aggregated daily data. Thus, failure to reject normality in these cases can not be seen as a rejection of the non-normal α-stable distribution.

Chapter 2 and appendix A contain a description of the α-stable distribution. The distribution of high frequency financial returns has tails that are fatter than would be expected by a normal distribution (i.e. $\alpha < 2$). The α-stable distribution appears to fit the data well. In an examination of the distribution of returns on 6 total return equity indices[2] I found values of α in the range 1.65 to 1.73 and small negative values for the skew parameter.

Section 4.2 gives details of the way the α-stable data were simulated and describes the six tests of normality that I have applied to sample sizes of 50, 100 and 200 and three values of each of the α β, γ and δ parameters. These sample sizes are typical of those that might be encountered when monthly data are derived from daily data. Detailed results are reported in Section 4.3 and in the Appendix to this chapter. These results are summarised in Section 4.4.

Section 4.3 also details the results of applying the normality tests to aggregated monthly series of 50, 100 and 200 observations derived from the daily returns used in the earlier analysis.

The values of the β, γ and δ parameters used do not have a large effect on the analysis. In general the tests wrongly accept normality far too often and results are satisfactory only for $\alpha = 1.6$. The Pearson and Cramer-von Mises tests are unsatisfactory in all cases while the Lilliefors (Kolmogorov-Smirnov)

[2]This study is the subject of Chapter 2. The total return equity indices examined included the ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite (DJAC) and S&P500. The estimation period was from October 1959 to September 2005 for the DAX30 and from the late 1970s to September 2005 for the other indices.

test is satisfactory only for a sample size of 200 and an α parameter of 1.6. The Jarque-Bera and Shapiro-Wilk test can differentiate with $\alpha = 1.6$ and a sample size of greater than 100, with $\alpha = 1.7$ and a sample size of 200. The Jarque-Bera can also detect the departure from normality for $\alpha = 1.8$ and a sample size of 200. The measured relative power of these normality tests are specific to the alternative of an α-stable distribution and should not be regarded as measures of the relative merit of the tests against other alternatives.

4.2 The Tests

4.2.1 Simulations

The α-stable random numbers used in this exercise were generated using the α-stable random number generator in the Rmetrics (Wuertz (2007)) package which is part of the R (R Development Core Team (2008)) statistical package. The method used is a variation of that proposed by Chambers et al. (1976) as extended by Weron (1996a,b). Let θ have a uniform distribution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and w have an exponential distribution with mean 1. If

$$X = C_{\alpha,\beta}\left(\frac{\sin(\alpha(\theta+\theta_0))}{(\cos\theta)^{\frac{1}{\alpha}}}\right)\left(\frac{\cos(\theta-\alpha(\theta+\theta_0))}{w}\right)^{\frac{1-\alpha}{\alpha}},$$

where

$$C_{\alpha,\beta} = \left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)^{\frac{1}{2\alpha}} \quad \text{and}$$
$$\theta_0 = \frac{\arctan(\beta\tan\frac{\pi\alpha}{2})}{\alpha},$$

then X has an α-stable distribution with stability parameter α for $\alpha \neq 1$, skewness parameter β, spread parameter 1 and location parameter 0.[3] The transformation of variables ($Y = \gamma X + \delta$) produces an α-stable variable with arbitrary spread (γ) and location (δ) parameters.

For each of three values[4] of the α-stable parameter (1.6, 1.7 and 1.8), three values of the skewness parameter, β, (0, -0.075 and -0.150), three values of the spread parameter, γ, (2.7, 3.6 and 4.5) and three values of the mean parameter δ (0.44, 0.88, 1.32) samples of 50, 100 and 200 observations were drawn. Each of these 243 experiments was replicated 1000 times. Six tests for normality

[3] When $\alpha = 1$ use

$$X = \frac{2}{\pi}\left[\left(\frac{\pi}{2}+\beta\theta\right)\tan\theta - \beta\log\left(\frac{\frac{\pi}{2}w\cos\theta}{\frac{\pi}{2}+\beta\theta}\right)\right].$$

[4] The ranges of values for each parameter are the monthly equivalent of those found in Chapter 2. See page 29.

were applied to each of the 243,000 samples. As a control on the process the simulations were repeated for a normal distribution with corresponding mean and variance.

The tests used were:

1. Anderson-Darling.
2. Cramer-von Mises.
3. Lilliefors (Kolmogorov-Smirnov).
4. Pearson (χ^2 Goodness of Fit).
5. Shapiro-Wilk.
6. Jarque-Bera.

A brief summary of each test follows. For an extended account of testing for normality see Thode (2002)

4.2.2 Lilliefors (Kolmogorov-Smirnov) Test

The first three normality tests considered here are based on the difference between the empirical distribution function (EDF) and the normal distribution function. If the order statistics of a random sample of size n are given by $x_{(1)}$, $x_{(2)}, \dots x_{(n)}$, the EDF is given by:

$$F_n(x) = \begin{cases} 0 & x < x_{(1)}, \\ i/n & x_{(i)} \leq x < x_{(i+1)} \quad i = 1, \dots, n-1, \\ 1 & x_{(n)} \leq x. \end{cases} \tag{4.1}$$

If $\Phi()$ is the standard normal distribution function and X has a normal distribution with mean μ and variance σ^2 the corresponding values of the distribution function are given by

$$q_i = \Phi([x_{(i)} - \mu]/\sigma) \tag{4.2}$$

The Kolmogorov-Smirnov test statistic is based on the maximum difference between the EDF and the q_i and is given by D in Equation (4.3).

$$
\begin{aligned}
D^+ &= \max_{i=1,\ldots,n}[i/n - q_i], \\
D^- &= \max_{i=1,\ldots,n}[q_i - i/n], \\
D &= \max[D^+, D^-].
\end{aligned}
\tag{4.3}
$$

The Kolmogorov-Smirnov test has been extended by Lilliefors (1967) to the case where the mean and variance are unknown and the estimated test statistic is based on the usual estimates of the mean and variance. See also Stephens (1974) and Thode (2002).

4.2.3 Cramer-von Mises Test

A class of EDF tests proposed by Anderson and Darling (1952) is defined by

$$
W_n^2 = n \int_{-\infty}^{\infty} |F_n(x) - F(x)|^2 \psi[F(x)] dF \tag{4.4}
$$

where $F()$ is the hypothesised distribution function and $\psi()$ is a non-negative weight function. For certain weight functions, including $\psi = 1$ and $\psi(t) = 1/[t(1-t)]$, it is possible to derive explicit limit distributions of this statistic. The Cramer-von Mises statistic uses the first of these weight functions and is given by

$$
W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(q(i) - \frac{2i-1}{2n} \right)^2 \tag{4.5}
$$

with the modification

$$
W^{2*} = (1.0 + 0.5/n) W^2
$$

accounting for differences in sample size when using tabulated critical values.

4.2.4 Anderson-Darling Test

The Anderson-Darling test uses the weighting function $\psi(t) = 1/[t(1-t)]$ in Equation (4.4). This gives the test statistic

$$A^2 = -n - n^{-1}\sum_{i=1}^{n}[2i-1][\log(p_{(i)}) + \log(1-p_{(n-i+1)})], \tag{4.6}$$

where $p_{(i)} = \Phi([x_{(i)} - \hat{\mu}]/\hat{\sigma})$ and $\hat{\mu}$ and $\hat{\sigma}$ are estimated values of the mean and standard deviation.

The modification

$$A^{2*} = (1.0 + 0.75/n + 2.25/n^2)A^2 \tag{4.7}$$

allows the standard critical values to be applied to all sample sizes. The Anderson-Darling test gives more weight to the tails of the distribution than the Cramer-von Mises test and may therefore be better able to differentiate between normal and α-stable distributions.

4.2.5 Pearson (χ^2 Goodness of Fit) Test

The Pearson test is the traditional test of goodness of fit. The observations are divided into k intervals. Let O_i and E_i be the observed and expected number in the i^{th} interval. The Pearson test statistic is

$$P = \sum_{i=1}^{k}\frac{(O_i - E_i)^2}{E_i}. \tag{4.8}$$

The test is implemented here by dividing the samples of 50, 100 and 200 into 10, 13 and 17 respectively of intervals which are of equal probability under the null of normality. P is distributed asymptotically as χ^2 with $k-3$ degrees of freedom, where k is the number of intervals used in the calculation of P. Since the advent of specific tests for a null of a normal distribution the Pearson test is not generally used.

4.2.6 Shapiro-Wilk Test

If the data are a good fit to a normal distribution then the plot of $x_{(i)}$ against $\Phi(i/n)$ will be close to a straight line. The Shapiro-Wilk test is a measure of this fit based on a generalised least squares regression using the covariance matrix of the order statistics. Due to difficulties in calculating this covariance matrix the Shapiro-Wilk test was originally available only for sample sizes up to 50. The difficulty being partially due to the fact that a separate covariance matrix had to be calculated for each sample size. Initially the Shapiro-Wilk test allowed smaller samples to be tested for normality than the previous Pearson test. Various approximations are now available that allow the test to be used for samples up to 5000. See Royston (1982a,b, 1995)

4.2.7 Jarque-Bera Test

The Jarque-Bera[5] test is probably the normality test best known to economists and is often used as a test of the normality of residuals. If m_i is the i^{th} moment about the mean of a sample then the skewness ($b_1^{1/2}$) and kurtosis (b_2) are defined by

$$b_1^{1/2} = \frac{m_3}{m_2^{3/2}} \quad \text{and} \quad b_2 = \frac{m_4}{m_2^2} \tag{4.9}$$

For a sample of size n from a normal distribution $b_1^{1/2}$ is asymptotically normal with mean zero and variance 6/n. For finite samples the variance of $b_1^{1/2}$ is better given by[6]

$$c_1 = \frac{6(n-2)}{(n+1)(n+3)}$$

In the same circumstances the distribution of b_2 is asymptotically normal with mean 3 and variance 24/n. For finite samples the mean c_2 and variance

[5] The small sample corrections to the Jarque-Bera test statistic used in this section are given in Wuertz and Katzgraber (2005) and are not the same as those described in Section 2.3 on page 26.

[6] For details see Thode (2002)

Table 4.1: Critical Values of Jarque-Bera Test of Normality

Sample Size	Simulated JB	Simulated AJB	Asymptotic
50	4.98	6.55	5.99
100	5.43	6.32	5.99
200	5.68	6.15	5.99

c_3 of b_2 are given by

$$c_2 = \frac{3(n-1)}{(n+1)}$$
$$c_3 = \frac{24n(n-2)(n-3)}{n+1)^2(n+3)(n+5)}$$

The Jarque-Bera statistic is given by

$$JB = n\left(\frac{(b_1^{1/2})^2}{6} + \frac{(b_2-3)^2}{24}\right)$$

which under the null hypothesis of normality has an asymptotic χ^2 distribution with 2 degrees of freedom. In finite samples the skewness and kurtosis are not independent and the JB statistic converges slowly to it asymptotic limit. Two solutions have been proposed. First the JB statistic may be modified by replacing the asymptotic means and variances by their values in finite samples and defining an adjusted Jarque-Bera (AJB) statistic:

$$AJB = \left(\frac{(b_1^{1/2})^2}{c_1} + \frac{(b_2-c_2)^2}{c_3}\right).$$

The AJB and JB statistics have the same asymptotic distribution. For both the JB and AJB statistics critical values have been estimated by Wuertz and Katzgraber (2005) using a large sample Monte Carlo simulation. A comparison of the simulated and asymptotic critical values for the sample sizes used here is given in the table below.

Thus inference based on the asymptotic distribution of the standard JB statistic will tend to accept normality to often. Inference based on the asymptotic distribution of the adjusted statistic tends to reject normality to often. In the simulations in this chapter inferences were based on the simulated distribution of the standard Jarque-Bera statistic. Tables 4.2, 4.3 and 4.4 contain both JB and AJB tests on monthly returns. The significance levels given there are based on those in Wuertz and Katzgraber (2005). In all cases both tests lead to the same conclusion.

4.3 Results

The results of the simulations of the tests on the α-stable samples are shown in Tables 4.5 to 4.13 and summarised in Figures 4.1, 4.2 and 4.3. The control tests on the normal distribution are given in Table 4.14. Each of these 729 experiments described in Section 4.2.1 was replicated 1000 times. Each replication consisted of the generation of a pseudo random sample of the selected size from an α-stable distribution with the appropriate parameters. The six tests detailed in Section 4.2 were then applied to the random sample. The number of times that the normality assumption was accepted, at the test size specified, over 1000 replications is recorded in each case.

Thus the figure of 318 at the top of column 5 of table 4.5 indicates that normality was accepted in 318 of the 1000 replications when an Anderson-Darling test of size 5% was used. The power of the test may be approximated as 68%.[7] Similarly in 363, 423, 530, 280 and 225 from the 1000 replications normality was accepted at the 5% size when, respectively, the Cramer-von Mises, Lilliefors, Pearson, Shapiro-Wilk and Jarque-Bera tests were applied. The numbers in these tables may be regarded as an estimate of the numbers of false acceptances of normality that may be found in applications of the test in the circumstances of the simulation i.e. they give the number of "wrong" answers. Thus, smaller numbers are better.

[7] These tables were computer generated from the output of the simulation programs. The tabulated data are proportional to the probability of a type II error rather than the more usual results in terms of power of the test. The power of the test is estimated as $1 - \frac{\text{number normality accepted}}{1000}$

The results of applying the tests to simulated data drawn from a normal distribution are given in Table 4.14. The results in this table show that there are no significant size distortions in any of the tests examined at the sample sizes considered.

4.3.1 Discussion of Results

The data in the tables show that the power of the tests varies with α, the sample size and the test size as may be expected. In the ranges examined the other three parameters are not as important. We have adopted the somewhat arbitrary definition of a satisfactory test as one of size 5% with power greater than 90%. A stricter definition would restrict the number of satisfactory tests while a more liberal approach would lead to a greater number of satisfactory outcomes.

Using this definition no test is satisfactory for a sample size of 50. The Jarque-Bera test outperforms the others with an average power of 76% for $\alpha = 1.6$ dropping to an average of under 50% for $\alpha = 1.8$

For a sample size of 100 the Jarque-Bera test is again best in all cases. For $\alpha = 1.6$ the average power of the test is 94%. This figure falls to 86% and 70% for α of 1.7 and 1.8, respectively. The Shapiro-Wilk and Anderson-Darling tests have power close to 90% when $\alpha = 1.6$ and the size of the test is 5%.

For a sample size of 200 and $\alpha = 1.6$ the power of the Jarque-Bera, Shapiro-Wilk, Anderson-Darling and Lilliefors (Kolmogorov-Smirnov) tests are good, with average powers of 1.00, 0.99, 0.99, and 0.96 respectively. In this case the average power of the Pearson and Cramer-von Mises tests are 0.89 and 0.71 respectively.

For a sample size of 200 and $\alpha = 1.7$ the Jarque-Bera, Shapiro-Wilk and Anderson-Darling tests have powers of 0.98, 0.96, and 0.92 respectively. For a sample size of 200 and $\alpha = 1.8$ the average power the Jarque-Bera test is just under 0.90.

The Pearson and Cramer-von Mises tests are not satisfactory in any case. The Jarque-Bera test is the most satisfactory.

The measured relative power of these normality tests are specific to the alternative of an α-stable distribution and should not be regarded as measures of the relative merit of the tests against other alternatives or forms of non-normality.

4.3.2 Application of tests to monthly Total Return Equity Indices

Tables 4.2, 4.3 and 4.4 show the results of applying the 6 tests examined to monthly returns on equity total return indices for periods of 50, 100 and 200 months, respectively, up to end August 2005. The total return equity indices included are those for the CAC40, DAX30, FTSE100, ISEQ, Dow Composite (DCI) and the S&P500. Corresponding calculations for daily data show an overwhelming rejection of normality in all cases. For the samples of 50, 100 and 200 months there are ,respectively, 11, 15 and 9 acceptances of the null hypothesis of normality from the 36 tests completed in each case. Given the possible common trends in the series one can not regard them as independent samples but as an illustration of the application of the earlier results in this Chapter.

Of the 9 acceptances of normality in the 200 month samples all but one are in the Pearson or Lilliefors tests which have been shown to have poor power. For the 100 month samples again the majority of rejections are six and three acceptances of normality, respectively in the Pearson and Lilliefors tests.

4.4 Summary and Conclusions

If one regards a satisfactory test as one of size 5% with a power of 90% then the conclusions are:

Sample size 50 No test is satisfactory.

Sample size 100 :

Figure 4.1: Power of Normality Tests when the Alternative is α-stable in Sample size 50

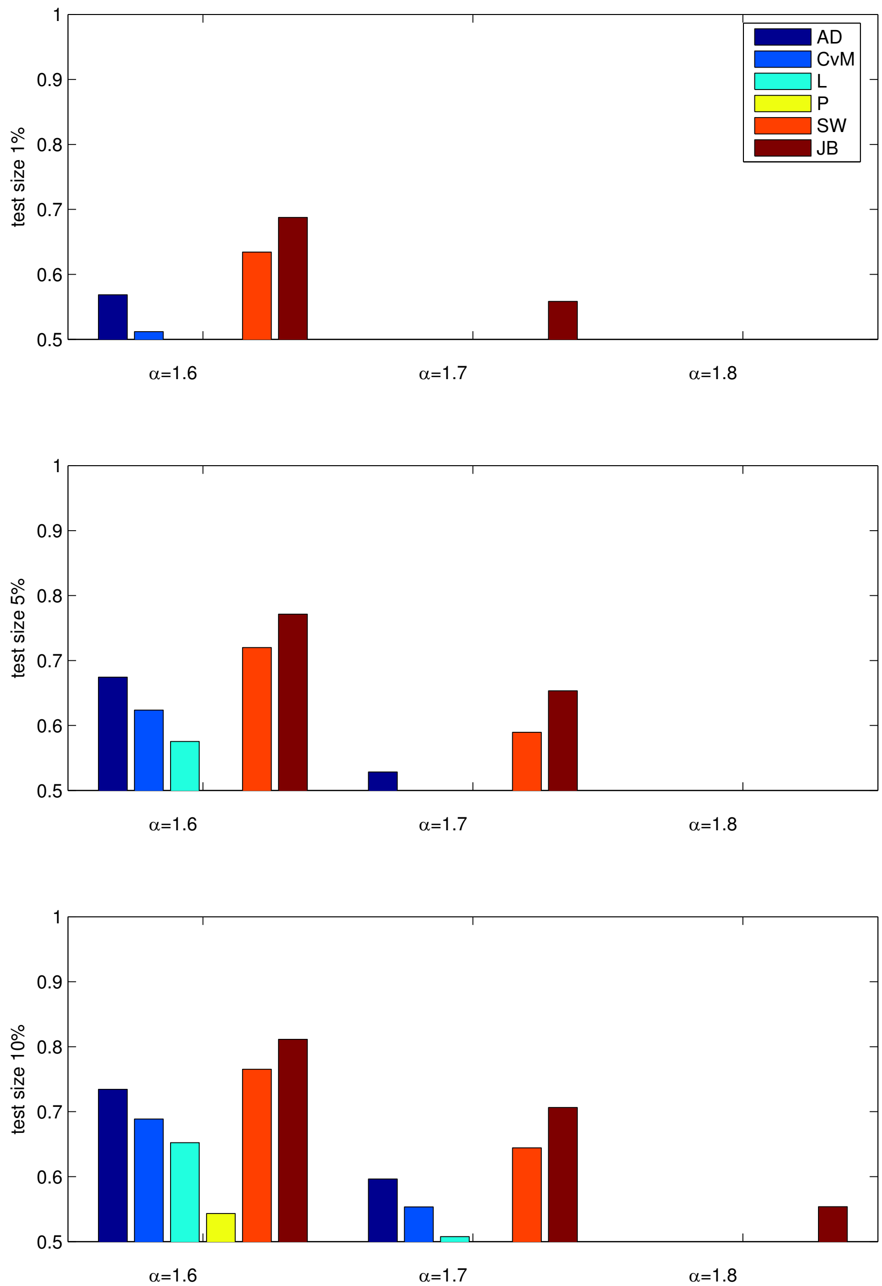


Figure 4.2: Power of Normality Tests when the Alternative is α-stable in Sample size 100

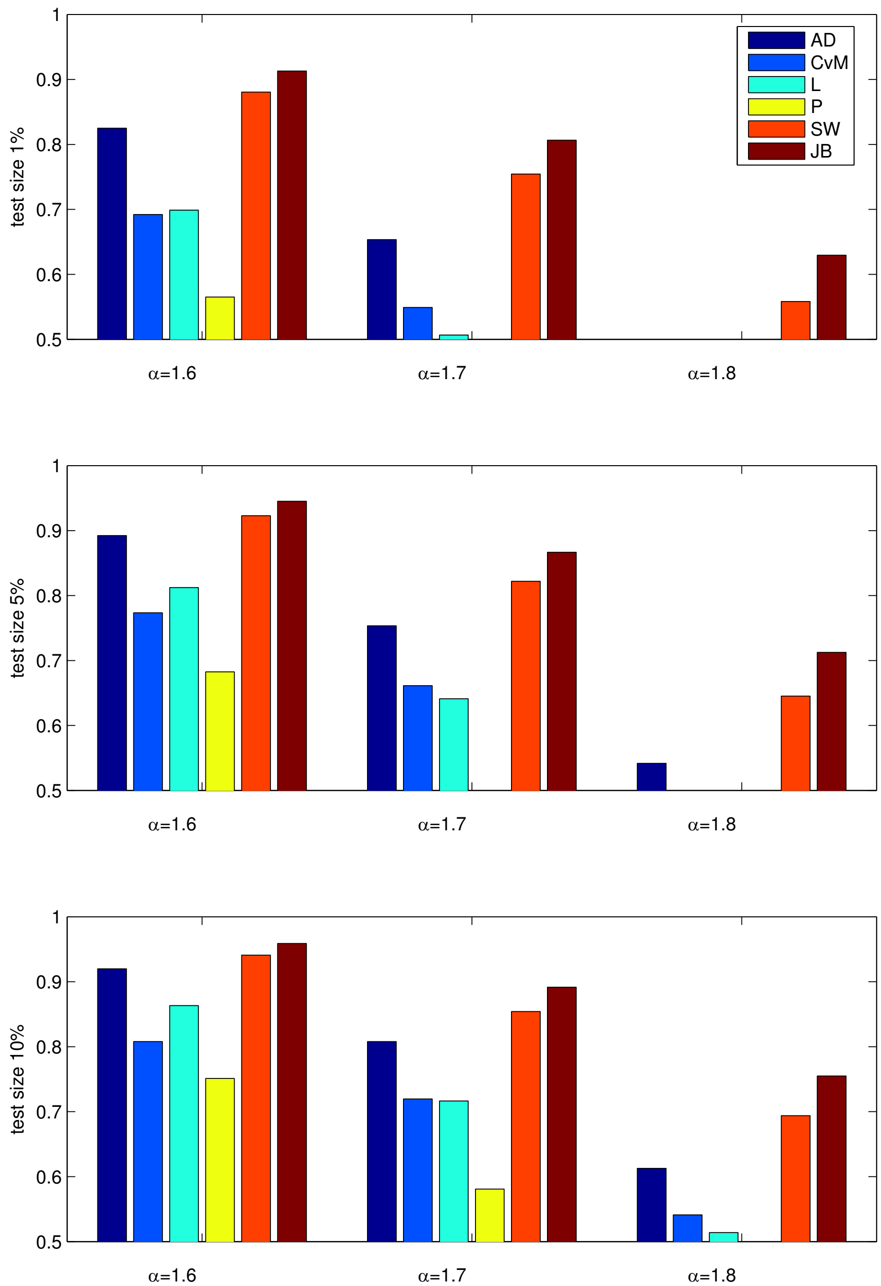
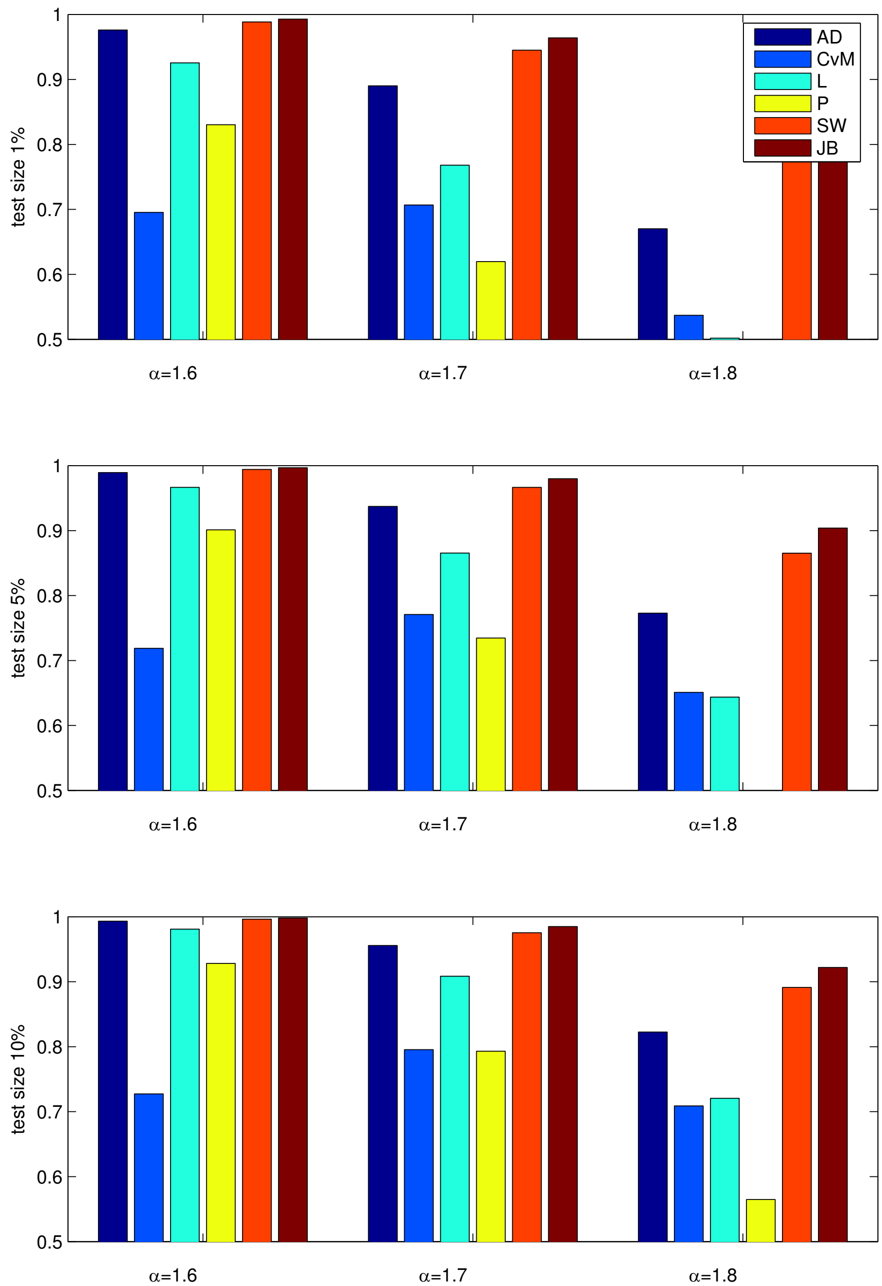


Figure 4.3: Power of Normality Tests when the Alternative is α-stable in Sample size 200



- For $\alpha = 1.6$ Jarque-Bera and Shapiro-Wilk tests are satisfactory.
- For $\alpha = 1.7$ no test is satisfactory.
- For $\alpha = 1.8$ no test is satisfactory.

Sample size 200 :

- For $\alpha = 1.6$ Jarque-Bera, Shapiro-Wilk, Anderson-Darling and Lilliefors tests are satisfactory.
- For $\alpha = 1.7$ Jarque-Bera, Shapiro-Wilk and Anderson-Darling tests are satisfactory.
- For $\alpha = 1.8$ The Jarque-Bera test was satisfactory in more than half the simulations at this level and close to satisfactory in the remainder.

At the parameter values likely to fit total return on equity indices a sample size of the order of 200 is required in order to reliably detect departures from normality using common normality tests.

The measured relative powers of these normality tests are specific to the alternative of an α-stable distribution and should not be regarded as measures of the relative merit of the tests against other alternatives.

4.5 Appendix – Tables of Detailed Results

Table 4.2: Normality Tests on Monthly Returns on Total Return Equity Indices for a 50 Month Period ending August 2005

Equity Index	Summary Statistics					Normality Statistics						
	Obs.	Mean	St. dev	Skewness	Kurtosis	Anderson-Darling	Craner-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera (JB)	Jarque-Bera (AJB)
CAC40	50	-0.469	8.291	-0.876	2.485	**0.732**	**0.116**	**0.094**	**9.200**	0.949	15.435	20.877
						(.053)	(.065)	(.327)	-0.239	(.032)	(.006)	(.006)
DAX30	50	-0.124	6.071	-0.775	1.512	1.067	0.193	0.154	23.200	0.943	7.952	10.523
						(.008)	(.006)	(.004)	(.001)	(.018)	(.023)	(.023)
FTSE	50	0.137	4.236	-1.024	1.592	1.845	0.285	0.169	36.700	0.900	11.842	15.037
						(.000)	(.000)	(.001)	(.000)	(.000)	(.011)	(.012)
ISEQ	50	0.271	5.490	-0.947	0.326	1.486	0.246	0.152	16.800	0.916	7.093	8.153
						(.001)	(.001)	(.005)	(.019)	(.002)	(.028)	(.036)
DCI	50	0.361	4.392	-1.106	2.186	**0.736**	**0.095**	**0.106**	**8.000**	0.933	16.755	21.696
						(.052)	(.129)	(.167)	(.333)	(.007)	(.005)	(.006)
S&P	50	-0.091	4.380	-0.336	0.827	0.833	0.139	0.129	16.400	**0.963**	**1.711**	**2.560**
						(.030)	(.032)	(.036)	(.022)	(0.121)	(.284)	(0.189)

(Data in bold face indicate acceptance of normality hypothesis at 5% level)

Table 4.3: Normality Tests on Monthly Returns on Total Return Equity Indices for a 100 Month Period ending August 2005

Equity Index	Summary Statistics					Normality Statistics						
	Obs.	Mean	St. dev	Skewness	Kurtosis	Anderson-Darling	Craner-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera (JB)	Jarque-Bera (AJB)
CAC40	100	0.720	6.275	-0.628	0.527	0.901	0.160	0.102	**14.140**	0.970	7.186	7.979
						(.021)	(.017)	(.012)	(.167)	(.021)	(.030)	(.033)
DAX30	100	0.332	7.747	-0.750	1.800	0.825	0.126	**0.071**	**11.800**	0.966	20.467	23.819
						(.032)	(049)	(.247)	(.299)	(.010)	(.003)	(.003)
FTSE100	100	0.409	4.325	-0.711	0.600	1.121	0.163	0.091	**12.320**	0.961	9.240	10.242
						(.006)	(.016)	(.041)	(.264)	(.005)	(.019)	(.021)
ISEQ	100	0.938	5.519	-0.822	0.993	1.165	0.195	0.105	**12.060**	0.960	14.188	15.909
						(005)	(.006)	(.009)	(.281)	(.004)	(.007)	(.008)
DCI	100	0.626	4.424	-0.815	1.305	**0.674**	**0.073**	**0.057**	**4.780**	0.959	16.559	18.850
						(.076)	(.253)	(.597)	(.905)	(.004)	(.005)	(.006)
S&P500	100	0.483	5.055	-0.499	0.293	**0.536**	**0.072**	**0.077**	**15.440**	**0.978**	**4.222**	**4.644**
						(.166)	(.260)	(.153)	(.117)	(.094)	(.074)	(.083)

(Data in bold face indicate acceptance of normality hypothesis at 5% level)

Table 4.4: Normality Tests on Monthly Returns on Total Return Equity Indices for a 200 Month Period ending August 2005

Equity Index	Summary Statistics					Normality Statistics						
	Obs.	Mean	St. dev	Skewness	Kurtosis	Anderson-Darling	Craner-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera (JB)	Jarque-Bera (AJB)
CAC40	200	0.745	5.689	-0.550	0.519	1.085	0.195	0.081	**17.600**	0.979	11.832	12.544
						(.007)	(.006)	(.002)	(.226)	(.004)	(.010)	(.011)
DAX30	200	0.640	6.58	-0.908	2.710	2.073	0.319	0.086	24.400	0.952	83.942	90.586
						(.000)	(.000)	(.001)	(.041)	(.000)	(.000)	(.000)
FTSE100	200	0.850	4.271	-0.303	0.646	1.083	0.181	**0.062**	**16.580**	0.986	6.014	6.668
						(.008)	(.009)	(.061)	(.279)	(.040)	(.044)	(.043)
ISEQ	200	1.013	5.287	-0.455	1.411	1.236	0.188	**0.062**	**13.180**	0.974	21.889	23.996
						(.003)	(.007)	(.061)	(.512)	(.001)	(.002)	(.002)
DCI	200	0.924	4.060	-0.763	1.387	1.177	0.193	0.066	**16.580**	0.978	33.681	36.110
						(.004)	(.007)	(.032)	(.279)	(.000)	(.000)	(.000)
S&P500	200	0.871	4.311	-0.548	0.911	0.809	**0.118**	**0.053**	**19.779**	0.980	15.929	17.174
						(.036)	(0.062)	(.187)	(.137)	(.006)	(.005)	(.005)

(Data in bold face indicate acceptance of normality hypothesis at 5% level)

Table 4.5: Simulation of 5% Normality Tests on α-stable Samples of Size 50 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	318	363	423	530	280	225
1.6	0	2.7	0.88	325	386	448	555	286	238
1.6	0	2.7	1.32	353	399	430	565	303	251
1.6	0	3.6	0.44	301	348	400	532	262	207
1.6	0	3.6	0.88	362	416	461	576	303	234
1.6	0	3.6	1.32	329	377	418	526	288	244
1.6	0	4.5	0.44	354	397	458	554	283	226
1.6	0	4.5	0.88	323	377	431	546	274	225
1.6	0	4.5	1.32	313	372	433	529	265	208
1.6	-0.075	2.7	0.44	309	371	420	522	282	232
1.6	-0.075	2.7	0.88	305	359	390	512	265	215
1.6	-0.075	2.7	1.32	344	395	432	535	298	231
1.6	-0.075	3.6	0.44	316	370	418	539	264	216
1.6	-0.075	3.6	0.88	343	388	436	562	305	250
1.6	-0.075	3.6	1.32	339	394	433	541	289	228
1.6	-0.075	4.5	0.44	334	378	432	544	305	249
1.6	-0.075	4.5	0.88	323	372	407	538	275	223
1.6	-0.075	4.5	1.32	337	378	430	551	283	242
1.6	-0.15	2.7	0.44	322	372	430	517	278	232
1.6	-0.15	2.7	0.88	345	394	434	558	306	237
1.6	-0.15	2.7	1.32	305	348	416	535	268	225
1.6	-0.15	3.6	0.44	340	390	434	543	280	241
1.6	-0.15	3.6	0.88	311	372	420	522	274	226
1.6	-0.15	3.6	1.32	308	372	409	517	270	221
1.6	-0.15	4.5	0.44	300	351	392	529	243	200
1.6	-0.15	4.5	0.88	327	379	425	528	263	225

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Table 4.5: Simulation of 5% Normality Tests on α-stable Samples of Size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	-0.15	4.5	1.32	305	345	407	522	267	221
1.7	0	2.7	0.44	500	538	583	656	415	345
1.7	0	2.7	0.88	477	522	576	693	423	351
1.7	0	2.7	1.32	464	510	560	668	394	343
1.7	0	3.6	0.44	440	498	536	650	392	328
1.7	0	3.6	0.88	473	508	559	664	414	338
1.7	0	3.6	1.32	470	521	591	671	419	351
1.7	0	4.5	0.44	464	529	576	667	403	336
1.7	0	4.5	0.88	481	523	588	674	417	346
1.7	0	4.5	1.32	477	526	577	696	435	360
1.7	-0.075	2.7	0.44	470	505	569	669	410	332
1.7	-0.075	2.7	0.88	479	532	574	664	410	345
1.7	-0.075	2.7	1.32	454	496	549	662	407	348
1.7	-0.075	3.6	0.44	468	514	581	664	406	332
1.7	-0.075	3.6	0.88	442	483	553	655	386	330
1.7	-0.075	3.6	1.32	479	522	582	680	399	342
1.7	-0.075	4.5	0.44	496	535	581	677	429	360
1.7	-0.075	4.5	0.88	498	551	604	693	419	354
1.7	-0.075	4.5	1.32	465	511	558	680	399	334
1.7	-0.15	2.7	0.44	468	511	567	675	423	366
1.7	-0.15	2.7	0.88	463	514	574	676	405	328
1.7	-0.15	2.7	1.32	462	517	579	675	400	353
1.7	-0.15	3.6	0.44	478	529	574	690	417	354
1.7	-0.15	3.6	0.88	499	541	605	695	427	382
1.7	-0.15	3.6	1.32	458	511	571	673	415	346
1.7	-0.15	4.5	0.44	451	493	535	665	392	336
1.7	-0.15	4.5	0.88	476	527	590	693	407	354

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Table 4.5: Simulation of 5% Normality Tests on α-stable Samples of Size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	-0.15	4.5	1.32	482	530	575	694	422	364
1.8	0	2.7	0.44	601	650	707	773	534	469
1.8	0	2.7	0.88	625	661	704	767	566	506
1.8	0	2.7	1.32	649	691	727	785	573	512
1.8	0	3.6	0.44	631	685	721	782	559	497
1.8	0	3.6	0.88	658	693	717	797	589	534
1.8	0	3.6	1.32	634	676	718	782	568	525
1.8	0	4.5	0.44	626	668	707	774	561	490
1.8	0	4.5	0.88	624	671	699	783	549	470
1.8	0	4.5	1.32	640	681	720	789	582	524
1.8	-0.075	2.7	0.44	639	679	722	805	563	485
1.8	-0.075	2.7	0.88	619	657	704	772	561	492
1.8	-0.075	2.7	1.32	638	674	713	783	556	502
1.8	-0.075	3.6	0.44	647	680	709	763	563	509
1.8	-0.075	3.6	0.88	626	661	709	781	570	492
1.8	-0.075	3.6	1.32	643	686	718	806	574	524
1.8	-0.075	4.5	0.44	623	665	719	795	562	517
1.8	-0.075	4.5	0.88	654	688	728	782	578	520
1.8	-0.075	4.5	1.32	650	690	714	789	582	521
1.8	-0.15	2.7	0.44	654	679	724	798	584	530
1.8	-0.15	2.7	0.88	598	646	692	769	542	483
1.8	-0.15	2.7	1.32	657	690	727	785	583	507
1.8	-0.15	3.6	0.44	623	656	691	762	555	492
1.8	-0.15	3.6	0.88	641	685	746	786	578	510
1.8	-0.15	3.6	1.32	648	688	714	770	580	497
1.8	-0.15	4.5	0.44	638	681	728	789	571	510
1.8	-0.15	4.5	0.88	619	662	704	786	556	494

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Table 4.5: Simulation of 5% Normality Tests on α-stable Samples of Size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.8	-0.15	4.5	1.32	628	653	701	773	573	512

Table 4.6: Simulation of 5% Normality Tests on α-stable Samples of Size 100 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	134	229	217	343	98	67
1.6	0	2.7	0.88	110	215	177	307	74	52
1.6	0	2.7	1.32	103	216	181	314	78	55
1.6	0	3.6	0.44	93	201	177	319	68	51
1.6	0	3.6	0.88	113	229	189	309	81	51
1.6	0	3.6	1.32	117	227	198	324	77	49
1.6	0	4.5	0.44	104	242	195	332	76	52
1.6	0	4.5	0.88	94	209	159	281	68	49
1.6	0	4.5	1.32	95	225	176	313	75	55
1.6	-0.075	2.7	0.44	119	249	186	317	90	68
1.6	-0.075	2.7	0.88	110	216	199	323	78	58
1.6	-0.075	2.7	1.32	100	225	180	318	69	47
1.6	-0.075	3.6	0.44	121	242	187	304	97	71
1.6	-0.075	3.6	0.88	116	243	195	350	82	57
1.6	-0.075	3.6	1.32	98	220	182	305	63	45
1.6	-0.075	4.5	0.44	115	242	193	303	81	61
1.6	-0.075	4.5	0.88	115	240	196	348	74	52
1.6	-0.075	4.5	1.32	89	196	200	333	62	46
1.6	-0.15	2.7	0.44	119	248	198	330	83	64

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Table 4.6: Simulation of 5% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	-0.15	2.7	0.88	98	221	185	297	73	52
1.6	-0.15	2.7	1.32	101	225	181	298	79	51
1.6	-0.15	3.6	0.44	101	218	176	307	73	56
1.6	-0.15	3.6	0.88	97	202	182	320	68	55
1.6	-0.15	3.6	1.32	106	229	190	324	73	50
1.6	-0.15	4.5	0.44	110	239	199	329	74	61
1.6	-0.15	4.5	0.88	114	227	199	328	85	52
1.6	-0.15	4.5	1.32	113	241	169	297	82	52
1.7	0	2.7	0.44	254	351	366	506	202	146
1.7	0	2.7	0.88	273	354	371	505	202	141
1.7	0	2.7	1.32	215	317	333	482	155	116
1.7	0	3.6	0.44	246	351	367	497	175	122
1.7	0	3.6	0.88	240	320	352	483	158	109
1.7	0	3.6	1.32	231	332	358	513	169	130
1.7	0	4.5	0.44	240	338	353	493	170	127
1.7	0	4.5	0.88	264	342	373	528	180	135
1.7	0	4.5	1.32	262	348	375	498	183	128
1.7	-0.075	2.7	0.44	234	318	347	486	164	120
1.7	-0.075	2.7	0.88	257	317	338	498	191	138
1.7	-0.075	2.7	1.32	250	342	377	519	188	153
1.7	-0.075	3.6	0.44	246	334	350	494	187	148
1.7	-0.075	3.6	0.88	240	344	344	504	181	137
1.7	-0.075	3.6	1.32	263	353	359	497	170	126
1.7	-0.075	4.5	0.44	235	330	343	502	163	127
1.7	-0.075	4.5	0.88	246	340	365	509	186	140
1.7	-0.075	4.5	1.32	235	322	345	496	171	132
1.7	-0.15	2.7	0.44	247	337	361	519	185	144

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Table 4.6: Simulation of 5% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	-0.15	2.7	0.88	219	320	353	491	174	134
1.7	-0.15	2.7	1.32	230	334	337	489	150	109
1.7	-0.15	3.6	0.44	272	365	387	518	191	145
1.7	-0.15	3.6	0.88	232	322	345	482	179	134
1.7	-0.15	3.6	1.32	268	360	385	523	194	145
1.7	-0.15	4.5	0.44	248	346	366	503	181	144
1.7	-0.15	4.5	0.88	257	352	381	503	191	144
1.7	-0.15	4.5	1.32	254	358	360	494	169	124
1.8	0	2.7	0.44	453	531	572	670	345	274
1.8	0	2.7	0.88	472	538	577	672	364	269
1.8	0	2.7	1.32	419	505	548	656	331	270
1.8	0	3.6	0.44	465	525	563	686	365	304
1.8	0	3.6	0.88	445	511	543	665	341	287
1.8	0	3.6	1.32	472	546	579	670	361	284
1.8	0	4.5	0.44	463	533	575	682	370	298
1.8	0	4.5	0.88	447	509	556	669	346	291
1.8	0	4.5	1.32	487	551	590	693	373	305
1.8	-0.075	2.7	0.44	465	537	582	672	359	309
1.8	-0.075	2.7	0.88	441	523	561	668	339	267
1.8	-0.075	2.7	1.32	463	534	579	674	352	271
1.8	-0.075	3.6	0.44	479	536	595	695	375	310
1.8	-0.075	3.6	0.88	461	525	565	680	344	267
1.8	-0.075	3.6	1.32	434	509	581	678	350	288
1.8	-0.075	4.5	0.44	461	553	579	690	359	305
1.8	-0.075	4.5	0.88	447	521	559	668	352	288
1.8	-0.075	4.5	1.32	445	535	578	671	349	270
1.8	-0.15	2.7	0.44	472	534	581	699	365	285

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Table 4.6: Simulation of 5% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.8	-0.15	2.7	0.88	452	525	578	677	348	278
1.8	-0.15	2.7	1.32	457	534	585	693	368	297
1.8	-0.15	3.6	0.44	449	525	549	661	334	267
1.8	-0.15	3.6	0.88	472	543	581	708	359	295
1.8	-0.15	3.6	1.32	461	530	559	663	353	294
1.8	-0.15	4.5	0.44	475	550	594	691	342	274
1.8	-0.15	4.5	0.88	448	525	562	682	370	308
1.8	-0.15	4.5	1.32	470	548	574	695	366	304

Table 4.7: Simulation of 5% Normality Tests on α-stable Samples of Size 200 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	11	271	47	102	5	1
1.6	0	2.7	0.88	16	283	34	112	11	6
1.6	0	2.7	1.32	15	287	37	99	9	8
1.6	0	3.6	0.44	9	280	33	84	5	3
1.6	0	3.6	0.88	10	288	30	106	7	4
1.6	0	3.6	1.32	15	294	34	96	6	3
1.6	0	4.5	0.44	13	270	42	104	9	6
1.6	0	4.5	0.88	14	288	26	83	9	5
1.6	0	4.5	1.32	12	270	34	98	3	3
1.6	-0.075	2.7	0.44	10	279	38	102	4	3
1.6	-0.075	2.7	0.88	9	276	30	106	7	3
1.6	-0.075	2.7	1.32	7	286	22	88	4	1

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Table 4.7: Simulation of 5% Normality Tests on α-stable Samples of Size 200 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	-0.075	3.6	0.44	9	288	31	91	6	4
1.6	-0.075	3.6	0.88	12	276	27	87	7	4
1.6	-0.075	3.6	1.32	12	289	32	104	7	4
1.6	-0.075	4.5	0.44	14	294	45	124	9	6
1.6	-0.075	4.5	0.88	14	295	46	113	7	2
1.6	-0.075	4.5	1.32	7	262	28	88	3	2
1.6	-0.15	2.7	0.44	8	277	31	105	3	2
1.6	-0.15	2.7	0.88	8	272	27	87	5	3
1.6	-0.15	2.7	1.32	9	291	32	93	4	4
1.6	-0.15	3.6	0.44	8	262	43	116	5	3
1.6	-0.15	3.6	0.88	10	260	34	102	7	2
1.6	-0.15	3.6	1.32	5	295	23	91	2	0
1.6	-0.15	4.5	0.44	13	285	35	90	5	1
1.6	-0.15	4.5	0.88	10	283	34	103	3	1
1.6	-0.15	4.5	1.32	6	294	28	93	4	2
1.7	0	2.7	0.44	56	224	128	259	30	19
1.7	0	2.7	0.88	58	244	141	264	35	20
1.7	0	2.7	1.32	64	256	132	272	33	20
1.7	0	3.6	0.44	61	233	133	275	29	21
1.7	0	3.6	0.88	59	238	151	267	26	20
1.7	0	3.6	1.32	59	217	125	234	28	25
1.7	0	4.5	0.44	68	236	138	253	34	21
1.7	0	4.5	0.88	57	241	131	259	26	18
1.7	0	4.5	1.32	61	244	128	269	30	18
1.7	-0.075	2.7	0.44	75	211	156	308	44	29
1.7	-0.075	2.7	0.88	66	242	147	288	41	26
1.7	-0.075	2.7	1.32	72	226	140	272	38	18

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Table 4.7: Simulation of 5% Normality Tests on α-stable Samples of Size 200 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	-0.075	3.6	0.44	66	224	135	258	30	13
1.7	-0.075	3.6	0.88	66	228	130	245	37	21
1.7	-0.075	3.6	1.32	52	211	111	246	28	15
1.7	-0.075	4.5	0.44	68	228	133	265	41	23
1.7	-0.075	4.5	0.88	71	214	143	272	41	24
1.7	-0.075	4.5	1.32	67	238	133	258	32	16
1.7	-0.15	2.7	0.44	52	218	129	266	30	20
1.7	-0.15	2.7	0.88	66	214	136	275	36	20
1.7	-0.15	2.7	1.32	55	203	134	264	35	19
1.7	-0.15	3.6	0.44	57	240	124	253	35	21
1.7	-0.15	3.6	0.88	55	225	112	232	27	13
1.7	-0.15	3.6	1.32	79	255	157	296	40	23
1.7	-0.15	4.5	0.44	68	222	141	276	36	27
1.7	-0.15	4.5	0.88	60	234	133	270	32	20
1.7	-0.15	4.5	1.32	57	219	131	267	28	14
1.8	0	2.7	0.44	225	348	364	505	128	84
1.8	0	2.7	0.88	247	375	360	508	151	107
1.8	0	2.7	1.32	230	365	349	515	133	94
1.8	0	3.6	0.44	241	351	370	551	149	101
1.8	0	3.6	0.88	230	345	366	541	133	98
1.8	0	3.6	1.32	232	347	360	513	156	114
1.8	0	4.5	0.44	240	367	363	503	131	101
1.8	0	4.5	0.88	243	354	349	519	142	101
1.8	0	4.5	1.32	203	323	344	507	114	79
1.8	-0.075	2.7	0.44	233	359	365	523	139	97
1.8	-0.075	2.7	0.88	233	363	357	493	115	84
1.8	-0.075	2.7	1.32	242	380	366	536	146	110

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Table 4.7: Simulation of 5% Normality Tests on α-stable Samples of Size 200 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.8	-0.075	3.6	0.44	227	340	345	524	134	89
1.8	-0.075	3.6	0.88	239	353	369	498	137	101
1.8	-0.075	3.6	1.32	238	377	379	511	161	108
1.8	-0.075	4.5	0.44	234	346	375	523	135	107
1.8	-0.075	4.5	0.88	208	335	306	477	118	84
1.8	-0.075	4.5	1.32	200	315	342	504	119	90
1.8	-0.15	2.7	0.44	217	341	366	523	133	101
1.8	-0.15	2.7	0.88	226	362	354	515	138	96
1.8	-0.15	2.7	1.32	195	309	313	474	123	90
1.8	-0.15	3.6	0.44	191	312	343	509	106	72
1.8	-0.15	3.6	0.88	219	347	347	496	126	93
1.8	-0.15	3.6	1.32	239	354	354	526	151	107
1.8	-0.15	4.5	0.44	224	341	367	519	135	92
1.8	-0.15	4.5	0.88	219	333	360	518	137	91
1.8	-0.15	4.5	1.32	257	384	389	537	146	100

Table 4.8: Simulation of 1% Normality Tests on α-stable Samples of Size 50 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	417	489	541	642	368	312
1.6	0	2.7	0.88	440	505	570	695	381	327
1.6	0	2.7	1.32	447	493	555	674	389	335
1.6	0	3.6	0.44	403	456	540	647	345	300
1.6	0	3.6	0.88	470	525	593	697	401	340

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Table 4.8: Simulation of 1% Normality Tests on α-stable Samples of Size 50 (1000 Replications) *continued*

α-Stable Parameters				No. of replications where normality hypothesis accepted					
α	β	γ	δ	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
1.6	0	3.6	1.32	417	481	543	652	365	319
1.6	0	4.5	0.44	448	494	549	652	371	318
1.6	0	4.5	0.88	438	499	563	667	360	302
1.6	0	4.5	1.32	432	481	552	668	351	298
1.6	-0.075	2.7	0.44	426	489	560	662	366	316
1.6	-0.075	2.7	0.88	411	465	538	668	335	291
1.6	-0.075	2.7	1.32	443	503	546	660	383	323
1.6	-0.075	3.6	0.44	432	485	551	662	358	293
1.6	-0.075	3.6	0.88	439	491	570	685	388	337
1.6	-0.075	3.6	1.32	445	510	571	670	383	319
1.6	-0.075	4.5	0.44	438	488	565	677	372	330
1.6	-0.075	4.5	0.88	420	477	540	665	350	305
1.6	-0.075	4.5	1.32	439	488	559	664	379	315
1.6	-0.15	2.7	0.44	425	485	548	636	364	322
1.6	-0.15	2.7	0.88	447	507	563	676	391	342
1.6	-0.15	2.7	1.32	419	464	535	649	341	299
1.6	-0.15	3.6	0.44	446	507	555	672	372	311
1.6	-0.15	3.6	0.88	429	482	554	646	351	308
1.6	-0.15	3.6	1.32	416	481	549	649	353	295
1.6	-0.15	4.5	0.44	404	461	515	634	342	269
1.6	-0.15	4.5	0.88	443	497	558	643	366	309
1.6	-0.15	4.5	1.32	415	476	535	637	349	300
1.7	0	2.7	0.44	590	640	694	764	511	436
1.7	0	2.7	0.88	589	646	704	806	531	456
1.7	0	2.7	1.32	571	614	676	766	478	420
1.7	0	3.6	0.44	567	613	660	758	490	434
1.7	0	3.6	0.88	566	611	684	787	503	441

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Table 4.8: Simulation of 1% Normality Tests on α-stable Samples of Size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	0	3.6	1.32	585	642	715	780	504	435
1.7	0	4.5	0.44	584	635	698	764	499	434
1.7	0	4.5	0.88	596	645	698	777	512	440
1.7	0	4.5	1.32	597	649	708	805	520	460
1.7	-0.075	2.7	0.44	576	624	676	773	512	444
1.7	-0.075	2.7	0.88	580	638	687	772	490	439
1.7	-0.075	2.7	1.32	562	600	671	767	484	432
1.7	-0.075	3.6	0.44	585	636	690	779	515	438
1.7	-0.075	3.6	0.88	561	598	672	762	486	411
1.7	-0.075	3.6	1.32	584	638	689	783	500	435
1.7	-0.075	4.5	0.44	600	647	700	777	527	458
1.7	-0.075	4.5	0.88	612	658	704	784	517	451
1.7	-0.075	4.5	1.32	585	634	696	787	500	421
1.7	-0.15	2.7	0.44	581	632	711	772	503	454
1.7	-0.15	2.7	0.88	581	624	699	789	507	441
1.7	-0.15	2.7	1.32	591	640	706	790	498	439
1.7	-0.15	3.6	0.44	605	640	719	807	522	460
1.7	-0.15	3.6	0.88	608	660	708	793	527	473
1.7	-0.15	3.6	1.32	592	637	699	774	500	446
1.7	-0.15	4.5	0.44	552	603	666	773	480	431
1.7	-0.15	4.5	0.88	597	647	706	790	507	440
1.7	-0.15	4.5	1.32	588	641	693	788	507	456
1.8	0	2.7	0.44	725	763	816	874	629	570
1.8	0	2.7	0.88	731	765	812	858	664	594
1.8	0	2.7	1.32	751	792	833	876	663	607
1.8	0	3.6	0.44	750	792	820	871	664	591
1.8	0	3.6	0.88	754	781	813	864	675	618

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Table 4.8: Simulation of 1% Normality Tests on α-stable Samples of Size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-	Cramer-			Shapiro-	Jarque-
α	β	γ	δ	Darling	von Mises	Lilliefors	Pearson	Wilk	Bera
1.8	0	3.6	1.32	738	773	819	861	666	601
1.8	0	4.5	0.44	741	773	817	854	648	594
1.8	0	4.5	0.88	720	757	811	864	640	578
1.8	0	4.5	1.32	734	774	827	882	665	609
1.8	-0.075	2.7	0.44	749	786	836	899	671	592
1.8	-0.075	2.7	0.88	717	751	806	853	646	590
1.8	-0.075	2.7	1.32	734	776	817	887	659	592
1.8	-0.075	3.6	0.44	748	781	829	872	668	594
1.8	-0.075	3.6	0.88	721	767	816	863	648	601
1.8	-0.075	3.6	1.32	751	774	833	881	677	616
1.8	-0.075	4.5	0.44	740	779	828	882	652	591
1.8	-0.075	4.5	0.88	741	777	821	870	673	607
1.8	-0.075	4.5	1.32	748	768	808	864	664	622
1.8	-0.15	2.7	0.44	758	789	835	879	675	613
1.8	-0.15	2.7	0.88	715	745	793	857	625	568
1.8	-0.15	2.7	1.32	755	784	829	873	678	622
1.8	-0.15	3.6	0.44	716	749	803	863	642	583
1.8	-0.15	3.6	0.88	746	783	831	876	670	604
1.8	-0.15	3.6	1.32	733	776	807	858	665	596
1.8	-0.15	4.5	0.44	747	778	824	866	660	599
1.8	-0.15	4.5	0.88	739	775	814	880	658	587
1.8	-0.15	4.5	1.32	723	767	817	872	652	594

Table 4.9: Simulation of 1% Normality Tests on α-stable Samples of Size 100 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	6	191	20	268	3	52
1.6	0	2.7	0.88	14	181	27	247	7	42
1.6	0	2.7	1.32	10	184	27	245	7	37
1.6	0	3.6	0.44	3	167	14	256	2	45
1.6	0	3.6	0.88	7	189	20	240	3	39
1.6	0	3.6	1.32	10	195	24	260	4	38
1.6	0	4.5	0.44	6	204	18	257	4	38
1.6	0	4.5	0.88	7	179	13	218	7	38
1.6	0	4.5	1.32	9	193	21	251	3	35
1.6	-0.075	2.7	0.44	6	206	21	250	3	49
1.6	-0.075	2.7	0.88	7	181	22	258	4	43
1.6	-0.075	2.7	1.32	4	193	16	250	2	36
1.6	-0.075	3.6	0.44	5	209	16	233	4	50
1.6	-0.075	3.6	0.88	9	212	18	271	6	46
1.6	-0.075	3.6	1.32	7	191	20	242	5	32
1.6	-0.075	4.5	0.44	9	213	28	235	5	45
1.6	-0.075	4.5	0.88	10	198	23	273	5	39
1.6	-0.075	4.5	1.32	5	160	16	272	1	35
1.6	0.15	2.7	0.44	4	207	13	264	1	51
1.6	0.15	2.7	0.88	6	180	13	226	3	39
1.6	0.15	2.7	1.32	4	191	17	235	3	38
1.6	0.15	3.6	0.44	5	194	20	229	4	42
1.6	0.15	3.6	0.88	7	177	15	249	4	41
1.6	0.15	3.6	1.32	4	190	13	254	2	41
1.6	0.15	4.5	0.44	8	206	25	252	4	49
1.6	0.15	4.5	0.88	10	192	21	262	1	33

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Table 4.9: Simulation of 1% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-	Cramer-			Shapiro-	Jarque-
α	β	γ	δ	Darling	von Mises	Lilliefors	Pearson	Wilk	Bera
1.6	0.15	4.5	1.32	4	206	16	225	3	39
1.7	0	2.7	0.44	45	288	92	430	23	123
1.7	0	2.7	0.88	42	286	100	430	24	108
1.7	0	2.7	1.32	42	263	82	396	24	95
1.7	0	3.6	0.44	38	285	90	418	27	102
1.7	0	3.6	0.88	40	275	96	408	22	86
1.7	0	3.6	1.32	38	255	78	437	21	108
1.7	0	4.5	0.44	50	283	94	421	24	99
1.7	0	4.5	0.88	35	293	89	444	19	117
1.7	0	4.5	1.32	42	289	88	413	23	105
1.7	-0.075	2.7	0.44	55	273	109	398	36	97
1.7	-0.075	2.7	0.88	46	269	103	413	31	106
1.7	-0.075	2.7	1.32	53	285	97	437	26	113
1.7	-0.075	3.6	0.44	41	285	84	396	20	123
1.7	-0.075	3.6	0.88	51	279	89	417	28	115
1.7	-0.075	3.6	1.32	31	293	79	411	21	99
1.7	-0.075	4.5	0.44	53	272	98	402	31	100
1.7	-0.075	4.5	0.88	49	279	101	423	30	115
1.7	-0.075	4.5	1.32	52	260	91	417	19	102
1.7	0.15	2.7	0.44	31	290	85	438	19	116
1.7	0.15	2.7	0.88	47	259	95	413	25	111
1.7	0.15	2.7	1.32	39	276	87	392	22	92
1.7	0.15	3.6	0.44	43	307	87	448	26	116
1.7	0.15	3.6	0.88	41	265	78	404	16	112
1.7	0.15	3.6	1.32	54	300	111	440	30	122
1.7	0.15	4.5	0.44	50	279	89	422	29	119
1.7	0.15	4.5	0.88	45	286	92	423	25	121

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Table 4.9: Simulation of 1% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	0.15	4.5	1.32	43	298	87	426	21	104
1.8	0	2.7	0.44	175	456	285	595	101	222
1.8	0	2.7	0.88	200	471	283	606	125	228
1.8	0	2.7	1.32	182	423	288	573	107	234
1.8	0	3.6	0.44	189	455	308	605	122	271
1.8	0	3.6	0.88	185	448	293	587	107	240
1.8	0	3.6	1.32	182	472	278	594	125	235
1.8	0	4.5	0.44	187	464	297	611	110	254
1.8	0	4.5	0.88	197	442	271	586	116	237
1.8	0	4.5	1.32	148	483	257	610	95	262
1.8	-0.075	2.7	0.44	190	454	290	605	105	260
1.8	-0.075	2.7	0.88	175	456	274	595	91	231
1.8	-0.075	2.7	1.32	183	457	289	588	119	226
1.8	-0.075	3.6	0.44	173	462	263	612	106	262
1.8	-0.075	3.6	0.88	183	459	290	595	105	223
1.8	-0.075	3.6	1.32	192	432	303	577	133	245
1.8	-0.075	4.5	0.44	170	482	287	607	111	263
1.8	-0.075	4.5	0.88	168	451	246	597	97	251
1.8	-0.075	4.5	1.32	167	457	255	602	98	230
1.8	0.15	2.7	0.44	158	458	289	614	110	243
1.8	0.15	2.7	0.88	184	454	272	619	111	237
1.8	0.15	2.7	1.32	156	465	231	616	103	261
1.8	0.15	3.6	0.44	146	448	257	579	84	227
1.8	0.15	3.6	0.88	171	462	263	634	103	249
1.8	0.15	3.6	1.32	188	457	291	598	126	248
1.8	0.15	4.5	0.44	164	476	289	611	104	239
1.8	0.15	4.5	0.88	169	462	278	601	107	269

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Table 4.9: Simulation of 1% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

α-Stable Parameters				No. of replications where normality hypothesis accepted					
α	β	γ	δ	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
1.8	0.15	4.5	1.32	207	481	319	618	114	268

Table 4.10: Simulation of 1% Normality Tests on α-stable Samples of Size 200 (1000 Replications)

α-Stable Parameters				No. of replications where normality hypothesis accepted					
α	β	γ	δ	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
1.6	0	2.7	0.44	24	298	90	177	11	6
1.6	0	2.7	0.88	28	313	80	183	16	10
1.6	0	2.7	1.32	31	302	79	160	17	10
1.6	0	3.6	0.44	24	301	69	148	8	5
1.6	0	3.6	0.88	26	314	74	184	15	11
1.6	0	3.6	1.32	29	313	74	159	7	6
1.6	0	4.5	0.44	28	304	85	164	20	15
1.6	0	4.5	0.88	24	302	75	155	13	9
1.6	0	4.5	1.32	22	290	74	170	13	3
1.6	-0.075	2.7	0.44	25	307	82	165	13	9
1.6	-0.075	2.7	0.88	19	300	75	177	10	9
1.6	-0.075	2.7	1.32	19	309	63	161	6	3
1.6	-0.075	3.6	0.44	18	306	58	161	10	7
1.6	-0.075	3.6	0.88	23	291	62	154	12	7
1.6	-0.075	3.6	1.32	22	311	73	186	13	9
1.6	-0.075	4.5	0.44	31	322	91	191	16	7
1.6	-0.075	4.5	0.88	30	326	89	188	14	9
1.6	-0.075	4.5	1.32	23	286	62	165	6	4
1.6	-0.15	2.7	0.44	25	302	70	179	11	8

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Table 4.10: Simulation of 1% Normality tests on α-stable samples of size 200 (1000 replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-	Cramer-			Shapiro-	Jarque-
α	β	γ	δ	Darling	von Mises	Lilliefors	Pearson	Wilk	Bera
1.6	-0.15	2.7	0.88	17	293	75	165	11	9
1.6	-0.15	2.7	1.32	23	317	70	155	8	5
1.6	-0.15	3.6	0.44	34	295	86	192	16	7
1.6	-0.15	3.6	0.88	20	277	82	176	11	9
1.6	-0.15	3.6	1.32	14	317	70	151	3	2
1.6	-0.15	4.5	0.44	24	309	69	181	13	6
1.6	-0.15	4.5	0.88	26	304	72	170	9	3
1.6	-0.15	4.5	1.32	18	316	62	164	10	5
1.7	0	2.7	0.44	106	294	228	371	51	33
1.7	0	2.7	0.88	116	304	227	392	59	37
1.7	0	2.7	1.32	109	317	242	381	56	34
1.7	0	3.6	0.44	103	307	246	387	51	35
1.7	0	3.6	0.88	111	304	247	377	47	29
1.7	0	3.6	1.32	107	279	212	355	55	33
1.7	0	4.5	0.44	115	296	219	377	56	42
1.7	0	4.5	0.88	103	306	219	364	47	27
1.7	0	4.5	1.32	109	306	219	372	60	35
1.7	-0.075	2.7	0.44	128	273	259	428	62	47
1.7	-0.075	2.7	0.88	131	316	256	390	63	46
1.7	-0.075	2.7	1.32	121	297	235	377	60	40
1.7	-0.075	3.6	0.44	114	284	228	380	50	32
1.7	-0.075	3.6	0.88	93	279	224	357	52	39
1.7	-0.075	3.6	1.32	97	274	212	367	47	29
1.7	-0.075	4.5	0.44	108	294	226	379	71	39
1.7	-0.075	4.5	0.88	127	286	247	389	72	42
1.7	-0.075	4.5	1.32	108	297	223	378	62	31
1.7	-0.15	2.7	0.44	99	288	237	377	50	32

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Table 4.10: Simulation of 1% Normality tests on α-stable samples of size 200 (1000 replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-	Cramer-			Shapiro-	Jarque-
α	β	γ	δ	Darling	von Mises	Lilliefors	Pearson	Wilk	Bera
1.7	-0.15	2.7	0.88	111	279	220	381	53	36
1.7	-0.15	2.7	1.32	94	269	235	389	47	34
1.7	-0.15	3.6	0.44	104	305	231	370	52	40
1.7	-0.15	3.6	0.88	95	280	197	359	49	33
1.7	-0.15	3.6	1.32	127	330	262	404	62	46
1.7	-0.15	4.5	0.44	111	282	252	394	53	41
1.7	-0.15	4.5	0.88	104	290	240	381	50	30
1.7	-0.15	4.5	1.32	112	279	220	395	45	30
1.8	0	2.7	0.44	336	468	497	622	187	129
1.8	0	2.7	0.88	351	474	520	633	213	156
1.8	0	2.7	1.32	343	477	518	626	193	139
1.8	0	3.6	0.44	344	470	506	663	193	151
1.8	0	3.6	0.88	350	476	530	674	201	145
1.8	0	3.6	1.32	337	456	498	638	211	159
1.8	0	4.5	0.44	343	466	501	616	197	140
1.8	0	4.5	0.88	331	462	499	646	195	146
1.8	0	4.5	1.32	300	437	486	619	170	116
1.8	-0.075	2.7	0.44	336	454	494	647	191	135
1.8	-0.075	2.7	0.88	330	476	492	612	195	125
1.8	-0.075	2.7	1.32	346	497	530	662	199	154
1.8	-0.075	3.6	0.44	322	464	500	639	194	135
1.8	-0.075	3.6	0.88	335	463	495	623	185	137
1.8	-0.075	3.6	1.32	364	491	504	636	214	162
1.8	-0.075	4.5	0.44	341	472	505	647	197	141
1.8	-0.075	4.5	0.88	301	438	452	608	176	123
1.8	-0.075	4.5	1.32	305	449	492	636	171	129
1.8	-0.15	2.7	0.44	330	472	511	631	190	145

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Table 4.10: Simulation of 1% Normality tests on α-stable samples of size 200 (1000 replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.8	-0.15	2.7	0.88	330	463	498	625	192	138
1.8	-0.15	2.7	1.32	277	415	440	595	169	124
1.8	-0.15	3.6	0.44	299	429	484	643	164	120
1.8	-0.15	3.6	0.88	316	457	481	630	174	137
1.8	-0.15	3.6	1.32	337	467	501	657	206	153
1.8	-0.15	4.5	0.44	321	459	493	630	179	134
1.8	-0.15	4.5	0.88	318	459	503	642	182	130
1.8	-0.15	4.5	1.32	361	488	520	637	218	150

Table 4.11: Simulation of 10% Normality Tests on α-stable Samples of Size 50 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	259	311	332	442	233	195
1.6	0	2.7	0.88	271	317	374	467	253	197
1.6	0	2.7	1.32	285	338	367	476	253	210
1.6	0	3.6	0.44	234	287	327	445	221	185
1.6	0	3.6	0.88	286	336	382	487	246	197
1.6	0	3.6	1.32	270	321	343	438	249	211
1.6	0	4.5	0.44	283	335	372	486	232	182
1.6	0	4.5	0.88	266	302	353	471	234	189
1.6	0	4.5	1.32	257	304	341	449	220	168
1.6	-0.075	2.7	0.44	255	295	350	447	231	189
1.6	-0.075	2.7	0.88	252	299	324	431	224	172
1.6	-0.075	2.7	1.32	279	322	359	463	237	188

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Table 4.11: Simulation of 10% Normality tests on α-stable samples of size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	-0.075	3.6	0.44	253	293	356	460	221	176
1.6	-0.075	3.6	0.88	289	325	357	466	262	215
1.6	-0.075	3.6	1.32	284	332	354	458	238	192
1.6	-0.075	4.5	0.44	275	314	349	461	257	207
1.6	-0.075	4.5	0.88	259	303	334	459	229	180
1.6	-0.075	4.5	1.32	271	315	359	464	243	198
1.6	-0.15	2.7	0.44	263	307	346	446	243	200
1.6	-0.15	2.7	0.88	287	339	365	487	246	191
1.6	-0.15	2.7	1.32	261	303	337	454	232	179
1.6	-0.15	3.6	0.44	277	320	358	456	243	195
1.6	-0.15	3.6	0.88	255	307	335	443	223	187
1.6	-0.15	3.6	1.32	262	307	329	443	224	178
1.6	-0.15	4.5	0.44	235	275	318	435	195	152
1.6	-0.15	4.5	0.88	252	306	338	462	226	179
1.6	-0.15	4.5	1.32	250	291	330	435	223	181
1.7	0	2.7	0.44	428	483	513	572	361	298
1.7	0	2.7	0.88	405	446	490	611	359	302
1.7	0	2.7	1.32	392	437	473	582	354	289
1.7	0	3.6	0.44	386	431	470	572	343	283
1.7	0	3.6	0.88	407	443	495	574	339	275
1.7	0	3.6	1.32	401	445	501	589	366	308
1.7	0	4.5	0.44	385	425	490	583	346	278
1.7	0	4.5	0.88	412	459	496	600	364	294
1.7	0	4.5	1.32	419	460	501	609	372	309
1.7	-0.075	2.7	0.44	410	451	489	591	353	279
1.7	-0.075	2.7	0.88	403	441	502	605	365	298
1.7	-0.075	2.7	1.32	387	431	474	580	360	309

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Table 4.11: Simulation of 10% Normality tests on α-stable samples of size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	-0.075	3.6	0.44	408	448	493	584	346	279
1.7	-0.075	3.6	0.88	375	406	469	571	337	275
1.7	-0.075	3.6	1.32	401	455	505	613	349	290
1.7	-0.075	4.5	0.44	426	463	508	612	361	300
1.7	-0.075	4.5	0.88	414	463	525	629	370	299
1.7	-0.075	4.5	1.32	401	444	476	613	342	287
1.7	-0.15	2.7	0.44	409	454	481	580	373	315
1.7	-0.15	2.7	0.88	395	441	489	592	346	273
1.7	-0.15	2.7	1.32	387	425	487	577	349	299
1.7	-0.15	3.6	0.44	412	449	494	600	360	289
1.7	-0.15	3.6	0.88	423	472	521	606	376	317
1.7	-0.15	3.6	1.32	397	437	478	592	346	289
1.7	-0.15	4.5	0.44	386	427	462	585	344	275
1.7	-0.15	4.5	0.88	407	456	510	602	353	317
1.7	-0.15	4.5	1.32	421	465	502	607	371	300
1.8	0	2.7	0.44	532	562	625	702	471	418
1.8	0	2.7	0.88	561	593	627	705	507	453
1.8	0	2.7	1.32	574	612	647	715	515	448
1.8	0	3.6	0.44	551	601	643	703	502	428
1.8	0	3.6	0.88	583	621	641	717	530	471
1.8	0	3.6	1.32	568	605	633	692	522	465
1.8	0	4.5	0.44	561	603	632	701	504	435
1.8	0	4.5	0.88	550	595	627	718	493	425
1.8	0	4.5	1.32	583	614	651	714	535	463
1.8	-0.075	2.7	0.44	538	586	625	729	502	435
1.8	-0.075	2.7	0.88	553	587	630	698	502	443
1.8	-0.075	2.7	1.32	555	608	633	694	505	445

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Table 4.11: Simulation of 10% Normality tests on α-stable samples of size 50 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.8	-0.075	3.6	0.44	569	610	642	698	511	444
1.8	-0.075	3.6	0.88	558	597	639	713	515	447
1.8	-0.075	3.6	1.32	577	614	633	719	528	461
1.8	-0.075	4.5	0.44	563	585	629	713	515	468
1.8	-0.075	4.5	0.88	577	617	660	725	517	450
1.8	-0.075	4.5	1.32	573	610	640	721	523	457
1.8	-0.15	2.7	0.44	587	619	650	731	525	472
1.8	-0.15	2.7	0.88	535	574	611	711	480	424
1.8	-0.15	2.7	1.32	577	607	649	703	521	451
1.8	-0.15	3.6	0.44	554	596	628	691	502	419
1.8	-0.15	3.6	0.88	575	606	650	727	514	448
1.8	-0.15	3.6	1.32	581	619	642	702	522	446
1.8	-0.15	4.5	0.44	569	611	655	717	525	447
1.8	-0.15	4.5	0.88	548	585	634	713	501	437
1.8	-0.15	4.5	1.32	554	589	629	702	508	447

Table 4.12: Simulation of 10% Normality Tests on α-stable Samples of Size 100 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	99	191	163	268	69	52
1.6	0	2.7	0.88	86	181	129	247	58	42
1.6	0	2.7	1.32	81	184	130	245	58	37
1.6	0	3.6	0.44	70	167	128	256	50	45
1.6	0	3.6	0.88	79	189	137	240	60	39

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Table 4.12: Simulation of 10% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

α-Stable Parameters				No. of replications where normality hypothesis accepted					
α	β	γ	δ	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
1.6	0	3.6	1.32	82	195	137	260	61	38
1.6	0	4.5	0.44	75	204	145	257	52	38
1.6	0	4.5	0.88	72	179	124	218	54	38
1.6	0	4.5	1.32	73	193	125	251	59	35
1.6	-0.075	2.7	0.44	88	206	145	250	70	49
1.6	-0.075	2.7	0.88	76	181	137	258	60	43
1.6	-0.075	2.7	1.32	75	193	133	250	52	36
1.6	-0.075	3.6	0.44	95	209	141	233	79	50
1.6	-0.075	3.6	0.88	97	212	144	271	67	46
1.6	-0.075	3.6	1.32	70	191	135	242	50	32
1.6	-0.075	4.5	0.44	88	213	129	235	64	45
1.6	-0.075	4.5	0.88	78	198	144	273	53	39
1.6	-0.075	4.5	1.32	60	160	142	272	48	35
1.6	-0.15	2.7	0.44	90	207	150	264	68	51
1.6	-0.15	2.7	0.88	71	180	117	226	58	39
1.6	-0.15	2.7	1.32	76	191	129	235	53	38
1.6	-0.15	3.6	0.44	81	194	132	229	55	42
1.6	-0.15	3.6	0.88	69	177	130	249	52	41
1.6	-0.15	3.6	1.32	76	190	142	254	54	41
1.6	-0.15	4.5	0.44	82	206	145	252	61	49
1.6	-0.15	4.5	0.88	89	192	149	262	68	33
1.6	-0.15	4.5	1.32	85	206	131	225	63	39
1.7	0	2.7	0.44	203	288	287	430	165	123
1.7	0	2.7	0.88	193	286	288	430	162	108
1.7	0	2.7	1.32	167	263	266	396	126	95
1.7	0	3.6	0.44	190	285	291	418	142	102
1.7	0	3.6	0.88	184	275	268	408	130	86

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Table 4.12: Simulation of 10% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.7	0	3.6	1.32	187	255	279	437	136	108
1.7	0	4.5	0.44	192	283	283	421	139	99
1.7	0	4.5	0.88	211	293	290	444	155	117
1.7	0	4.5	1.32	196	289	295	413	148	105
1.7	-0.075	2.7	0.44	179	273	254	398	128	97
1.7	-0.075	2.7	0.88	201	269	281	413	147	106
1.7	-0.075	2.7	1.32	191	285	297	437	155	113
1.7	-0.075	3.6	0.44	212	285	290	396	152	123
1.7	-0.075	3.6	0.88	189	279	276	417	143	115
1.7	-0.075	3.6	1.32	196	293	297	411	142	99
1.7	-0.075	4.5	0.44	183	272	278	402	136	100
1.7	-0.075	4.5	0.88	197	279	282	423	147	115
1.7	-0.075	4.5	1.32	185	260	278	417	139	102
1.7	-0.15	2.7	0.44	199	290	278	438	152	116
1.7	-0.15	2.7	0.88	181	259	269	413	143	111
1.7	-0.15	2.7	1.32	178	276	270	392	130	92
1.7	-0.15	3.6	0.44	207	307	301	448	160	116
1.7	-0.15	3.6	0.88	188	265	269	404	154	112
1.7	-0.15	3.6	1.32	196	300	309	440	159	122
1.7	-0.15	4.5	0.44	186	279	284	422	147	119
1.7	-0.15	4.5	0.88	203	286	299	423	163	121
1.7	-0.15	4.5	1.32	191	298	298	426	138	104
1.8	0	2.7	0.44	376	456	483	595	288	222
1.8	0	2.7	0.88	381	471	500	606	308	228
1.8	0	2.7	1.32	360	423	455	573	287	234
1.8	0	3.6	0.44	380	455	470	605	316	271
1.8	0	3.6	0.88	386	448	468	587	302	240

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Table 4.12: Simulation of 10% Normality Tests on α-stable Samples of Size 100 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.8	0	3.6	1.32	393	472	500	594	305	235
1.8	0	4.5	0.44	393	464	502	611	314	254
1.8	0	4.5	0.88	373	442	460	586	297	237
1.8	0	4.5	1.32	411	483	495	610	326	262
1.8	-0.075	2.7	0.44	404	454	498	605	314	260
1.8	-0.075	2.7	0.88	380	456	482	595	281	231
1.8	-0.075	2.7	1.32	391	457	502	588	302	226
1.8	-0.075	3.6	0.44	403	462	510	612	320	262
1.8	-0.075	3.6	0.88	389	459	481	595	304	223
1.8	-0.075	3.6	1.32	374	432	474	577	299	245
1.8	-0.075	4.5	0.44	384	482	490	607	322	263
1.8	-0.075	4.5	0.88	371	451	479	597	305	251
1.8	-0.075	4.5	1.32	392	457	495	602	296	230
1.8	-0.15	2.7	0.44	400	458	498	614	307	243
1.8	-0.15	2.7	0.88	392	454	477	619	302	237
1.8	-0.15	2.7	1.32	392	465	495	616	315	261
1.8	-0.15	3.6	0.44	364	448	470	579	286	227
1.8	-0.15	3.6	0.88	392	462	495	634	320	249
1.8	-0.15	3.6	1.32	374	457	477	598	309	248
1.8	-0.15	4.5	0.44	397	476	505	611	296	239
1.8	-0.15	4.5	0.88	391	462	463	601	320	269
1.8	-0.15	4.5	1.32	414	481	499	618	324	268

Table 4.13: Simulation of 10% Normality Tests on α-stable Samples of Size 200 (1000 Replications)

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	0	2.7	0.44	6	262	20	84	3	1
1.6	0	2.7	0.88	14	276	27	89	7	3
1.6	0	2.7	1.32	10	278	27	73	7	6
1.6	0	3.6	0.44	3	269	14	65	2	2
1.6	0	3.6	0.88	7	278	20	77	3	1
1.6	0	3.6	1.32	10	287	24	68	4	1
1.6	0	4.5	0.44	6	263	18	74	4	3
1.6	0	4.5	0.88	7	282	13	58	7	4
1.6	0	4.5	1.32	9	267	21	77	3	2
1.6	-0.075	2.7	0.44	6	269	21	75	3	2
1.6	-0.075	2.7	0.88	7	271	22	73	4	1
1.6	-0.075	2.7	1.32	4	284	16	67	2	1
1.6	-0.075	3.6	0.44	5	282	16	63	4	2
1.6	-0.075	3.6	0.88	9	272	18	69	6	4
1.6	-0.075	3.6	1.32	7	282	20	77	5	3
1.6	-0.075	4.5	0.44	9	281	28	80	5	5
1.6	-0.075	4.5	0.88	10	281	23	81	5	1
1.6	-0.075	4.5	1.32	5	252	16	60	1	1
1.6	-0.15	2.7	0.44	4	266	13	74	1	2
1.6	-0.15	2.7	0.88	6	265	13	62	3	2
1.6	-0.15	2.7	1.32	4	283	17	72	3	3
1.6	-0.15	3.6	0.44	5	251	20	86	4	2
1.6	-0.15	3.6	0.88	7	250	15	69	4	2
1.6	-0.15	3.6	1.32	4	284	13	61	2	0
1.6	-0.15	4.5	0.44	8	273	25	65	4	0
1.6	-0.15	4.5	0.88	10	265	21	79	1	0

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Table 4.13: Simulation of 10% Normality Tests on α-stable Samples of Size 200 (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
α-Stable Parameters				Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
α	β	γ	δ						
1.6	-0.15	4.5	1.32	4	288	16	63	3	1
1.7	0	2.7	0.44	45	204	92	199	23	18
1.7	0	2.7	0.88	42	217	100	195	24	15
1.7	0	2.7	1.32	42	225	82	211	24	14
1.7	0	3.6	0.44	38	208	90	213	27	16
1.7	0	3.6	0.88	40	211	96	215	22	17
1.7	0	3.6	1.32	38	200	78	186	21	14
1.7	0	4.5	0.44	50	215	94	206	24	16
1.7	0	4.5	0.88	35	220	89	203	19	12
1.7	0	4.5	1.32	42	215	88	200	23	14
1.7	-0.075	2.7	0.44	55	175	109	243	36	19
1.7	-0.075	2.7	0.88	46	208	103	218	31	19
1.7	-0.075	2.7	1.32	53	206	97	217	26	15
1.7	-0.075	3.6	0.44	41	200	84	207	20	9
1.7	-0.075	3.6	0.88	51	204	89	195	28	13
1.7	-0.075	3.6	1.32	31	193	79	192	21	14
1.7	-0.075	4.5	0.44	53	211	98	205	31	16
1.7	-0.075	4.5	0.88	49	192	101	215	30	15
1.7	-0.075	4.5	1.32	52	216	91	203	19	10
1.7	-0.15	2.7	0.44	31	189	85	208	19	18
1.7	-0.15	2.7	0.88	47	185	95	213	25	17
1.7	-0.15	2.7	1.32	39	179	87	219	22	14
1.7	-0.15	3.6	0.44	43	217	87	196	26	18
1.7	-0.15	3.6	0.88	41	206	78	164	16	13
1.7	-0.15	3.6	1.32	54	234	111	230	30	16
1.7	-0.15	4.5	0.44	50	197	89	212	29	22
1.7	-0.15	4.5	0.88	45	203	92	224	25	11

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Table 4.13: Simulation of 10% Normality Tests on α-stable Samples of Size 200 (1000 Replications) *continued*

α-Stable Parameters				No. of replications where normality hypothesis accepted					
α	β	γ	δ	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
1.7	-0.15	4.5	1.32	43	193	87	199	21	11
1.8	0	2.7	0.44	175	295	285	433	101	66
1.8	0	2.7	0.88	200	311	283	426	125	89
1.8	0	2.7	1.32	182	297	288	443	107	77
1.8	0	3.6	0.44	189	297	308	469	122	84
1.8	0	3.6	0.88	185	282	293	465	107	77
1.8	0	3.6	1.32	182	298	278	425	125	86
1.8	0	4.5	0.44	187	306	297	419	110	86
1.8	0	4.5	0.88	197	306	271	434	116	86
1.8	0	4.5	1.32	148	266	257	419	95	68
1.8	-0.075	2.7	0.44	190	306	290	439	105	72
1.8	-0.075	2.7	0.88	175	312	274	432	91	69
1.8	-0.075	2.7	1.32	183	322	289	460	119	85
1.8	-0.075	3.6	0.44	173	278	263	427	106	75
1.8	-0.075	3.6	0.88	183	289	290	427	105	83
1.8	-0.075	3.6	1.32	192	320	303	435	133	94
1.8	-0.075	4.5	0.44	170	280	287	448	111	87
1.8	-0.075	4.5	0.88	168	285	246	405	97	72
1.8	-0.075	4.5	1.32	167	267	255	422	98	73
1.8	-0.15	2.7	0.44	158	281	289	445	110	81
1.8	-0.15	2.7	0.88	184	304	272	437	111	84
1.8	-0.15	2.7	1.32	156	262	231	396	103	71
1.8	-0.15	3.6	0.44	146	248	257	424	84	52
1.8	-0.15	3.6	0.88	171	285	263	436	103	70
1.8	-0.15	3.6	1.32	188	296	291	448	126	87
1.8	-0.15	4.5	0.44	164	271	289	437	104	78
1.8	-0.15	4.5	0.88	169	274	278	429	107	73

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Table 4.13: Simulation of 10% Normality Tests on α-stable Samples of Size 200 (1000 Replications) *continued*

α-Stable Parameters				No. of replications where normality hypothesis accepted					
α	β	γ	δ	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
1.8	-0.15	4.5	1.32	207	322	319	470	114	85

Table 4.14: Simulation of Normality Tests on a Normal Distribution (1000 replications)

Simulation details				No. of replications where normality hypothesis accepted					
				Test					
sample size	test size	st.dev	mean	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
50	5	3.8	0.44	946	950	942	946	934	947
50	5	3.8	0.88	946	939	955	955	951	940
50	5	3.8	1.32	938	938	947	944	933	928
50	5	5.1	0.44	935	934	932	933	934	933
50	5	5.1	0.88	946	942	945	958	944	946
50	5	5.1	1.32	957	951	953	944	964	949
50	5	6.4	0.44	937	935	937	944	940	941
50	5	6.4	0.88	938	926	942	938	948	951
50	5	6.4	1.32	951	951	953	954	958	951
100	5	3.8	0.44	953	955	958	952	951	944
100	5	3.8	0.88	948	948	944	950	952	943
100	5	3.8	1.32	949	948	952	947	945	938
100	5	5.1	0.44	954	954	954	949	945	936
100	5	5.1	0.88	953	954	965	953	958	941
100	5	5.1	1.32	953	953	944	961	956	949
100	5	6.4	0.44	944	942	941	942	945	933
100	5	6.4	0.88	942	931	936	960	941	935
100	5	6.4	1.32	952	947	947	948	957	950

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Table 4.14: Simulation of Normality Tests on a Normal Distribution (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
Simulation details				Test					
sample size	test size	st.dev	mean	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
200	5	3.8	0.44	949	953	954	956	953	948
200	5	3.8	0.88	940	947	943	946	948	943
200	5	3.8	1.32	949	951	953	939	949	937
200	5	5.1	0.44	952	952	956	941	945	954
200	5	5.1	0.88	952	953	953	944	957	948
200	5	5.1	1.32	970	967	961	951	952	953
200	5	6.4	0.44	956	954	949	949	963	955
200	5	6.4	0.88	947	945	950	938	952	949
200	5	6.4	1.32	947	946	948	943	953	950
50	1	3.8	0.44	983	982	983	985	987	983
50	1	3.8	0.88	990	992	994	993	986	989
50	1	3.8	1.32	988	990	991	992	987	982
50	1	5.1	0.44	981	982	986	986	981	984
50	1	5.1	0.88	985	986	989	993	992	982
50	1	5.1	1.32	992	991	992	989	994	989
50	1	6.4	0.44	985	983	989	988	984	984
50	1	6.4	0.88	986	981	987	987	990	989
50	1	6.4	1.32	991	990	991	994	992	990
100	1	3.8	0.44	993	993	994	992	992	987
100	1	3.8	0.88	993	991	991	993	993	985
100	1	3.8	1.32	990	990	989	993	986	986
100	1	5.1	0.44	989	989	988	988	989	977
100	1	5.1	0.88	992	992	990	994	991	985
100	1	5.1	1.32	985	986	987	993	988	987
100	1	6.4	0.44	989	986	992	988	983	980
100	1	6.4	0.88	988	987	985	990	982	988

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Table 4.14: Simulation of Normality Tests on a Normal Distribution (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
Simulation details				Test					
sample size	test size	st.dev	mean	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
100	1	6.4	1.32	988	992	989	989	990	986
200	1	3.8	0.44	992	993	991	992	992	987
200	1	3.8	0.88	988	991	993	993	987	985
200	1	3.8	1.32	993	990	993	993	991	986
200	1	5.1	0.44	989	989	986	988	992	977
200	1	5.1	0.88	992	992	992	994	992	985
200	1	5.1	1.32	994	986	998	993	990	987
200	1	6.4	0.44	994	986	996	988	994	980
200	1	6.4	0.88	991	987	991	990	988	988
200	1	6.4	1.32	989	992	991	989	991	986
50	10	3.8	0.44	891	893	906	895	885	893
50	10	3.8	0.88	896	894	898	896	902	889
50	10	3.8	1.32	870	868	881	892	880	877
50	10	5.1	0.44	880	885	873	879	884	869
50	10	5.1	0.88	887	881	905	912	878	889
50	10	5.1	1.32	898	906	895	900	907	907
50	10	6.4	0.44	890	877	886	893	886	886
50	10	6.4	0.88	886	881	870	885	896	906
50	10	6.4	1.32	906	907	910	906	906	915
100	10	3.8	0.44	909	909	904	901	906	888
100	10	3.8	0.88	894	891	897	896	906	888
100	10	3.8	1.32	894	895	894	892	894	885
100	10	5.1	0.44	899	907	901	892	894	879
100	10	5.1	0.88	916	917	921	909	917	887
100	10	5.1	1.32	903	893	891	910	912	910
100	10	6.4	0.44	898	891	877	895	899	889

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Table 4.14: Simulation of Normality Tests on a Normal Distribution (1000 Replications) *continued*

				No. of replications where normality hypothesis accepted					
Simulation details				Test					
sample size	test size	st.dev	mean	Anderson-Darling	Cramer-von Mises	Lilliefors	Pearson	Shapiro-Wilk	Jarque-Bera
100	10	6.4	0.88	885	884	881	908	894	889
100	10	6.4	1.32	895	901	891	886	905	903
200	10	3.8	0.44	913	909	905	901	902	888
200	10	3.8	0.88	893	891	879	896	896	888
200	10	3.8	1.32	900	895	892	892	889	885
200	10	5.1	0.44	909	907	903	892	904	879
200	10	5.1	0.88	908	917	913	909	913	887
200	10	5.1	1.32	922	893	911	910	913	910
200	10	6.4	0.44	895	891	899	895	896	889
200	10	6.4	0.88	895	884	893	908	905	889
200	10	6.4	1.32	905	901	894	886	908	903

CHAPTER 5

Value at Risk (VaR) and the α-stable Distribution[1]

5.1 Introduction

Today Value at Risk (VaR) is the most common measure of risk used in many financial institutions. VaR at a $p\%$ level is estimated as the loss that might be exceeded $p\%$ of the time. Like many other models in finance it is often based on an assumption that losses follow a normal distribution. It is now well known that extreme losses are greater than, and occur much more often than, a normal distribution would predict. To allow for this, VaR measures are sometimes based on a t-distribution or on ARCH/GARCH systems with innovations having a normal or t-distribution. The emphasis here is on the use of an α-stable distribution.[2] Several other distributions or mixtures of distributions have been proposed but none has received universal acceptance

[1] This Chapter is based on a paper (Frain (2008b)) presented at:

- TCD Graduate Seminar, December 2007.
- Seminar, Kemmy Business School, University of Limerick, April 2008.
- IEA Annual Conference April 2008.

[2] For details of the α-stable distribution see Section 2.2 and Appendix A.

and it is probable that none ever will.

The purpose of this exercise is to evaluate the merit of calculating VaR on the assumption of an underlying non-normal α-stable distribution. Thus, we calculate VaR at various levels assuming that losses follow either a static α-Stable distribution or a TS-GARCH type process with α-stable innovations. The resulting estimates are compared with estimates obtained from static normal and t-distributions and GARCH(1,1) processes with normal and t-innovations. VaR is estimated for six total return[3] equity indices (ISEQ, CAC40, DAX30, FTSE100, S&P500, Dow Jones Composite (DJAC)) at 10%, 5%, 1%, 0.5% and 0.1% levels.

Section 5.2 of the Chapter gives a brief outline of the development and definition of VaR. The main results of the analysis are in Section 5.3. All parameter estimates are maximum likelihood estimates. Technical details and results of the estimations along with descriptions of the data and software used are in the appendices to this chapter. The results may be summarised as follows.

First, the normal distribution performs very badly even at the conventional 5% and 1% levels. It tends to overestimate VaR at the higher probability levels and underestimate at the lower. This is what one would expect given the observed fat tails of the returns distribution and the exponential decay in the tails of the normal distribution. It is misleading to management to the extent that they may agree to some investments that would not be accepted if a more accurate assessment of risk was used.

The t-distribution appears to perform very well, particularly in the tails of the distribution. Empirically it is marginally (but not statistically significantly) better than the α-stable. The simplicity of the t-distribution makes it an attractive alternative. While it appears to work well empirically there is no good theoretical or other economic explanation as to why this is so. Many time series models perform well but they can usually be regarded as a reduced form of some structural model. We do not know how a t-distribution can be a reduced form of a structural model. We would fear that results based on a t-distribution might not be robust.

[3]In calculating these indices it is assumed that dividends are reinvested in the portfolio.

The GARCH distributions with normal innovations performs somewhat better than the static normal distribution. Curiously the GARCH distribution with t-innovations does not perform as well as the static t-distribution but is better than the GARCH with normal innovations.

The static α-stable performance is about equivalent to the t-distribution but is excellent at the conventional levels. Extreme VaR at levels less than 1% tends to be conservative. Perhaps this is due to institutional factors within the exchange, takeovers of failing companies or to actions by supervisory authorities aimed at avoiding contagion. In such cases it is unlikely that any recorded loss is the private loss and does not include elements of the public cost of a failure.[4] The α-stable GARCH(1,1) model for losses provides the best measure of VaR. It gives good estimates at all VaR levels for all the indices considered. The theoretical justification for the good results is given in Appendix A. We have shown that the estimates of VaR derived from an α-stable distribution are feasible and are a useful addition to the toolbox of a risk manager or a financial regulator.

Section 5.4 summarises the analysis and sets out the conclusions that may be drawn from the analysis.

5.2 Value at Risk (VaR)

The worldwide equity crisis in 1987, the fall in the Japanese equity market in 1990, the Mexican peso crisis in 1994/95 and the severe losses suffered in various derivative transactions in the 1990s were a strong incentive to both market participants and regulators to measure and monitor market and other exposures to risk. Jorian (2007, p. 32) estimates losses in the 1990s publicly attributed to derivatives at over \$ 30 billion. Given the overall volume of derivative trading this is not an enormous sum. It is extremely problematic to the individual companies that incurred the losses. Financial regulators would also fear that losses such as these might have knock-on effects that would

[4] The estimates in this chapter were completed before the current period of volatility. If this period were included in the analysis it is likely that the stable distribution would be even closer to reality.

effect the efficient functioning of markets. Jorian (2007) lists five firms that each had losses of more than $ 1 billion attributed to derivative trading:[5]

- Orange County, California, December 1994, Reverse Repos, loss $1810 billion.
- Showa Shell Sekiyu, Japan, February 1993, Currency Forwards, loss $1580 billion.
- Kashima Oil, Japan, April 1994, Currency Forwards, loss $1450 billion.
- Metallgesellschaft, January 1994, Germany, Oil Futures, loss $1340 billion.
- Barings, U.K., February 1995, Stock Index Futures, loss $1330 billion.

One lesson to be learned from these and similar events was the need to introduce better methods of risk assessment and monitoring. At that time often simple rules based on guidelines like “high liquidity”, “low” interest rate risk, hedging, limits on amount invested, sectors etc. were often used. Such rules were often ambiguous or could easily be circumvented by “resourceful” traders. Many losses of the type outlined above were due to inadequate and/or circumvented supervisory controls. In the US the Sarbanes-Oxley Act of 2002 created a more rigourous legal environment for the board, the management committee, internal and external auditors, and the chief risk officer.

[5] Other recent large financial losses, not covered in this list, include:

- Society Generale, $ 7.2 bn. European Index Futures, (New York Times, January 25 2008).
- Amaranth Advisors, $ 6.5 bn. Leveraged Gas Futures, (MoneyWeek, 6 October 2006).
- LTCM, $4 bn. Convergence Arbitrage Hedge Fund, (Hull (2007)).
- Sumitomo, $2.6 bn. Copper Futures (The Japan Times, 26 March, 1998).
- DAIWA $1.1 bn. Illegal bond trading in 11 years from 1984 to 1995.

Dowd (2002), page 12, reports that LTCM had a risk model that suggested that the loss it suffered in the summer and autumn of 1998 was 14 times the standard deviation of its expected profit/loss and that a 14-sigma event should not occur once in the entire history of the universe. So LTCM were either very unlucky or their risk model was faulty. Each of the other four losses was attributed to the actions of one individual in the company.

These regulations apply to all companies with a quotation on a US exchange and thus apply to several large Irish companies. Management and directors of such institutions are now required to have risk measurement, audit and control systems in place and to report regularly on these. The financial regulatory authorities have now adopted the Basel II Capital Adequacy Directive which allows institutions to use, subject to approval, their internal risk measurement systems to determine capital adequacy for regulatory purposes.

Value-at-Risk (VaR) is a commonly used measure of the risk of an investment or a portfolio or even an entire institution. A p% VaR is the lower limit on the proportion of a portfolio that can be lost p% of the time. Thus a p% VaR is the $(100 - p)$% quantile of the loss distribution i.e. the p% VaR, V_p is given by

$$Prob[\text{ loss} \geq V_p] = p.$$

This is illustrated in figure 5.1 where the value at the left boundary of the shaded area represents the 5% VaR.

Thus if the daily loss on a portfolio is normally distributed with an expected value of 0.005% and a standard deviation of 0.010, one would expect to lose:

- more than 0.0114% 5% of the time.
- more than 0.0183% 1% of the time.

The daily VaR of the portfolio is then 0.0114% and 0.0183% at the 5% and 1% levels respectively. A properly implemented VaR includes all sources of risk and should encompass market, operational, credit, liquidity and model risk. VaR may be calculated at enterprise level, at various sector levels within the organisation and at individual trader level — the VaR at lower levels being aggregated to estimate VaR at the higher levels. Operational VaR levels may be set for individual traders. VaR limits for individual traders should also facilitate control of operations as a dealer operating outside his limits will

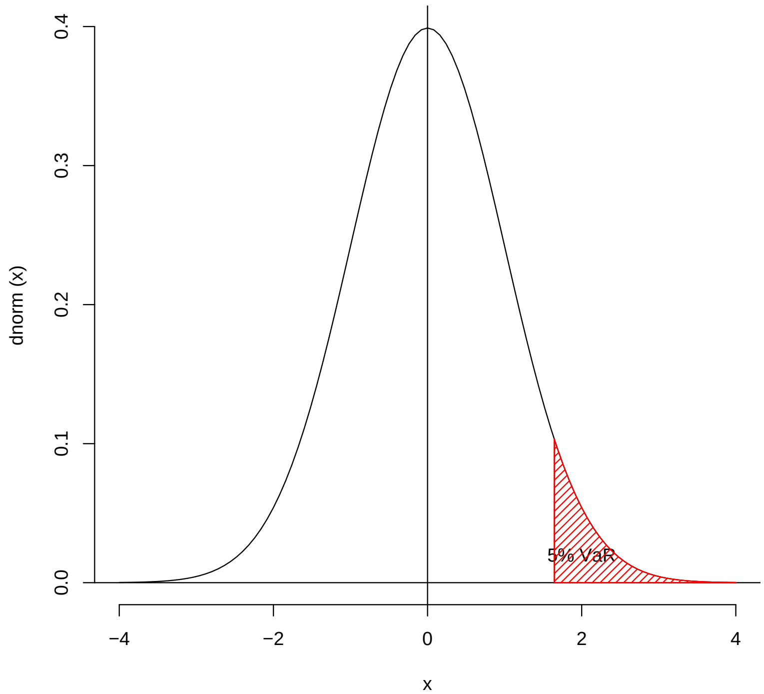

Figure 5.1: Loss Distribution and 5% Value at Risk

be detected if his dealings are properly recorded by the system.[6] It should be added that the risk management function in an organisation should not depend solely on a VaR system but should have a range of tools available to it. If one looks at many of the derivative disasters, a proper VaR implementation might have saved a lot of embarrassment

Risk is a very complex subject which we are not going to examine in detail here. In brief, it is the uncertainty in forecasted future returns. As such, like utility, it is an ordinal concept. Any one-one (strictly) monotonic transformation of a risk measure is an equivalent risk measure. The statement that one investment is 10% more risky than another simply does not make sense.

Artzner et al. (1999) set out a list of desirable properties that a measure of risk should have. Let X and Y be two assets. A risk measure $\rho()$ is coherent if it has the following four properties

Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (diversification reduces risk).

[6] A dealer making large profits but operating outside his limits should of course be subject to the same disciplinary action as his colleague who loses money in such circumstances.

Homogeneity: For any number $\alpha > 0$, $\rho(\alpha X) = \alpha\rho(X)$.

Monotonicity: $\rho(X) \geq \rho(Y)$ if $X \leq Y$.

Risk Free Condition: $\rho(X + k) = \rho(X) - k$ for any constant k.

VaR satisfies three of these conditions but may fail on subadditivity. To cope with this shortcoming alternative measures of risk have been proposed. Expected Shortfall (ES) is one such measure. Expected Shortfall is defined as the expected loss given that the VaR threshold has been exceeded. Daníelsson et al. (2005) has shown that subadditivity holds for VaR in all the distributions considered here. Subadditivity of VaR fails for assets which have super-fat tails (e.g. α-stable distributions where $\alpha \leq 1$, return/loss distributions which show very little variation apart from occasional jumps (e.g. "fixed" exchange rates) and some transactions involving derivatives). In all the cases considered here VaR is a monotonic transformation of ES and thus an equivalent measure of risk. The difference is in the explanation given to each measure. Presenting and explaining both measures to management gives a more complete picture of the situation and may allow a deeper understanding of the actual risk. Where there is doubt both measures should be calculated.

The main advantage of VaR is that it is a simple idea, may be relatively easy to calculate[7] and is easily explained to non-technical persons in management. In 1994, at 4.15 p.m. each evening, J.P. Morgan started to take a snapshot of their global trading positions to estimate, for management, their Daily-Earnings-at-Risk . This system was based on estimated correlation matrices, IGARCH systems and innovations with a normal distribution. In 1996 they made the relevant data and programs (Riskmetrics) available to all other users. This move allowed many smaller users to implement VaR systems without the required investment in data and programmes. The current version of the Riskmetrics package allows innovations to follow a t-distribution.

One problem with VaR is the apparent precision of the measurements which may lead management to underestimate the true risk or to miss some

[7]For a large financial institution dealing with a large number of exotic options, the calculation of VaR is not easy but it is difficult to think of a simpler alternative.

aspect of risk. Even in the simple cases considered here one can see that the estimates are subject to considerable margins of error. Risk managers must be aware of the limitations of VaR and avoid creating false impressions. VaR or for that matter any measure of risk that tries to express risk as a single number can never be a complete measure of risk.[8] This may be inconvenient but it is true.

A second criticism of VaR is that it takes no account of the shape of the distribution beyond the VaR point. Strictly speaking VaR estimates of two portfolios may be comparable only if the distributions of losses arising from the two portfolios are similar. A dealer may be able to increase returns by selling derivatives which might hedge the purchaser against some extreme risk. If the probability of the extreme event was small this would have very little effect of his calculated VaR. He has, however, changed the distribution of his losses. This is a serious problem with VaR systems and demonstrates the need to keep watch on the entire loss distribution. Risk management is a dynamic process and not simply a black box. Risk managers need to be extremely competent and be aware of the ability of traders to adapt to various constraints imposed on them. The risk manager needs to oversee the entire loss profile and not depend solely on an individual measure such as VaR. The combined use of VaR and ES might prove useful in such circumstances.

Any risk assessment should be supplemented by scenario analysis. In this procedure the loss is estimated for various scenarios which may be stylised or derived from historical events. If this procedure is done properly it provides a valuable addition to VaR and similar risk assessment.

Frain and Meegan (1996) contains an account of the concepts and analytics of Value-at Risk. For more details see also Dowd (1998, 2002), Jorian (2007) and Crouhy et al. (2006).

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[8] If all returns distributions were normal, then the estimated mean and variance are sufficient statistics and the variance or standard deviation or VaR as a multiple of the standard deviation would along with the mean returns give a complete picture of all that could be learned from the data.

5.3 Empirical Results

In this section we calculate and evaluate static and dynamic estimates of VaR. The four static estimates are based on:

1. a normal distribution,
2. a t-distribution, or
3. an α-stable distribution and
4. a non-parametric quantile estimation procedure.

Our initial evaluation of the parametric estimates is based on a comparison of the parametric and non-parametric estimators.

The dynamic VaR estimates are based on GARCH(1,1) processes with normal, t, and α-stable innovations. If an estimate of VaR at $p\%$ is good then it should be exceeded in the sample close to $p\%$ of the time. For each of the VaR estimates we calculate the exceedances and test the difference between the observed and predicted exceedances. This same test is also applied to the static estimates.

All distribution parameters are estimated by maximum likelihood. The Tables in Appendix 5.5 give details of these estimates. Data sources and software used are described in Appendix 5.5.4

5.3.1 VaR Estimates

Tables 5.1 to 5.5 set out static estimates of the VaR at 10%, 5%, 1%, 0.5% and 0.1% levels, respectively, for an investment in each of the six total return equity indices:

- ISEQ (daily from 4 January 1988 to 31 January 2008).
- CAC40 (daily from 31 December 1987 to 31 January 2008).
- DAX30 (daily from 28 September 1959 to 31 January 2008).
- FTSE100 (daily from 31 December 1985 to 31 January 2008).
- Dow Jones Composite (DJAC) (daily from 30 September 1987 to 31 January 2008).
- S&P500 (daily from 29 December 1989 to 31 January 2008).

The quantiles are calculated on the basis of returns following:

- An α-stable distribution with parameters estimated by maximum likelihood.
- A normal distribution with parameters estimated by maximum likelihood.
- A t-distribution[9] with nonzero mean, nonunitary scale and degrees of freedom to be estimated by maximum likelihood

[9] The standard t-distribution with ν degrees of freedom has zero mean and variance $\nu/(\nu-2)$ if $\nu > 2$. The t-distribution considered here is a generalisation of this distribution with mean μ, scale σ and degrees of freedom ν. Its probability density function is given by

$$f(x|\mu,\sigma,\nu) = \frac{\Gamma[(\nu+1)/2]}{(\pi\nu)^{1/2}\Gamma(\nu/2)} \frac{\sigma^{-1}}{[1+(x-\mu)^2/(\sigma^2\nu)]^{(\nu+1)/2}}, \quad -\infty < x < \infty.$$

where $\Gamma(\cdot)$ is the gamma function. Note that the standard deviation of x is $\sigma\sqrt{\frac{\nu}{\nu-2}}$. If $\mu = 0$ and $\sigma = 1$ this reduces to the standard Student's t-distribution with ν degrees of freedom. The heavier tails of the t-distribution are often used in economics and finance to model the fat tails that are often observed. Often the justification is empirical. A Bayesian justification involves a mixture of normal distributions with known mean and a prior inverse gamma distribution for the variance. For more details and references see Weitzman (2007).

Table 5.1: 10% VaR for each Equity Index for α-stable, Normal and t- distributions

Index	Distribution			(1) Sample	Quantile
	Stable	Normal	t	Quantile	s.e. (2)
ISEQ	**1.04**	1.23	1.13	1.03	0.03
CAC40	**1.48**	1.61	1.52	1.43	0.03
DAX30	**1.27**	1.49	1.31	1.23	0.02
FTSE100	**1.15**	1.30	1.22	1.13	0.02
DJAC	**1.05**	1.26	1.11	1.04	0.03
S&P500	**1.10**	1.20	**1.14**	1.09	0.03
(1) Harrell and Davis (1982)					
(2) Bootstrap estimate					

- A distribution free estimate of each quantile based on Harrell and Davis (1982). A bootstrapped standard error of each non-parametric quantile estimate was also calculated.

The estimates of the parametric distributions, in bold case, in the first three columns are within two standard deviations of the non-parametric estimates. If we regard the nonparametric estimates and their bootstrapped standard errors as accurate such estimates are then, at least, consistent with the non-parametric estimates and may be regarded as "good".

On this criterion the estimates based on a normal distribution are of little value. They overestimate VaR at 10%, are a little high at 5% and underestimate risk at the lower levels.

The estimates for the α-stable distribution are very good at the 10% and 5% levels and not that bad at the 1% levels. At the 0.5% and 0.1% levels they appear to overestimate the quantiles.

The t-distribution appears to perform well at the 1%, 0.5% and 0.1% levels and not that bad at the 5% level.

Table 5.2: 5% VaR for each Equity Index for α-stable, Normal and t- distributions

Index	Distribution Stable	Normal	t	Sample Quantile (1)	Quantile s.e. (2)
ISEQ	**1.48**	1.60	**1.57**	1.50	0.05
CAC40	**2.02**	**2.08**	**2.07**	2.04	0.05
DAX30	**1.75**	1.92	**1.81**	1.76	0.03
FTSE100	**1.59**	1.68	1.66	1.55	0.03
DJAC	**1.46**	1.63	**1.53**	1.47	0.05
S&P500	**1.54**	**1.55**	**1.56**	1.55	0.04
(1) Harrell and Davis (1982)					
(2) Bootstrap estimate					

Table 5.3: 1% VaR for each Equity Index for α-stable, Normal and t- distributions

Index	Distribution Stable	Normal	t	Sample Quantile (1)	Quantile s.e. (2)
ISEQ	**3.19**	2.28	**2.86**	2.99	0.14
CAC40	**3.89**	2.95	**3.49**	3.59	0.17
DAX30	**3.46**	2.73	**3.18**	3.19	0.11
FTSE100	**3.09**	2.40	**2.81**	2.92	0.12
DJAC	2.97	2.32	**2.69**	2.59	0.10
S&P500	3.25	2.21	**2.75**	2.73	0.10
(1) Harrell and Davis (1982)					
(2) Bootstrap estimate					

Table 5.4: 0.5% VaR for each Equity Index for α-stable, Normal and t- distributions

Index	Distribution Stable	Normal	t	Sample Quantile (1)	Quantile s.e. (2)
ISEQ	4.67	2.53	**3.58**	3.66	0.17
CAC40	5.44	3.27	**4.22**	4.34	0.16
DAX30	4.91	3.02	**3.91**	4.12	0.21
FTSE100	4.33	2.66	**3.40**	3.48	0.23
DJAC	4.25	2.57	**3.31**	3.25	0.20
S&P500	4.70	3.45	3.40	3.10	0.09
(1) Harrell and Davis (1982))					
(2) Bootstrap estimate					

Table 5.5: 0.1% VaR for each Equity Index for α-stable, Normal and t- distributions

Index	Distribution			Sample Quantile (1)	Quantile s.e. (2)
	Stable	Normal	t		
ISEQ	12.00	3.04	**5.91**	5.55	0.55
CAC40	12.98	3.94	**6.33**	6.15	0.48
DAX30	11.98	3.63	**6.12**	6.44	0.42
FTSE100	10.37	3.20	**5.12**	5.61	0.71
DJAC	10.52	3.10	**5.22**	6.27	1.45
S&P500	11.83	2.95	**5.40**	4.68	0.68
(1) Harrell -Davis (1982)					
(2) Bootstrap estimate					

5.3.2 Exceedances of VaR Estimates

If a p% VaR estimate is reasonable we would expect that losses should exceed it approximately p% of the time. In these circumstances, the distribution of the number of times that the p% VaR is exceeded (the exceedances) can be approximated by a Poisson[10] distribution with parameter given by p% of the sample size. Tables 5.6 to 5.10 present details of such counts of exceedances and an estimate of the probability of a higher value than that found based on the assumption of this Poisson distribution. Exceedances which accept the null at 95% level are set in bold font.

The measures of VaR in the these tables include:

- Static normal distribution with parameters estimated by maximum likelihood.

- GARCH(1,1) with normal innovations estimated by maximum likelihood. This gives rise to a dynamic VaR estimate which may be seen as a generalisation of the traditional Riskmetrics Group (1999) methodology. See appendix 5.5.2 for details of estimates and specification tests of the GARCH(1,1) models.

- t-distribution with mean, scale and degrees of freedom estimated by maximum likelihood (see footnote 9 on page 126)

- GARCH(1,1) with t-errors estimated by maximum likelihood. The resulting VaR may be compared to the Riskmetrics 2006 methodology (Zumbach (2006). See appendix 5.5.2 for details of estimates and specification tests.

- α-stable distribution - parameters estimated by maximum likelihood. See appendix 5.5.1 for details of estimates and specification tests.

- α-stable GARCH(1,1) - This is a variation of a TS-GARCH(1,1) with α-stable innovations. See Appendix 5.5.3 for details.

[10]The Poisson approximation to the binomial is sufficient here.

Table 5.6: % Exceedances for 10% VaR for each Equity Index for Normal, Normal GARCH, t, t GARCH, α-stable and α-stable GARCH

	Total Return Index					
	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
Observations	5037	5056	12098	5578	5158	4559
Normal	7.35	8.13	7.20	7.48	6.79	8.64
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
GARCH(1,1) with	9.18	10.88	9.18	**9.77**	**9.36**	**10.13**
Normal Errors	(0.98)	(0.00)	(1.00)	(0.70)	(0.92)	(0.38)
t	8.11	8.48	8.57	7.76	7.83	8.62
	(1.00)	(0.99)	(1.00)	(1.00)	(1.00)	(1.00)
d.f.	3.4	4.5	3.9	4.4	3.8	3.7
GARCH(1,1) with	6.19	**9.14**	8.12	8.78	7.15	8.36
t Errors	(1.00)	(0.02)	(1.00)	(1.00)	(1.00)	(1.00)
α-Stable	**9.77**	**9.39**	**9.52**	**9.56**	**9.87**	**9.87**
	(0.69)	(0.91)	(0.95)	(0.84)	0.61	(0.60)
α-Stable	**10.18**	**10.48**	**10.18**	**10.44**	**10.61**	**10.91**
GARCH(1,1)	(0.32)	(0.13)	(0.26)	(0.15)	(0.08)	(0.03)
Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.						

Table 5.7: % Exceedances for 5% VaR for each Equity Index for Normal, Normal GARCH, t, t GARCH, α-stable and α-stable GARCH

	Total Return Index					
	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
Observations	5037	5056	12098	5578	5158	4559
Normal	4.40	**4.79**	4.07	4.10	4.11	**4.98**
	(1.00)	(0.74)	(1.00)	(1.00)	(1.00)	(0.51)
GARCH(1,1) with	4.39	**4.47**	4.40	3.87	4.32	**4.47**
Normal Errors	(1.00)	(0.95)	(1.00)	(1.00)	(0.99)	(0.94)
t	4.17	**4.27**	4.40	3.87	**4.65**	**4.47**
	(1.00)	(0.95)	(1.00)	(1.00)	(0.86)	(0.94)
d.f.	3.4	4.5	3.9	4.4	3.8	3.7
GARCH(1,1) with	3.87	4.27	3.87	4.12	3.28	3.90
t Errors	(1.00)	(0.99)	(1.00)	(1.00)	(1.00)	(1.00)
α-Stable	**5.18**	**4.98**	**5.00**	**4.66**	**5.02**	**5.13**
	(0.27)	(0.50)	(0.47)	(0.87)	(0.46)	(0.33)
α-Stable	**5.30**	**5.58**	**5.24**	**5.45**	**5.20**	**5.51**
GARCH(1,1)	(0.16)	(0.03)	(0.11)	(0.06)	(0.25)	(0.06)
Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.						

Table 5.8: % Exceedances for 1% VaR for each Equity Index for Normal, Normal GARCH, t, t GARCH, α-stable and α-stable GARCH

	Total Return Index					
	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
Observations	5037	5056	12098	5578	5158	4559
Normal	2.12	1.76	1.61	1.70	1.47	1.97
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
GARCH(1,1) with Normal Errors	1.32 (0.00)	1.52 (0.00)	1.32 (0.00)	1.47 (0.00)	1.82 (0.00)	1.78 (0.00)
t	**1.03**	**1.01**	**1.00**	**1.08**	**0.83**	**0.92**
	(0.37)	(0.44)	(0.51)	(0.26)	(0.87)	(.67)
d.f.	3.4	4.5	3.9	4.4	3.8	3.7
GARCH(1,1) with t Errors	0.65 (1.00)	0.69 (0.99)	0.65 (1.00)	**0.86** (0.83)	0.52 (1.00)	0.68 (0.99)
α-Stable	**0.79**	**0.83**	0.81	**0.82**	0.62	0.35
	(0.92)	(0.87)	(0.98)	(0.90)	(1.00)	(1.00)
α-Stable GARCH(1,1)	**0.97** (0.54)	**1.11** (0.20)	**1.13** (0.07)	**1.09** (0.22)	**1.09** (0.24)	**1.12** (0.19)
Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.						

Table 5.9: % Exceedances for 0.5% VaR for each Equity Index for Normal, Normal GARCH, t, t GARCH, α-stable and α-stable GARCH

	Total Return Index					
	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
Observations	5037	5056	12098	5578	5158	4559
Normal	1.55	1.31	1.17	1.34	1.00	1.43
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
GARCH(1,1) with	0.81	0.91	0.82	0.95	1.16	1.16
Normal Errors	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
t	**0.50**	**0.53**	**0.57**	**0.52**	**0.43**	0.20
	(0.46)	(0.32)	(0.12)	(0.37)	(0.74)	(1.00)
d.f.	3.4	4.5	3.9	4.4	3.8	3.7
GARCH(1,1) with	0.33	**0.32**	0.34	**0.52**	0.29	**0.35**
t Errors	(0.99)	(0.97)	(0.99)	(0.50)	0.98	(0.91)
α-Stable	0.18	0.18	0.31	0.25	0.21	0.09
	(1.00)	(1.00)	(1.00)	(1.00)	1.00	(1.00)
α-Stable	**0.32**	**0.47**	**0.51**	**0.56**	**0.47**	**0.81**
GARCH(1,1)	(0.96)	(0.55)	(0.39	(0.24)	(0.59)	(0.81)
Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.						

Table 5.10: % Exceedances for 0.1% VaR for each Equity Index for Normal, Normal GARCH, t, t GARCH, α-stable and α-stable GARCH

	Total Return Index					
	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
Observations	5037	5056	12098	5578	5158	4559
Normal	0.95	0.83	0.69	0.66	0.58	0.61
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
GARCH(1,1) with Normal Errors	0.34	0.33	0.34	0.43	0.54	0.55
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
t	**0.11**	**0.08**	**0.12**	**0.13**	**0.14**	**0.09**
	(0.32)	(0.57)	(0.24)	(0.20)	(0.15)	(0.48)
d.f.	3.4	4.5	3.9	4.4	3.8	3.7
GARCH(1,1) with t Errors	0.06	**0.12**	**0.11**	**0.11**	**0.11**	**0.11**
	(1.00)	(0.25)	(0.32)	(0.48)	(0.26)	(0.31)
α-Stable	0.00	0.00	0.01	**0.04**	**0.02**	0.00
	(0.99)	(0.99)	(1.00)	(0.92)	(0.96)	(0.99)
α-Stable GARCH(1,1)	**0.06**	**0.06**	**0.05**	**0.04**	**0.06**	**0.04**
	(0.74)	(0.74)	(0.96)	(0.92)	(0.76)	(0.083)
Figures in brackets are the estimated probability of a greater % than found based on a Poisson distribution for the number of exceedances.						

Table 5.11 provides a summary of Tables 5.6 to 5.10 For each VaR level and for each index it give details of:

- the number of times the proportion of exceedances was significantly less than the VaR level. In these cases the estimate of the risk is too high.
- the number of times that exceedances were not significantly different to the VaR level. In these cases the measure of risk can not be rejected.
- the number of times the proportion of exceedances was significantly more than the VaR level. In these cases risk has been under estimated.

On the basis of these results the VaR estimates derived from an α-stable GARCH(1,1) are more accurate than the other estimates examined. The observed exceedances are not statistically different from the expected for any of the equity indices at any of the five levels considered. Table 5.25 on page 158 sets out, for the returns on each of the equity indices, the estimates for the parameters of this α-stable GARCH process. It should be noted that while they are greater than the corresponding static estimates, the α parameters are significantly less than 2 in all cases. Table 5.26 on page 159 gives exceedances and percentage exceedances for this process.

Figure 5.2 shows a typical example of VaR estimated in this way and the corresponding losses

The static α-stable and the t-distribution are next in order of merit. The α-stable distribution performs best at the 10% and 5% levels and is somewhat conservative at the 1% and 0.1% level and very conservative at the 0.5% level. The t-distribution performs extremely well in the extreme tails of the distribution.

The ease of implementation of a VaR system based on a t-distribution, compared to the equivalent based on an α-stable , combined with the results here would incline many people to favour the t-distribution. However the sum or average of independent observations from a t-distribution does not follow a t-distribution. Thus if returns on individual components of an investment portfolio have a t-distribution the return on the portfolio will not

Table 5.11: Summary Exceedances

VaR Level	Result	Distribution						
		Normal	GARCH (Normal)	t-distr.	GARCH (t-distr)	α-Stable	α-Stable GARCH	All
10%	low	6	2	6	5	0	0	19
	equal	0	3	0	1	6	6	16
	high	0	1	0	0	0	0	1
5%	low	4	4	3	6	0	0	17
	equal	2	2	3	0	6	6	19
	high	0	0	0	0	0	0	0
1%	low	0	0	0	5	3	0	8
	equal	0	0	6	1	3	6	16
	high	6	6	0	1	0	0	12
0.5%	low	0	0	1	3	6	0	10
	equal	0	1	5	3	0	6	15
	high	6	5	0	1	0	0	11
0.1%	low	0	0	0	1	4	0	5
	equal	0	0	6	5	2	6	19
	high	6	6	0	0	0	0	12
All	low	10	6	10	20	13	0	59
	equal	2	6	20	10	17	30	85
	high	18	18	0	0	0	0	36

Result -
low : % exceedances < VaR level - conservative view
equal : % exceedances not significantly different from VaR level
high : % exceedances > VaR level - liberal view

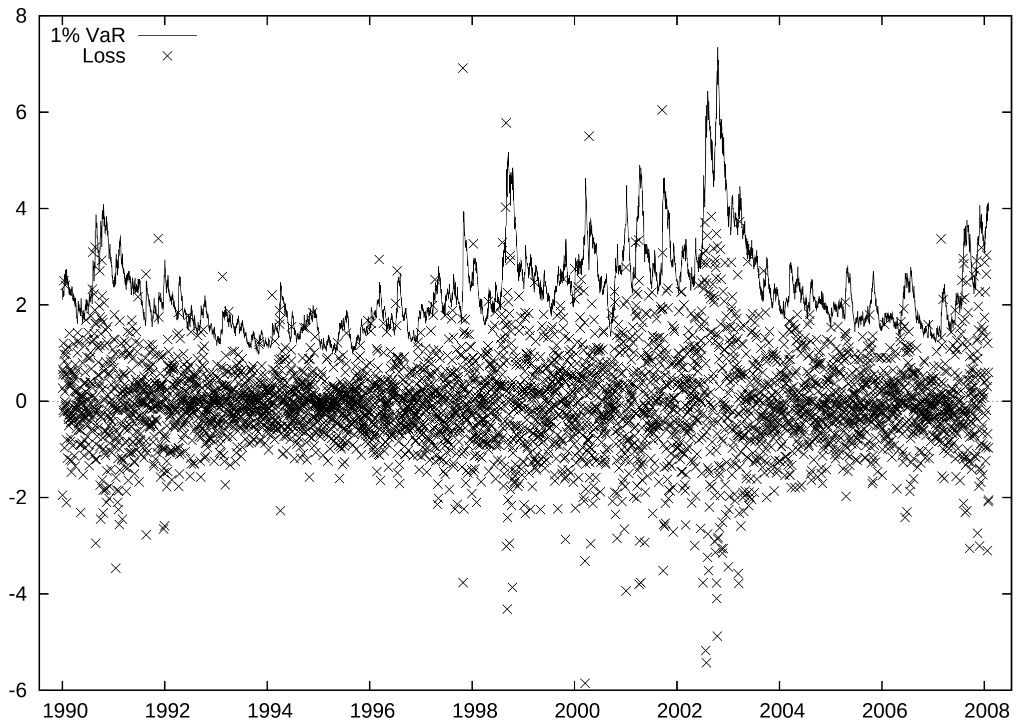

Figure 5.2: Losses on S&P 500 and 1% VaR Based on an α-stable GARCH Process

have a t-distribution. If the return on the portfolio has a t-distribution then the returns on the components will not have a t-distribution. If returns on an asset or portfolio have a t-distribution at one frequency they will not have a t-distribution at other frequencies. This lack of the stability property of the t-distribution is a significant problem in using it to model returns. If the degrees of freedom of a t-distribution are less than two the t-distribution is in the domain of attraction of a non-normal α-stable distribution with stability parameter, α, equal to the degrees of freedom of the t-distribution. If the degrees of freedom are two or greater the t-distribution is in the domain of attraction of a normal distribution. The effect of aggregation of independent observations on a t-distribution may be gauged from the following simulation study. Six independent random samples, of size 1000, were drawn from a standard t-distribution with 1.75 degrees of freedom and their mean across each of the 1000 observations calculated. A t-distribution was then fitted to the sample of 1000 means, using the methods described in this section. The analysis was replicated 1000 times. In 856 cases the estimated degrees of freedom was greater than 2. The mean degrees of freedom was 2.21 with a standard deviation of 0.02. Thus approximating the distribution of the mean (or sum of independent t-distributions as a t-distribution is likely to wrongly place the distribution of the mean within the domain of attraction of a normal distribution.

I also do not know of any theory in economics or finance that would lead to a t-distribution for returns. The idea that a t-distribution for asset returns results from a mixture of normal random variables with variance following an inverse gamma distribution has been argued in Weitzman (2007) is mathematically correct and as he admits has been well known to Bayesian statisticians but had no sound basis in economic theory. Many econometric models that fall down fail, not because there are problems with their econometrics, but because the economics behind the model is faulty or non-existent. The t-distribution may provide a good measure of what has been going on in the tails of the distribution but the results may be very sensitive to policy actions.

The normal distribution is conservative at the 10% level and greatly underestimates risk at the 1% and lower levels. These quantile estimates based on the normal distribution are further evidence of the poor fit of the normal distribution to the data.

Exceedances for the two GARCH models are not good with approximately three quarters of the measures exceedances being significantly different from their expected values.

5.4 Conclusions

The relative performance of the various measures of VaR considered may be summarised as follows

The static normal distribution performs very badly even at the conventional 5% and 1% levels. It tends to over estimate VaR at the higher probability levels and under estimate at the lower. This is what one would expect given the exponential decay in the tails of the normal distribution. A normal VaR at 1% may be extremely misleading if given to management.

The static t-distribution performs very well, particularly in the tails of the distribution. In contrast to the normal and α-stable distributions the t-distribution lacks the stability property and does not possess a domain of attraction. The sometimes quoted justification for a t-distribution as a normal mixture with variances following an inverse gamma distribution is not very convincing.

The GARCH distributions with normal innovations performs somewhat better than the static normal distribution. Curiously the GARCH distribution with t-innovations does not performs better than the static t-distribution but is better than the GARCH with normal innovations.

Then α-stable distribution performance is about equivalent to the t-distribution but is good at conventional VaR levels. Extreme VaR at levels less than 1% tends to be somewhat conservative. While it is likely that the α-stable distribution can be applied to all risk assessments it is an important measure that provides a good measure of VaR at conventional levels and perhaps conservative estimates at extreme levels. Given the likely effects of losses at these extreme levels this is probably not a bad idea.

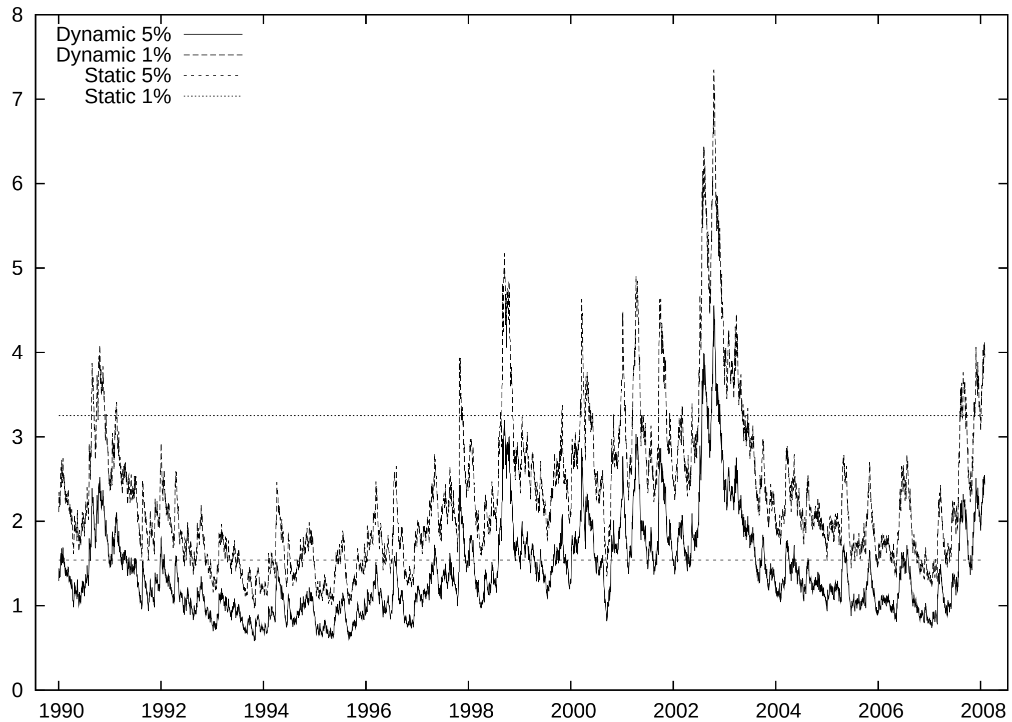

Figure 5.3: 5% and 1% Static and Dynamic VaR of Losses on S&P 500

The α-stable GARCH(1,1) model for returns provides the best measure of VaR. It gives good estimates at all VaR levels for all the indices considered. The null hypothesis of a different rate of exceedances can not be rejected in a single case.

For the Risk manager or the supervisor I have shown that accurate measures of VaR can be obtained using an α-stable distribution. Theoretical justification can be provided by the generalised central limit theorem and the time aggregation and domain of attraction properties which define or are unique to this distribution. Figure 5.3 compares the static and dynamic (GARCH) α-stable 1% and 5% VaR for the S&P 500.

The volatility of the dynamic VaR may give rise to problems. Daníelsson et al. (2001) have asked if the adoption of dynamic VaR systems of risk management lead to constrains on the financial system during times of liquidity shortage. Masschelein (2007) has argued that, up to recent times, regulatory VaR requirements have not been binding. It can be argued that if regulatory requirements had been more severe in less volatile times we may not have encountered the severe liquidity crisis that exists today. The use of the kind of static α-stable VaR estimates provided here might form a useful basis for deciding appropriate levels for such an arrangement.

I have also shown that α-stable estimates of VaR are feasible. They are a valuable and more accurate measure of VaR and would provide additional information to a risk manager. They are, of course only one aspect of risk management.

5.5 Appendices

5.5.1 Maximum Likelihood estimates of α-stable parameters

Table 5.12 gives results of maximum likelihood estimates of the α-stable parameters of the distribution of losses on total return indices for the ISEQ, CAC40, DAX30, FTSE100, DJAC and S&P500. Estimation is by maximum likelihood computed in C++ using the stable library functions of Nolan (2005a). The results here cover a longer period than those reported in table 2.2.[11] Comparing the two tables we note that the results are basically similar apart from the fact that the estimate of the skew statistic is significant for all indices in the later table rather than for three in the earlier table. As we are reporting results in terms of losses in this Chapter the signs of the skewness and location statistic have been reversed.

5.5.2 GARCH estimates

Tables 5.13 to 5.24 gives results of maximum likelihood estimates of GARCH models of the distribution of losses on total return indices for the ISEQ, CAC40, DAX30, FTSE100, DJAC and S&P500. I estimate ARMA(p,q)-GARCH(1,1) models for $(p, q) \in \{(0,0), (1,0), (2,0), (1,1)\}$. Although there are some problems of autocorrelation in the more parsimonious models, the number

[11] Table 2.2 is based on data to 26 September 2005. Table 5.12 is basee on data to end January 2008.

Table 5.12: Estimates of Parameters of Stable distributions of Equity Total Return Indices (complete period)

	ISEQ	CAC40	DAX 30	FTSE100	DJAC	S&P500
start date	04/01/88	31/12/87	28/09/59	31/12/85	30/09/87	03/01/89
end date	31/01/08	31/01/08	31/01/08	31/01/08	31/01/08	31/01/08
observations	5037	5056	12098	5578	5158	4559
α[a]	1.650	1.740	1.709	1.736	1.697	1.672
	(0.043)	(0.040)	(0.027)	(0.039)	(0.041)	(0.044)
β	0.103	0.172	0.092	0.170	0.129	0.173
	(0.107)	(0.130)	(0.078)	(0.123)	(0.115)	(0.116)
γ	0.539	0.761	0.657	0.601	0.545	0.551
	(0.015)	(0.020)	(0.011)	(0.015)	(0.014)	(0.016)
δ	-0.047	-0.030	-0.020	-0.035	-0.038	-0.028
KS (stable)	0.013	0.012	0.008	0.007	0.016	0.022
p-value	0.326	0.473	0.481	0.938	0.111	0.026
LR[b] test of Normality	831.6	463.3	2088.2	790.4	1217.5	471.9

[a] Figures in brackets under each coefficient estimate are the 95% confidence interval half-width estimates

[b] Likelihood ratio test of the joint restriction $\alpha = 2$ and $\beta = 0$. The test statistic is asymptotically $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

of exceedances appears to be robust with respect to the choice of ARMA components and the analysis is based on a constant mean. Specification tests in bold case are not statistically significant. Estimation and testing of GARCH processes, with normal and t innovations was completed using R (R Development Core Team (2008)) and the Rmetrics library (Wuertz (2007)).

Table 5.13: Estimated ARMA(p,q) GARCH(1,1), Normal Innovations (CAC40)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.066	-0.066	-0.066	-0.091
	(0.015)	(0.015)	(0.015)	(0.025)
ϕ_1		0.179	0.019	0.363
		(0.015)	(0.015)	(0.239)
ϕ_2			0.019	
			(0.015)	
θ_1				0.380
				(0.232)
ω	0.032	0.032	0.032	0.032
	(0.006)	(0.006)	(0.006)	(0.006)
α_1	0.086	0.086	0.086	0.086
	(0.009)	(0.009)	(0.009)	(0.009)
β_1	0.895	0.895	0.895	0.895
	(0.011)	(0.011)	0.011	(0.011)
Standardised Residual tests				
J-B test	1090.50	1098.59	1104.99	1093.58
Residual Q10	**17.66**	**15.70**	**15.46**	**16.16**
Residual Q15	**21.08**	**19.01**	**18.84**	**19.47**
Residual Q20	**24.33**	**22.52**	**22.37**	**23.04**
Residual ARCH tests				
ARCH Q10	**13.94**	**13.92**	**13.77**	**13.92**
ARCH Q15	**17.02**	**17.01**	**16.85**	**17.01**
ARCH Q20	**18.54**	**18.50**	**18.37**	**18.53**
Information Criterion Tests				
AIC	-3.115	-3.114	-3.112	-3.113
BIC	-3.110	-3.108	-3.105	-3.106
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.14: Estimated ARMA(p,q) GARCH(1,1), t Innovations (CAC40)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.074	-0.063	-0.061	-0.051
	(0.011)	(0.010)	(0.010)	(0.011)
ϕ_1		0.144	0.140	0.292
		(0.014)	(0.015)	(0.091)
ϕ_2			0.015	
			(0.014)	
θ_1				-0.153
				(0.094)
ω	0.022	0.021	0.020	0.021
	(0.006)	(0.006)	(0.006)	(0.006)
α_1	0.095	0.097	0.096	0.097
	(0.016)	(0.016)	(0.016)	(0.016)
β_1	0.886	0.885	0.886	0.885
	(0.019)	(0.019)	(0.019)	(0.019)
ν	5.236	5.258	5.245	5.262
	(0.373)	(0.372)	(0.372)	(0.374)
Standardised Residual tests				
Residual Q10	120.66	22.51	19.15	**17.57**
Residual Q15	128.44	28.66	25.15	**23.44**
Residual Q20	134.82	33.24	**29.97**	**28.16**
Residual ARCH tests				
ARCH Q10	**3.08**	**3.74**	**3.76**	**3.74**
ARCH Q15	**5.82**	**6.53**	**6.55**	**6.53**
ARCH Q20	**7.34**	**7.93**	**7.93**	**7.92**
Information Criterion Tests				
AIC	-2.521	-2.499	-2.498	-2.499
BIC	-2.514	-2.492	-2.489	-2.489
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.15: Estimated ARMA(p,q) GARCH(1,1) Normal Innovations (DAX 30)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.036	-0.032	-0.034	-0.044
	(0.008)	(0.008)	(0.008)	(0.011)
ϕ_1		0.099	0.104	0.241
		(0.010)	(0.010)	(0.072)
ϕ_2			-0.051	
			(0.010)	
θ_1				0.347
				(0.069)
ω	0.031	0.030	0.030	0.030
	(0.003)	(0.003)	(0.003)	(0.003)
α_1	0.130	0.131	0.132	0.132
	(0.008)	(0.008)	(0.008)	(0.007)
β_1	0.851	0.850	0.850	0.850
	(0.009)	(0.008)	0.008	(0.008)
Standardised Residual tests				
J-B test	24402	22917	21696	17038
Residual Q10	126.84	27.28	24.36	20.53
Residual Q15	133.29	32.64	30.09	25.95
Residual Q20	43.87	41.36	39.61	34.95
Residual ARCH tests				
ARCH Q10	**4.65**	**4.38**	**4.44**	**4.50**
ARCH Q15	**6.37**	**6.47**	**6.58**	**6.66**
ARCH Q20	**7.67**	**7.97**	**8.15**	**8.24**
Information Criterion Tests				
AIC	-2.842	-2.834	-2.814	-2.832
BIC	-2.840	-2.831	-2.827	-2.828
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.16: Estimated ARMA(p,q) GARCH(1,1) t Innovations (DAX 30)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.041	-0.037	-0.039	-0.050
	(0.008)	(0.008)	(0.008)	(0.011)
ϕ_1		0.093	0.098	-0.220
		(0.009)	(0.010)	(0.070)
ϕ_2			-0.048	
			(0.009)	
θ_1				0.320
				(0.068)
ω	0.022	0.022	0.021	0.022
	(0.003)	(0.003)	(0.003)	(0.003)
α_1	0.109	0.112	0.111	0.111
	(0.008)	(0.008)	(0.008)	(0.008)
β_1	0.876	0.873	0.874	0.873
	(0.008)	(0.008)	(0.008)	(0.008)
ν	10.814	10.875	10.773	10.804
	(0.826)	(0.831)	(0.821)	(0.823)
Standardised Residual tests				
Residual Q10	127.78	28.74	24.24	20.48
Residual Q15	134.36	34.26	30.15	26.08
Residual Q20	144.88	43.00	39.60	35.03
Residual ARCH tests				
ARCH Q10	**4.89**	**4.90**	**5.05**	**5.08**
ARCH Q15	**7.04**	**7.41**	**7.61**	**7.64**
ARCH Q20	**9.13**	**9.67**	**9.98**	**9.99**
Information Criterion Tests				
AIC	-2.800	-2.792	-2.790	-2.790
BIC	-2.797	-2.788	-2.785	=2.786
5% critical points for χ^2 distribution with 2 ,10, and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.17: Estimated ARMA(p,q) GARCH(1,1), Normal Innovations (FTSE100)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.065	-0.064	-0.064	-0.064
	(0.011)	(0.011)	(0.011)	(0.022)
ϕ_1		0.024	0.024	0.019
		(0.014)	(0.014)	(0.302)
ϕ_2			-.002	
			(0.014)	
θ_1				-0.004
				(0.303)
ω	0.018	0.018	0.018	0.018
	(0.004)	(0.004)	(0.004)	(0.004)
α_1	0.091	0.091	0.091	0.091
	(0.009)	(0.009)	(0.009)	(0.009)
β_1	0.893	0.893	0.893	0.893
	(0.011)	(0.011)	0.011	(0.011)
Standardised Residual tests				
J-B test	10791	10856	10885	10857
Residual Q10	**17.05**	**10.09**	**10.01**	**10.08**
Residual Q15	**23.38**	**16.10**	**16.04**	**16.09**
Residual Q20	**28.31**	**20.95**	**20.96**	**20.95**
Residual ARCH tests				
ARCH Q10	**8.79**	**8.10**	**8.09**	**8.10**
ARCH Q15	**11.49**	**10.80**	**10.80**	**10.80**
ARCH Q20	**15.83**	**15.21**	**15.20**	**15.21**
Information Criterion Tests				
AIC	-2.644	-2.643	-2.642	-2.642
BIC	-2.639	-2.637	-2.635	-2.635
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.18: Estimated ARMA(p,q) GARCH(1,1) t Innovations (FTSE100)

	ARMA model			
p q	0 0	1 0	2 0	1 1
μ	-0.068 (0.014)	-0.067 (0.011)	-0.068 (0.011)	-0.079 (0.025)
ϕ_1		0.021 (0.014)	0.022 (0.014)	-0.155 (0.313)
ϕ_2			-0.017 (0.014)	
θ_1				0.178 (0.312)
ω	0.014 (0.003)	0.014 (0.003)	0.014 (0.003)	0.014 (0.003)
α_1	0.080 (0.009)	0.078 (0.008)	0.079 (0.008)	0.080 (0.008)
β_1	0.906 (0.009)	0.906 (0.010)	0.907 (0.010)	0.906 (0.009)
ν	12.397 (1.549)	12.432 (1.558)	12.272 (1.532)	12.404 (1.553)
Standardised Residual tests				
Residual Q10	**17.04**	**10.53**	**11.277**	**10.26**
Residual Q15	**23.36**	**16.57**	**17.39**	**16.28**
Residual Q20	**28.36**	**21.49**	**22.51**	**21.24**
Residual ARCH tests				
ARCH Q10	**12.30**	**11.37**	**11.33**	**11.29**
ARCH Q15	**14.99**	**14.06**	**14.02**	**13.98**
ARCH Q20	**19.14**	**18.28**	**18.16**	**18.18**
Information Criterion Tests				
AIC	-2.608	-2.607	-2.606	-2.606
BIC	-2.602	-2.600	-2.598	-2.2.598
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.19: Estimated ARMA(p,q) GARCH(1,1), Normal Innovations (ISEQ)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.075	-0.062	-0.061	-0.048
	(0.012)	(0.012)	(0.012)	(0.012)
ϕ_1		0.150	0.146	0.341
		(0.016)	(0.016)	(0.099)
ϕ_2			0.018	
			(0.016)	
θ_1				-0.197
				(0.104)
ω	0.034	0.033	0.033	0.033
	(0.012)	(0.006)	(0.006)	(0.006)
α_1	0.090	0.089	0.089	0.089
	(0.011)	(0.011)	(0.011)	(0.011)
β_1	0.877	0.877	0.876	0.877
	(0.016)	(0.015)	0.015	(0.015)
Standardised Residual tests				
J-B test	14401.46	15342.95	15272.99	15182.22
Residual Q10	119.87	21.10	**17.56**	**15.47**
Residual Q15	127.60	27.28	**23.58**	**21.27**
Residual Q20	134.68	32.47	**29.06**	**27.72**
Residual ARCH tests				
ARCH Q10	**2.55**	**3.69**	**3.80**	**3.63**
ARCH Q15	**4.58**	**5.77**	**5.93**	**5.73**
ARCH Q20	**5.57**	**6.63**	**6.70**	**6.59**
Information Criterion Tests				
AIC	-2.633	-2.613	-2.612	-2.612
BIC	-2.628	-2.606	-2.604	-2.604
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.20: Estimated ARMA(p,q) GARCH(1,1) t Innovations (ISEQ)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.074	-0.063	-0.061	0.051
	(0.011)	(0.010)	(0.010)	(0.011)
ϕ_1		0.144	0.140	0.292
		(0.014)	(0.015)	(0.091)
ϕ_2			0.015	
			(0.014)	
θ_1				-0.153
				(0.094)
ω	0.022	0.021	0.020	0.021
	(0.006)	(0.006)	(0.006)	(0.006)
α_1	0.095	0.097	0.096	0.097
	(0.016)	(0.016)	(0.016)	(0.016)
β_1	0.886	0.885	0.886	0.885
	(0.019)	(0.019)	(0.019)	(0.019)
ν	5.236	5.258	5.245	5.262
	(0.373)	(0.372)	(0.372)	(0.374)
Standardised Residual tests				
Residual Q10	120.66	22.51	19.15	**17.57**
Residual Q15	128.44	28.66	25.15	**23.44**
Residual Q20	134.82	33.24	**29.97**	**28.16**
Residual ARCH tests				
ARCH Q10	**3.08**	**3.74**	**3.76**	**3.74**
ARCH Q15	**5.82**	**6.53**	**6.55**	**6.53**
ARCH Q20	**7.34**	**7.93**	**7.93**	**7.92**
Information Criterion Tests				
AIC	-2.521	-2.499	-2.498	-2.499
BIC	-2.514	-2.492	-2.489	-2.489

5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively

Table 5.21: Estimated ARMA(p,q) GARCH(1,1), Normal Innovations (S&P500)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.063	-0.059	-0.059	-0.062
	(0.011)	(0.011)	(0.011)	(0.018)
ϕ_1		0.059	0.059	0.009
		(0.016)	(0.016)	(0.227)
ϕ_2			-0.001	
			(0.016)	
θ_1				-0.049
				(0.227)
ω	0.009	0.009	0.009	0.009
	(0.002)	(0.002)	(0.002)	(0.002)
α_1	0.066	0.066	0.066	0.066
	(0.008)	(0.008)	(0.008)	(0.008)
β_1	0.925	0.926	0.926	0.926
	(0.009)	(0.008)	(0.009)	(0.009)
Standardised Residual tests				
J-B test	1129	1139	1144	1 1148
Residual Q10	31.02	**10.24**	**10.25**	**10.19**
Residual Q15	43.71	**21.86**	**21.79**	**21.79**
Residual Q20	44.43	**22.61**	**22.54**	**22.54**
Residual ARCH tests				
ARCH Q10	**5.01**	**4.85**	**4.85**	**4.85**
ARCH Q15	**7.28**	**7.12**	**7.12**	**7.11**
ARCH Q20	**8.91**	**8.49**	**8.49**	**8.48**
Information Criterion Tests				
AIC	-2.525	-2.521	-2.520	-2.520
BIC	-2.519	-2.514	-2.512	-2.512
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.22: Estimated ARMA(p,q) GARCH(1,1) t innovations (S&P500)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.075	-0.071	-0.073	-0.090
	(0.011)	(0.011)	(0.011)	(0.020)
ϕ_1		0.046	0.047	-0.201
		(0.015)	(0.015)	(0.188)
ϕ_2			-0.025	
			(0.015)	
θ_1				0.250
				(0.187)
ω	0.005	0.005	0.005	0.005
	(0.002)	(0.002)	(0.002)	(0.002)
α_1	0.059	0.060	0.060	0.060
	(0.008)	(0.008)	(0.008)	(0.008)
β_1	0.937	0.936	0.936	0.936
	(0.009)	(0.009)	(0.009)	(0.009)
ν	7.39	7.537	7.438	7.507
	(0.769)	(0.795)	(0.779)	(0.789)
Standardised Residual tests				
Residual Q10	31.85	**13.03**	**14.42**	**12.79**
Residual Q15	44.57	**24.88**	26.14	**24.52**
Residual Q20	45.31	**25.62**	**26.90**	**25.25**
Residual ARCH tests				
ARCH Q10	**5.93**	**5.59**	**5.58**	**5.60**
ARCH Q15	**8.26**	**8.01**	**7.90**	**7.96**
ARCH Q20	**10.32**	**9.86**	**9.71**	**9.79**
Information Criterion Tests				
AIC	-2.484	-2.480	-2.479	-2.479
BIC	-2.476	-2.472	-2.469	-2.469

5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively

Table 5.23: Estimated ARMA(p,q) GARCH(1,1), Normal Innovations (Dow Jones Composite)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.068	-0.066	-0.068	-0.107
	(0.011)	(0.011)	(0.011)	(0.021)
ϕ_1		0.026	0.026	-0.571
		(0.015)	(0.015)	(0.144)
ϕ_2			-0.020	
			(0.015)	
θ_1				0.603
				(0.141)
ω	0.022	0.023	0.023	0.022
	(0.003)	(0.004)	(0.004)	(0.004)
α_1	0.091	0.092	0.091	0.092
	(0.008)	(0.008)	(0.007)	(0.008)
β_1	0.889	0.889	0.889	0.889
	(0.010)	(0.010)	(0.010)	(0.010)
Standardised Residual tests				
J-B test	11353	11939	11887	11831
Residual Q10	24.41	**14.86**	**15.19**	**13.35**
Residual Q15	32.11	**22.91**	**23.21**	**21.32**
Residual Q20	37.65	**28.47**	**28.78**	**26.82**
Residual ARCH tests				
ARCH Q10	**2.08**	**2.70**	**2.74**	**1.97**
ARCH Q15	**4.08**	**5.29**	**5.35**	**4.31**
ARCH Q20	**5.76**	**6.84**	**6.90**	**5.87**
Information Criterion Tests				
AIC	-2.584	-2.584	-2.583	-2.582
BIC	-2.579	-2.577	-2.575	-2.575
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

Table 5.24: Estimated ARMA(p,q) GARCH(1,1), t Innovations (Dow Jones Composite)

	ARMA model			
p	0	1	2	1
q	0	0	0	1
μ	-0.071	-0.069	-0.072	-0.107
	(0.010)	(0.010)	(0.010)	(0.024)
ϕ_1		0.019	0.020	-0.513
		(0.014)	(0.014)	(0.250)
ϕ_2			-0.035	
			(0.014)	
θ_1				0.541
				(0.094)
ω	0.014	0.014	0.014	0.014
	(0.003)	(0.003)	(0.003)	(0.003)
α_1	0.059	0.059	0.059	0.059
	(0.007)	(0.008)	(0.008)	(0.008)
β_1	0.925	0.925	0.926	0.925
	(0.009)	(0.009)	(0.009)	(0.009)
ν	6.172	6.191	6.112	6.211
	(0.500)	(0.503)	(0.494)	(0.504)
Standardised Residual tests				
Residual Q10	24.22	**16.42**	19.03	**13.47**
Residual Q15	32.15	**24.89**	27.45	**21.82**
Residual Q20	37.56	**30.28**	32.88	**27.19**
Residual ARCH tests				
ARCH Q10	**5.77**	**6.45**	**6.77**	**5.99**
ARCH Q15	**7.12**	**8.25**	**8.58**	**7.61**
ARCH Q20	**8.46**	**9.51**	**9.83**	**8.85**
Information Criterion Tests				
AIC	-2.496	-2.499	-2.493	-2.494
BIC	-2.489	-2.492	-2.484	-2.485
5% critical points for χ^2 distribution with 2 ,10, 15 and 20 degrees of freedom are 5.99, 18.31, 25.00 and 31.41 respectively				

5.5.3 α-stable GARCH Estimates and VaR

In the usual GARCH(p,q) model the disturbance takes the form

$$\varepsilon_t = z_t \sigma_t,$$

where z_t is an *iid* process with zero mean and unit variance. The conditional variance of this process is σ_t^2. σ_t^2 is taken to follow various stochastic processes. The GARCH process is defined as the following process:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2.$$

In the GARCH estimates above z_t was taken to follow either a standard normal or a t-distribution. The residuals in both the normal and t-distributions for z_t showed considerable excess kurtosis.

It would be attractive to model the z_t with an α-stable distribution. The exact formulation can not be followed in the general case when $\alpha < 2$ as the second moment of the distribution of z_t does not exist. Following Panorska et al. (1995) or Rachev and Mittnik (2000) we say that X follows a stable GARCH(α,p,q) if X_t is α-stable with parameters α, β, γ_t and δ where

$$\gamma_t = \omega + \sum_{i=1}^{q} a_i |x_{t-i} - \delta| + \sum_{i=1}^{p} b_i \gamma_{t-i} \tag{5.1}$$

and $a_i,\ i = 1, \ldots, q$ and $b_j,\ i = 1, \ldots, p$ and $\omega > 0$. Panorska et al. (1995) establishes stationarity conditions for the process in Equation (5.1). For the stable GARCH(1,1) process, estimated here, we require that $b_1 + \lambda a_1 < 1$ where λ is a function of α and, for example, $\lambda = 1.5091,\ 1.3709,$ and 1.2687 for $\alpha = 1.6,\ 1.7,$ and 1.8 respectively. All α-stable processes estimates here satisfy these restrictions and are taken to be stationary.

Parameters were estimated by maximum likelihood using C++ and the STABLE library functions of Nolan (2005a). The optimisation[12] process was

[12]Maximisation was completed by minimising the negative of the log likelihood.

Table 5.25: Estimated Parameters of α-stable GARCH Loss Distributions

	ISEQ	CAC40	DAX30	FTSE100	DJAC	S&P500
α	1.80 (0.020)	1.95 (0.0084)	1.94 (0.0085)	1.95 (0.016)	1.88 (.023)	1.91 (0.0024)
β	0.175 (0.035)	0.727 (0.0041)	0.362 (0.0078)	0.851 (0.165)	0.438 (0.066)	0.703 (0.0014)
δ	-0.0581 (0.047)	-0.0657 (0.00026)	-0.0315 (0.00070)	-0.0522 (0.0098)	-0.513 (0.022)	-0.550 (2.7e-5)
ω	0.00984 (.00028)	0.0104 (2.4e-05)	0.0128 (0.00018)	.00862 (0.0054)	0.00761 (0.00088)	0.00463 (5.8e-6)
α_1	0.0599 (0.0024)	0.0570 (9.7e-05)	0.0738 (0.00040)	0.0584 (0.0085)	.0426 (0.00073)	0.0471 (5.0e-5)
β_1	0.911 (0.0033)	0.922 (0.0013)	0.897 (0.0012)	0.919 (0.173)	0.937 (.0016)	0.939 (0.00053)

initialised using the Nelder-Mead minimisation algorithm and continued to completion using the BFGS algorithm. Standard errors of the estimates were derived from the inverse Hessian matrix calculated during the minimisation process. Parameter estimates and standard errors are given in Table 5.25. Table 5.26 gives details of sample exceedances and percentage exceedances for α-stable GARCH VaR estimates. These are very close to expected values at all levels of VaR considered.

Table 5.26: Exceedances and Percentage Exceedances for α-stable GARCH VaR Estimates

Index		Observations	VaR Level				
			10.0%	5.0%	1.0%	0.5%	0.1%
ISEQ	Count	5037	513	267	49	16	3
	%		10.18	5.30	0.97	0.32	0.06
CAC40	Count	5056	530	282	56	24	3
	%		10.48	5.58	1.11	0.47	0.06
DAX30	Count	12098	1232	634	137	62	6
	%		10.18	5.24	1.13	0.51	0.05
DJAC	Count	5156	547	268	56	24	3
	%		10.61	5.20	1.09	0.47	0.06
FTSE100	Count	5575	582	304	61	31	2
	%		10.44	5.45	1.09	0.56	0.04
S&P500	Count	4557	497	251	51	18	2
	%		10.91	5.51	1.12	0.39	0.04
All	Count	37479	3901	2006	410	175	19
	%		10.41	5.35	1.09	0.47	0.05

5.5.4 Data and Software

Data

The total return indices used in this analysis were downloaded from the Reuters EcoWin database. The series used were:

- France, Paris SE, CAC 40 Index, Total Return, Close, EUR, (ew:fra15660).
- Germany, Deutsche Boerse, DAX 30, Index, Total Return, Close, EUR, (ew:deu15500).
- United States, Dow Jones, Averages, Composite Index, Total Return, Close, USD, (ew:usa15575200).
- United Kingdom, FTSE100, Index, Total Return, Close, GBP, (ew:gbr15500200).
- United States, Standard & Poors, 500 Composite, Equal Weighted Index, Total Return, Close, USD, (ew:usa15508200).
- Ireland, Irish SE, ISEQ Index, Total Return, Close, EUR, (ew:irl15550).

Software

The parameters of α-stable distributions were estimated by Maximum Likelihood using C++ and the Dynamic link Libraries of Nolan (2005a). Other statistical analysis was completed in R (R Development Core Team (2008)) (using the Rmetrics (Wuertz (2007)), QRMlib (McNeil and Ulman (2007)) and related R packages), Gretl (Cottrell and Lucchetti (2007)) and MATHEMATICA (Wolfram (2003)).

APPENDIX A

An Introduction to the α-stable distribution

This appendix outlines the properties of the α-stable family of distributions and compares these with the properties of a normal distribution. Proofs are not given. These and further details may be found in Feller (1966, 1971), Gnedenko and Kolmogorov (1954), Janicki and Weron (1994), Rachev and Mittnik (2000), Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999) or Zolotarev (1986).

A.1 Central Limit Theorems

An assumption of a normal distribution has formed part of almost all developments in theoretical and empirical finance in the last half century. In the introduction we have already referred to the Capital Asset Pricing Model,[1] optimal portfolio allocation and option pricing as depending on a normality assumption. Some form of central limit theorem[2] has been implicit in all

[1] See Section A.6 in this appendix for the extension of the CAPM to include returns with an α-stable distribution.

[2] According to Jaynes (2003, p. 242) the name "Central Limit Theorem" was first used in print in Pólya (1920). In the German the adjective central modifies the word theorem and

these developments.

One can look at the central limit theorem to see why it might be appropriate to returns distributions and to gain insight into how it might fail. The elementary version of the theorem quoted in many econometrics texts (e.g. Hayashi (2000, p. 96)) may be set out as follows:

Lindeberg-Levy Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be independent random variables with identical distributions with mean μ and finite variance σ^2, then

$$\frac{1}{\sqrt{n}\,\sigma}\sum_{k=1}^{n}(X_k - \mu) \xrightarrow{d} N(0,1). \tag{A.1}$$

These assumptions may be weakened considerably. If one keeps the independence assumption and allows the distributions of the X_i to vary but impose certain restrictions on the variances of the distributions we may state the following.

Lindeberg-Levy-Feller Theorem: Let $X_1, X_2, \ldots$ be independent random variables with finite variances, and set, for $k \geq 1$, $E\,X_k = \mu_k$, $Var\,X_k = \sigma_k^2$, and, for $n \geq 1$, $s_n^2 = \sum_{k=1}^{n}\sigma_k^2$. The Lindeberg conditions are:

$$L_1(n) = \max_{1\leq k\leq n}\frac{\sigma_k^2}{s_n^2} \to 0 \text{ as } n \to \infty. \tag{A.2}$$

$$L_2(n) = \frac{1}{s_n^2}\sum_{k=1}^{n} E[|X_k - \mu_k|^2] I\{|X_k - \mu_k| > \varepsilon s_n\} \to 0 \text{ as } n \to \infty, \tag{A.3}$$

where $I\{A\}$ is the indicator function of (the set) A.

not the word limit. The theorem is central to probability and statistics.

If Equation (A.3) is satisfied then so is Equation (A.2) and[3]

$$\frac{1}{s_n^2}\sum_{k=1}^{n}(X_k-\mu_k) \xrightarrow{d} N(0,1) \text{ as } n \to \infty. \tag{A.4}$$

A proof of this theorem is given in Gut (2005). Thus the sum of independent random variables is normal subject to some fairly unrestrictive conditions on the tails of the distribution. The value of many variables of interest in economics or finance are the result of the cumulative effect of a large number of shocks and are thus taken as having a normal distribution. Returns on equities or on a portfolio of equities may be regarded as the outcome of a large number of transactions or as the result of the accumulation of news. Returns would tend to normal if, in the limit, the effect of individual items in the accumulation does not have a significant effect on the sum. A first reaction is that individual items do not have a significant effect on the aggregate and that the only question outstanding is whether the number of effects in the accumulation is sufficient to justify using the asymptotic normal limit. However, how many times is it necessary, in econometrics, to insert a dummy variable for an outlier? Outliers can occur by chance even if a normal distribution is appropriate but they might also be an indication of a failure of the central limit theorem. In such a case we have the generalised central limit theorem which states conditions under which aggregates tend towards an α-stable distribution. In the generalised central limit theorem the restriction that individual items do not have a significant effect on the distribution of the aggregate is replaced by a condition that they do. We return to the generalised central limit theorem in A.3 after a review of the properties of the α-stable distribution.

[3]The notation $\xrightarrow{d}$ implies a limit in distribution. The notation $\stackrel{d}{=}$ implies that the variables on either side of the sign have the same distribution.

A.2 The α-stable Distribution

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables and let

$$(X_1 + X_2 + \cdots + X_n) \stackrel{d}{=} b_n X + a_n, \tag{A.5}$$

where

- $b_n > 0$ and a_n are real constants, a_n is a location parameter and b_n a scaling factor.
- $\lim_{n\to\infty} \max\{P(|X_j b_n^{-1}| > \varepsilon) : j = 1, 2, \ldots, n\} = 0$. A sufficient condition for this to hold is that $b_n \to \infty$ as $n \to \infty$. ($P\{A\}$ is the probability of the set A).
- n is any positive integer, and
- the distribution of X is not degenerate.

Then X is an α-stable random variable.[4] The term "stable" refers to the property that the sum of identically distributed independent random variables has the same distribution as the original, up to scale (b_n) and location (a_n) factors. The term "stable" here implies that the distribution of a random variable is invariant or stable under addition of independent copies. Neglecting possible time of day, day of week, seasonal and other effects of a similar nature, one might expect that returns on equities would have this property.

If the X's are normal with mean μ and variance σ^2 the appropriate scale and location factors are

$$b_n = \sqrt{n} = n^{\frac{1}{2}},$$
$$a_n = (1 - n^{\frac{1}{2}})\mu.$$

[4] When $a_n = 0$ we say that the distribution is strictly α-stable. Lévy (1954) referred to this strictly α-stable distribution as a "loi stable". See also footnote 7 on page 17.

It can also be shown (see Feller (1971, p. 170)) that the value of the scaling factor b_n is restricted and can only take a value given by

$$b_n = n^{\frac{1}{\alpha}}, \quad \text{for,} \quad 0 < \alpha \leq 2. \tag{A.6}$$

For the normal distribution $\alpha = 2$. The parameter α is referred to as the stability parameter or characteristic exponent of the α-stable distribution. The α-stable distribution is, in effect, a family of distributions indexed by the value of the stability parameter.

The characteristic function of an α-stable distribution can be deduced from the stability property and is given by

$$\begin{aligned}\phi(t) &= E[e^{itx}] \\ &= \int e^{itx} dS(x) \\ &= \begin{cases} \exp(-\gamma^{\alpha}|t|^{\alpha}[1 - i\beta(\tan\frac{\pi\alpha}{2}) \text{ sign } t] + i\delta t), & \text{if } \alpha \neq 1 \text{ ;} \\ \exp(-\gamma\,|t|\,[1 + i\beta\frac{2}{\pi}(\text{ sign } t) \log(|t|)] + i\delta t), & \text{if } \alpha = 1. \end{cases}\end{aligned} \tag{A.7}$$

(see Zolotarev (1986) or Samorodnitsky and Taqqu (1994)). E is the expectation function, $S(x)$ is the α-stable distribution function and $i = \sqrt{-1}$. The sign t function is defined as

$$\text{sign } t = \begin{cases} -1, & t < 0; \\ 0, & t = 0; \\ 1, & t > 0. \end{cases} \tag{A.8}$$

The distribution depends on four parameters α, β, γ and δ. These parameters[5] can be interpreted as follows:

- α is the basic stability parameter. It determines the weight in the tails. It is the same α as that in Equation (A.6).
- β is a skewness parameter and $-1 \le \beta \le 1$. A zero beta implies that the distribution is symmetric. Negative or positive β imply that the distribution is skewed to the left or right, respectively.
- The parameter γ is positive and measures dispersion. When the α-stable distribution is normal then $\gamma = \sigma/\sqrt{2}$, where σ is the standard deviation of the normal distribution.
- The parameter δ is a real number and may be thought of as a location measure. When the α-stable distribution is normal then $\delta = \mu$

[5]Note that different notation is adopted by various authorities. The principal differences include

- reversal of the sign of β
- Substitution of $c = \gamma^\alpha$
- Substitution of $\sigma/\sqrt{2}$ for γ
- The characteristic function in Equation (A.7) is not continuous at $\alpha = 1$. This may lead to problems in certain circumstances. If one makes the substitution

$$\delta_0 = \begin{cases} \delta + \beta\gamma \tan\frac{\pi\alpha}{2} & \alpha \neq 1 \\ \delta + \beta\frac{2}{\pi}\gamma\log\gamma & \alpha = 1, \end{cases}$$

following the notation of Nolan (2007) we may write the characteristic function of an α-stable function as

$$\int e^{itx} dS(x) = \begin{cases} \exp\left(-\gamma^\alpha|t|^\alpha[1 + i\beta(\tan\frac{\pi\alpha}{2})(\text{sign } u)(|\gamma t|^{1-\alpha} - 1)] + i\delta_0 t]\right) & \alpha \neq 1 \\ \exp\left(-\gamma|t|[1 + i\beta\frac{2}{\pi}(\text{sign } u)\log(\gamma|t|)] + i\delta_0 t\right) & \alpha = 1 \end{cases}$$

Because of the better behaviour of this parametrisation at $\alpha = 1$ it is the form most often used in numerical calculations. Nolan refers to this as an $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 0)$ distribution. The parametrisation in Equation (A.7) is referred to as an $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 1)$ distribution and is the form most often used here. In the $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 1)$ note that when $1 < \alpha \le 2$, $EX = \mu$. In the $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 0)$ this does not hold, in general. Note than if $\beta = 0$ or $\alpha = 2$ the two parameterisations coincide. Here, in the $\boldsymbol{S}(\alpha, \beta, \gamma, \delta; 1)$ parametrisation the density and distribution functions will be denoted by $s(x, \alpha, \beta, \gamma, \delta)$ and $S(x, \alpha, \beta, \gamma, \delta)$ respectively. If the variables are standardised ($\gamma = 1$ and $\delta = 0$) we may use the symbols $s(x, \alpha, \beta)$ and $S(x, \alpha, \beta)$ for the density and distribution respectively.

The characteristic function of the α-stable distribution is absolutely integrable and thus α-distributions are absolutely continuous with a bounded continuous density function which can be obtained by the standard inversion formula for characteristic functions. The density function of the α-stable distribution is differentiable on the real line (see Gnedenko and Kolmogorov (1954)). The α-stable density function is unimodal (Kanter (1976)). If $\beta = 0$ the characteristic function is

$$\phi(t) = \int e^{itx} dS(x) = e^{it\delta - \gamma^{\alpha}|t|^{\alpha}}. \tag{A.9}$$

If, in addition, $\delta = 0$, the characteristic function is real and the distribution is symmetric.

Except in three special cases the density function of the α-stable distribution can not be expressed in terms of elementary functions. The special cases are:

Normal Distribution If $\alpha = 2$ the characteristic function in Equation (A.7) reduces to

$$\phi(t) = \int e^{itx} dH(x) = e^{(i\delta t + \gamma^2 t^2)}, \tag{A.10}$$

which is the characteristic function of the normal distribution

$$\frac{1}{2\gamma\sqrt{\pi}} \exp \frac{(x-\delta)^2}{4\gamma^2}, \quad -\infty < x < \infty$$

with mean δ and variance $2\gamma^2$. Note that as $\tan \pi = 0$ the skewness parameter, β, does not appear in the characteristic function in this case. Conventionally it is taken to be zero.

Cauchy Distribution When $\alpha = 1$ and $\beta = 0$ the characteristic function reduces to

$$\phi(t) = \exp(-\gamma|t| + i\delta t),$$

which is the characteristic function of the Cauchy Distribution

$$\frac{1}{\pi(\gamma^2 + (x - \delta)^2)}, \quad -\infty < x < \infty.$$

Lévy Distribution When $\alpha = 1/2$ and $\beta = -1$ the characteristic function reduces to

$$\phi(t) = \exp(-\sqrt{\gamma|t|}(1 + \text{sign } t) + i\delta t),$$

which is the characteristic function of a Lévy distribution

$$\left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x - \mu)^{3/2}} \exp\left(-\frac{\sigma}{2(x - \mu)}\right), \quad \mu < x < \infty.$$

Figures A.1 to A.3 illustrate various properties of α-stable distributions. Figure A.1 shows the density functions for symmetric ($\beta = 0$) α stable distributions with $\alpha = 2$ (normal), $\alpha = 1.5$ and $\alpha = 1.0$ (Cauchy). As α is reduced, the peak gets higher and the tails get heavier. This process continues as α is reduced. Figure A.2 is an enlarged version of the left tail of the distribution and shows clearly the heavier tails.

Figure A.3 shows the effect of varying the symmetry parameter β for fixed α. With $\alpha = 1.5$ As β falls from 0 to -1 the left tail becomes heavier relative to the right tail and the mode of the distribution shifts to the left of the mean. Similar transformations occur in the opposite direction when β moves from 0 to 1. The skewness caused by a particular value of β is more marked as α is reduced.

A.3 A Generalised Central Limit Theorem

Consider a random variable X with density function $f(x)$ such that

$$f(x) \sim \begin{cases} B_-|x|^{-(1+a)} & \text{as} \quad x \to -\infty \\ B_+|x|^{-(1+a)} & \text{as} \quad x \to \infty, \end{cases} \tag{A.11}$$

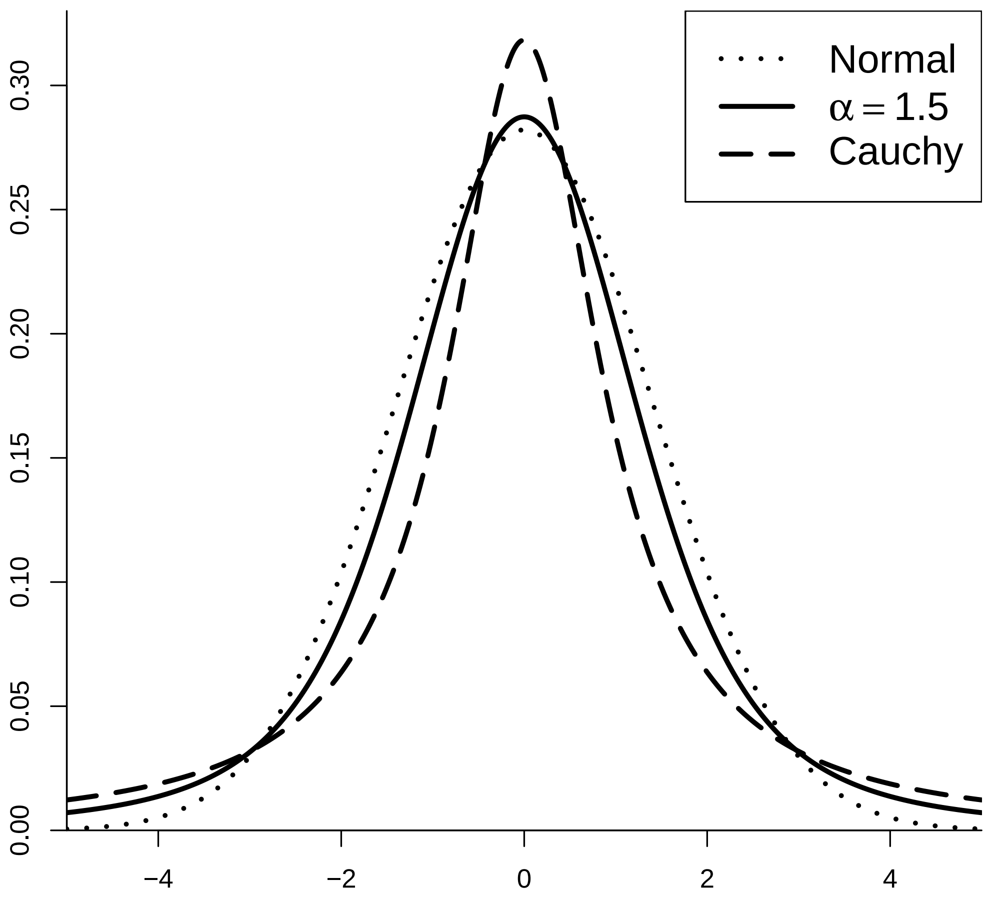

Figure A.1: Normal, α-stable ($\alpha = 1.5$) and Cauchy Distributions

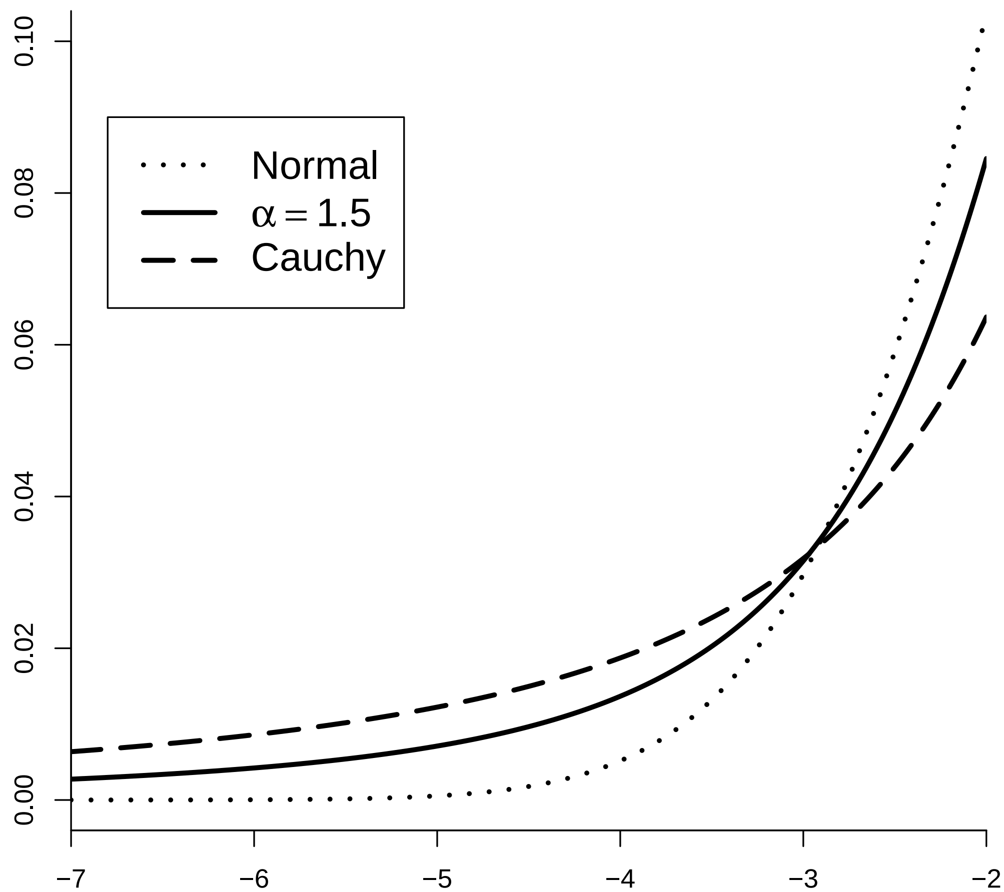

Figure A.2: Tails of Normal, α-stable ($\alpha = 1.5$) and Cauchy Distributions

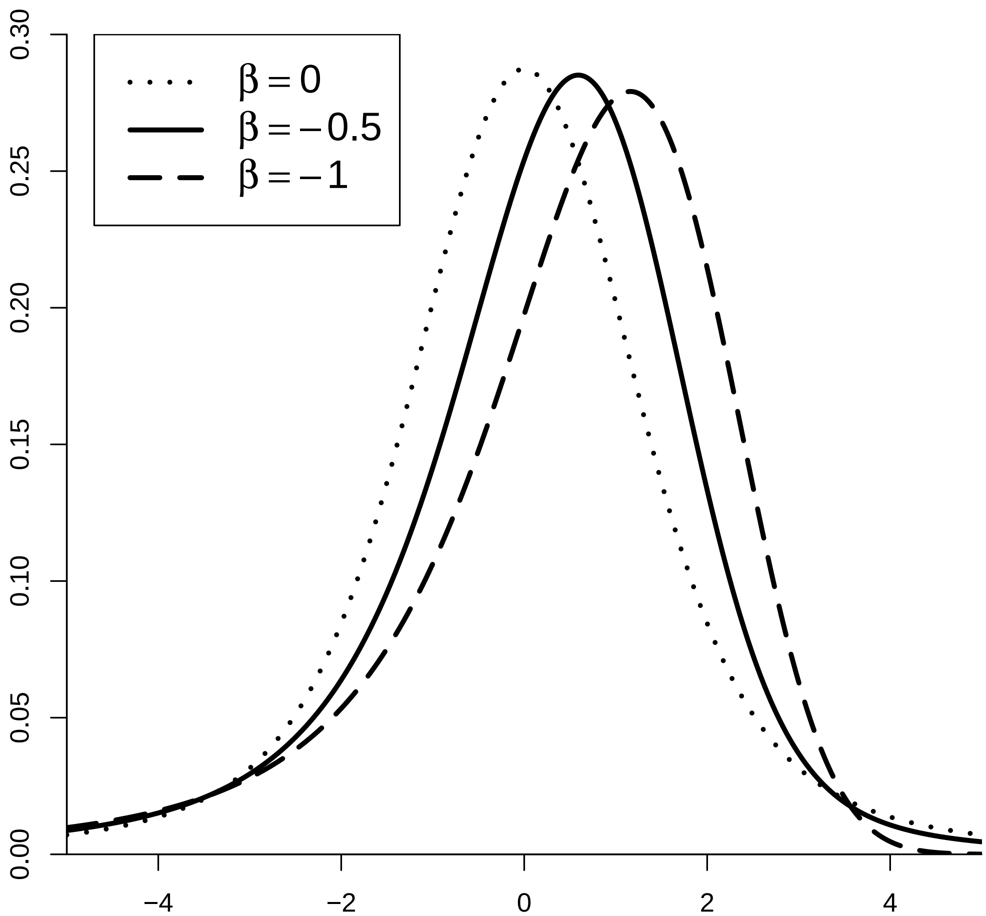

Figure A.3: α-stable Distribution, α = 1.5, β various where $0 < a < 2$ and B_- and B_+ are appropriate positive constants. Thus, asymptotically, the tails of the distribution are proportional to a Pareto distribution.[6] Put

$$B = \frac{B_+ - B_-}{B_+ + B_-} = \frac{B_+}{B_+ + B_-} - \frac{B_-}{B_+ + B_-} \tag{A.12}$$

B measures the excess proportional weight in the positive or negative tails respectively. A more exact statement of this condition would require the use of regularly varying functions (see Feller (1971), page 312).

If $X_1, X_2, \ldots X_n$ are independent, identically distributed random variables for which these conditions hold, then the random variable

$$S = \frac{1}{n^{\frac{1}{a}}} \sum_{i=1}^{n} X_i \tag{A.13}$$

has a limit in distribution which is α-stable with parameters $\alpha = a$ and $\beta = B$.

Returns on assets are often modelled with a binomial tree. In this process a return may move up with probability p and down with probability $1 - p$. The asymptotic limit of the sum of such a process is a normal distribution. To get a corresponding discrete process for an α stable process consider the

[6] The Pareto distribution was used by Pareto almost one hundred years ago to model the distribution of incomes above a certain threshold. A random variable has a Pareto distribution if its density function is of the form:

$$f(x; a, b) = ab^a x^{-(1+a)} \quad x > b,\ a > 0,\ b > 0.$$

This distribution has a remarkable property. If we increase the threshold the shape of the distribution remains the same apart from a scaling factor. For example, by integration, $P\{X \geq cb\} = c^{-a}$. Then the distribution of X given that $X \geq cb$, where $c > 1$ is given by

$$f_X(x; a, cb) = a(cb)^a x^{-(1+a)} \quad x > cb,\ a > 0,\ b > 0,\ c > 1.$$

Thus $P\{X \geq c^2 b | X \geq 2b\} = c^{-a}$. To illustrate let the distribution of the wealth of persons with wealth greater than say €1,000,000 be Pareto with parameter $a = 1.5$ Then the probability that a person in this group will have wealth of twice the threshold is about 0.35. Now let the threshold be €2,000,000 then the probability that a person above that threshold will have a wealth of twice that threshold (€4,000,000) is again 0.35. This is in complete contrast to the normal or lognormal distribution. Note that the mean of this distribution exists if $a > 1$ and the variance if $a > 2$.

Pareto distribution

$$f(x;\alpha,b) = \alpha b^{\alpha} x^{-(\alpha+1)}, \quad x \geq b,\ b > 0,\ \alpha > 0.$$

We may consider this as one tail of a process. Consider the following case where the tail of the distribution ($x > b > 0$) has a Pareto distribution and $\lambda > 1$.

$$\begin{aligned}
P\{(x < b\} &= p \\
P\{b \leq x < \lambda b\} &= (1-p)\left(1 - \left(\frac{1}{\lambda}\right)^{\alpha}\right) \\
&\dots \\
P\{\lambda^{k-1} b \leq x < \lambda^k b\} &= \frac{1}{\lambda^{\alpha(k-1)}}(1-p)\left(1 - \left(\frac{1}{\lambda}\right)^{\alpha}\right) \\
&\dots
\end{aligned}$$

This leads us to consider the following generalisation of a random walk. We have a process that at each increment of time takes a step the size of which may be $\pm b,\ \pm b\lambda, \dots,\ \pm b\lambda^{k-1}, \dots$ with probabilities given by

$$\begin{array}{ll}
\pm b & \frac{p}{2} \\
\pm \lambda b & \frac{(1-p)}{2}\left(1 - \left(\frac{1}{\lambda}\right)^{\alpha}\right) \\
\dots & \dots \\
\pm \lambda^k b & \frac{1}{\lambda^{\alpha(k-1)}}\frac{(1-p)}{2}\left(1 - \left(\frac{1}{\lambda}\right)^{\alpha}\right) \\
\dots & \dots
\end{array}$$

The tails of this process are asymptotically proportional to a Pareto distribution with parameter α. The k^{th} absolute moment of this distribution is given by

$$E[|X|^k] = b^k \left(p + \lambda(1-p)\left(1 - \left(\frac{1}{\lambda}\right)^{\alpha}\right)\left(\sum_{j=1}^{\infty} \frac{1}{\lambda^{(j-1)(\alpha-k)}}\right)\right).$$

For $\lambda > 1$ this converges only for $k < \alpha$. Thus if $\alpha > 2$ the variance exists and is finite and the standard central limit theorem, Equation (A.1), implies that the asymptotic distribution of sums of independent random variables drawn from this distribution will be normal. If $0 < \alpha \leq 2$, the variance does not exist. The asymptotic distribution of the tails of the distribution are such that the generalised central limit theorem applies. Thus the asymptotic distribution of sums of independent random variables drawn from this distribution will be α-stable with stability parameter α.

A.4 Some properties of α-stable distributions

Some of the more important properties of α-stable distributions are given below

- The only α-stable distribution for which moments of all orders exist is the normal distribution. When $1 < \alpha < 2$ the variance is not defined (infinite) and $E[X^k]$ exists only when $k < \alpha$. In this case in our notation the mean exists when $\alpha > 1$ and is given by $E[X] = \delta$. Apart from the lack of a simple form for the density function of an α-stable density function, the non-existence of a variance is the greatest barrier to its use. Put simply, measures of the variance of an α-stable process will increase with sample size and will not converge.[7]

  If $0 < \alpha \leq 1$ the mean does not exist. If $\alpha < 1$ the sampling distribution of the mean is even more dispersed than the individual measurements. In applications of the α-stable distribution to finance, values of α of the order of 1.5 to 1.8 are usually appropriate. The values estimated in table 2.2 vary from 1.63 to 1.73.

- The α-stable density is symmetric with respect to simultaneous changes of the sign of x and β, that is

$$s(x, \alpha, \beta, \gamma, \delta) = s(-x, -\beta, \gamma, \delta). \tag{A.14}$$

[7] See footnote 11 on page 19. For values of α close to those found in finance it may be difficult to observe this effect.

- If a and $b > 0$ are real constants and the density of x is given by $s(x, \alpha, \beta, \gamma, \delta)$ then the density of $\frac{x-a}{b}$ is given by

$$\frac{1}{b}s(\frac{x-a}{b}, \alpha, \beta, \frac{\gamma}{b}, \frac{\delta - a}{b}) \tag{A.15}$$

or in particular that of $\frac{x-\delta}{\gamma}$ by

$$\frac{1}{\gamma}s(\frac{x-\delta}{\gamma}, \alpha, \beta, 1, 0). \tag{A.16}$$

- Let X_1 and X_2 be α-stable random variables with densities $s(x, \alpha, \beta_i, \gamma_i, \delta_i)$, $i = 1, 2$. Then $X_1 + X_2$ is α-stable with

$$\beta = \frac{\beta_1\gamma_1^\alpha + \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, \quad \gamma = (\gamma_1^\alpha + \gamma_2^\alpha)^{\frac{1}{\alpha}}, \quad \delta = \delta_1 + \delta_2 \tag{A.17}$$

A.5 Domains of Attraction

Let $X_1,\ X_2,\ \ldots,\ X_n$ be independent identically distributed random variables. If these random variables have a finite variance the central limit theorem implies that one can find a_n and b_n such that

$$\frac{X_1 + X_2 + \cdots + X_n}{a_n} - b_n \xrightarrow{d} G$$

where G is the normal distribution function. The distribution of X is said to be in the domain of attraction of the normal distribution. The generalised central limit also implies that for each value of α, the α-stable distribution also has a domain of attraction. In Section A.3 we have already looked at one member of the domain of attraction of a stable distribution. It can be proven that the only distributions that have domains of attraction are the α-stable distributions (see Feller (1971), Chapter IX). The implication is that even if equity returns do not have an exact α-stable distribution then an α-stable distribution will provide an approximation to their distribution.

Consider the example in Section A.3. We have argued in Chapter 5 that an observed shortfall in the extreme tails of the observed data relative to the α-stable distribution may have two possible causes. First a supervisory authority or an exchange may intervene to alleviate extreme losses. Alternatively a severe loss on equities may involve externalities and one may be observing the private cost of the loss rather than the public cost. To measure public cost it is necessary to isolate the effects of the equity loss from other factors that may have diminished economic growth rates. The diminished growth rates may themselves have contributed to the equity loss. Two possible ways have been proposed to model the deficit in the tails of the observed distribution.

1. Abandon the idea of measuring the entire sample of returns and concentrate on modelling extremes. This is the topic of Extreme Value Theory which is comprehensively covered in the books such as McNeil et al. (2005) and Embrechts et al. (1997). Applications of Extreme Value Theory in Hydrology, Environmental Sciences, Finance and Insurance and Material and Life Sciences are covered in Reiss and Thomas (2007). It is possible that this methodology may give good VaR estimates on many occasions. Such measures of risk are based on an assumption that the tails of the distribution have an asymptotic distribution

   $$f(x,\alpha) = cx^{-(1+\alpha)} \text{ for } x > x_0 \text{ and } x_0 \text{ large, } \alpha > 0,$$

   but do not impose the condition $\alpha \leq 2$. In particular the t-distribution and Pareto distribution (trivially) have this property. If $0 < \alpha \leq 2$ we are still in the realm of an α-stable distribution. Many sources have estimated an α of the order of 4 and claim that this rules out an α-stable distribution. On the contrary Weron (2001) finds that if α is in the range typically estimated for returns data then extreme value theory may give estimates of $\alpha > 2$. We should also note that distributions such as the t-distribution are not scaling.[8]

2. Magenta and Stanley (1994) consider the case of a physical system in which there is a natural cutoff (e.g. the quantity of a compound can not

[8] Apart from the particular case where there is one degree of freedom and the t-distribution becomes a Cauchy distribution.

fall below a single molecule). There may also be a physical upper bound. In the example in Section A.3 this would correspond to the case where the discrete distribution is cut off above and below at some truncation point. The variance of the truncated discrete distribution is then finite, the ordinary central limit will apply and the asymptotic distribution of the sums will be normal. If the truncation points are far from the mean, the relevant α-stable distribution provides a better fit to the empirical distribution than the normal. As the number of summands is increased the fit of the normal will improve and the relative fit of the α-stable will deteriorate. There is a turnover point where the normal distribution becomes more accurate. However, as the truncation points become remote the turnover point increases rapidly. Menn and Rachev (2005) extend this idea to "Smoothly Truncated Stable Distributions" where the tails of the α-stable distribution are replaced by similarly sized tails of a normal distribution in such a way as to maintain continuity of the density function. These arguments are an alternative solution to the problems outlined in the previous point. It may be possible that the extremes of returns data are censored by acts of an exchange or regulatory body who wish to avoid contagion and step in to support or limit certain effects. Such actions may reduce the private cost of a "crash" but the returns data will not reflect the total social cost. Regulatory bodies should be more interested in the public cost. The additional safety margin which an α-stable distribution provides should find favour in a regulatory body.

A.6 CAPM Models and the α-stable Distribution

Fama (1965b, 1971) showed how the Sharpe-Lintner Capital Asset Pricing Models could be extended to take account of returns that might have an α-stable distribution. He assumes that the returns on n individual assets arise from the fact that there is a common market factor M that affects the returns on all n assets in addition to an individual specific effect ε_i that impinges only on the individual asset. He notes that additional variables may also be included in the regression without altering the main argument. The one-period return on asset i can be written

$$r_i = \alpha_i^{CAPM} + \beta_i^{CAPM} M + \varepsilon_i, \quad i = 1, 2, \ldots, n \tag{A.18}$$

where the superscript "$CAPM$" is used to distinguish the CAPM parameters α^{CAPM} and β^{CAPM} from the parameters of the α-stable distribution. Fama assumes that M and ε_i $i = 1, 2, \ldots, n$, have independent symmetric α-stable distributions with the same stability parameter α. Note that each return is a sum of independent effects but the returns are not independent as each has the common factor M. The return on asset i is thus symmetric α-stable with stability parameter α, $\beta = 0$ and scale and location parameters γ_i and δ_i given by

$$\gamma_i = \left(\gamma_M^{\alpha^{CAPM}} \left|x_i \beta_i^{CAPM}\right|^{\alpha^{CAPM}} + \gamma_{\varepsilon_i}^{\alpha^{CAPM}} |x_i|^{\alpha}\right)^{1/\alpha^{CAPM}} \quad \text{and}$$
$$\delta_i = \alpha_i^{CAPM} + \beta_i^{CAPM} E[M], \ \text{respectively.}$$

The return on the portfolio is then symmetric α-stable with the same α parameter as the individual assets. If x_i, $i = 1, 2, \ldots, n$, are the shares of each asset in the portfolio then the return, δ_P, expected return, $E[\delta_P]$, and scale parameters, γ_P, are given by

$$\begin{aligned}
\delta_P &= \sum_{i=1}^{n} x_i \alpha_i^{CAPM} + \sum_{i=1}^{n} x_i \beta_i^{CAPM} M + \sum_{i=1}^{n} x_i \varepsilon_i \\
E[\delta_P] &= \sum_{i=1}^{n} x_i \alpha_i^{CAPM} + \sum_{i=1}^{n} x_i \beta_i^{CAPM} E[M] \\
\gamma_P &= \left(\gamma_M^{\alpha^{CAPM}} \left|\sum_{i=1}^{n} x_i \beta_i^{CAPM}\right|^{\alpha^{CAPM}} + \sum_{i=1}^{n} \gamma_{\varepsilon_i}^{\alpha^{CAPM}} |x_i|^{\alpha^{CAPM}}\right)^{1/\alpha^{CAPM}}
\end{aligned}$$

Fama (1971) shows that if consumers are risk averse maximisers of expected utility with an inter-temporal utility function $u(c_1, c_2)$ of period 1

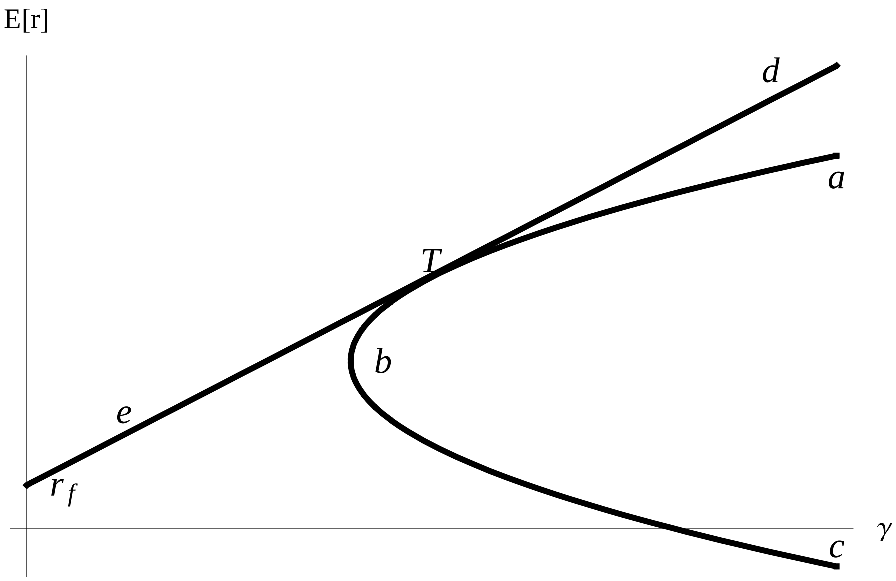

Figure A.4: CAPM Efficient Frontiers

(c_1) and period 2 (c_2) consumptions that is monotone increasing and strictly concave in (c_1, c_2) then, in the usual two-period model, the expected utility $E[u(c_1, c_2)]$ can be expressed as a function, $V(\delta_P, \gamma_P)$, of the α-stable location and scale parameters of the portfolio. With no risk-free asset the optimisation process continues in a very similar way to that of the corresponding Markovich portfolio optimisation except that the risk measure, standard deviation or equivalently variance, is replaced by the α-stable scale parameter γ. Thus, this portfolio optimisation problem is to find the values of x_i that minimise γ_P for fixed δ_P and $\sum x_i = 1$. With no risk free asset the efficient set is the curve ab in Figure A.4.

Assume that one may be long or short, to any extent, in a risk free asset with a rate of return of r_f. Let r_p and γ_p are the return and scale parameter of a portfolio on the boundary of the Markovich efficient curve ab in Figure A.4. By borrowing or investing at the risk-free rate we can achieve any point on the line joining the points where total investment is in the risk free asset $(0, r_f)$ and this boundary point. The slope of this line, $(r_p - r_f)/\gamma_p$ is the rate at which return is increased per unit of scale parameter. This slope is functionally the equivalent of the Sharpe ratio as that ratio is usually defined. With this in mind we will also refer to the revised ratio as the Sharpe ratio. One can always achieve a higher utility level if the Sharpe ratio is higher. In Figure A.4 The highest Sharpe ratio is achieved when the line dTe is tangent to the Markovich efficiency curve.

The CAPM theory makes various assumptions about markets. These generally include frictionless markets, ability to borrow or lend unlimited amounts at the risk free-rate, investors are price takers, neutral taxes and all investors have the same knowledge of returns and risk. The CAPM assumptions imply that in order for the market to clear the market portfolio must be on the minimum variance portfolio frontier.

This implies that the relationship between the expected return, $E[r_i]$, on the i_{th} component of the portfolio and the expected return of the market portfolio, $E[r_m]$, is given by

$$E[r_i] - r_f = \beta_i^{CAPM}(E[r_m] - r_f).$$

In the normal case

$$\beta_i^{CAPM} = \frac{Cov(r_i, r_M)}{\sigma^2(r_M)}.$$

In the non-normal α-stable case Fama (1971) showed that

$$\beta_i^{CAPM} = \frac{1}{\gamma_m}\frac{\partial \gamma_m}{\partial x_j}$$

Considerable work has gone into the estimation and testing of the normal CAPM and its many extensions. See, for example, Campbell et al. (1997) or Cochrane (2005). Four different estimates for the non-normal α-stable β^{CAPM} can be considered.

1. OLS estimates are consistent but are inefficient. Confidence intervals for the regression coefficients are not available.

2. Blattberg and Sargent (1971) proposed an unbiased minimum dispersion estimator of a univariate regression coefficient where the independent

variable is deterministic and the disturbances follow an α-stable distribution. Again confidence intervals are not available. Franke et al. (2000) compare the results of these first two estimates of the β^{CAPM}. They also give Monte-Carlo based standard deviations for both estimates. OLS relative efficiency fall as α is reduced. Their sample size of 250 is small for inference about α. It should be noted that the Blattberg and Sargent (1971) estimators were derived on the basis of non-stochastic regressors.

3. The methods described in Chapter 3 might be applied. The confidence limits are not appropriate as the theory there is based on non-stochastic regressors. We have encountered several problems in applying this methodology to a project "Evaluating the Trading and Risk Management Style of Fixed-Income Hedge Fund Managers". It may be possible to use the methods described in point 4 following.

4. It is possible to base estimates of β^{CAPM} on the theory of multivariate stable estimators. Hardin, Jr. et al. (1991a,b) cover the theory. We do not know of any application of this methodology to CAPM estimation but intend to pursue this line at a later stage.

Some may consider the assumption of a constant α to be restrictive but one should remember that the usual arguments based on a normal distribution include the assumption of a fixed $\alpha = 2$.

All the references to CAPM and the non-normal α-stable distribution include an assumption that the distribution is symmetric. This assumption can be relaxed to the extent of assuming that the returns distributions of all the candidate assets have the same skewness parameter. In this case the application of the α-stable properties described in Section A.4 imply that all portfolios which are linear combinations of these assets will have the same skew parameter as the original assets. The stability and skew parameters of the α-stable distribution are not effected by portfolio size or by leverage. The CAPM argument outlined above still holds.

If the portfolio contains assets with different skewness parameters it may be possible to construct two portfolios with the same expected return and spread but different skew. An agent is likely to prefer the return which has the smaller relative negative losses. This problem applies to all skewed distributions.

A.7 Numerical Analysis

A.7.1 Evaluation of Density and Likelihood functions

Nolan (1997) reviews the numerical inversion of the characteristic function of an α-stable distribution. He also describes the methods used in the program STABLE (Nolan (2005b, 2006)). The numerical method used is based on the representation of α-stable distributions by integrals in Zolotarev (1986). The STABLE program also offers a faster approximation to the functions which was not used in this analysis. The library version of the STABLE routines used in conjunction with C++ provides a fast and convenient way of working with α-stable distributions if one has some knowledge of C++. Here the STABLE program was used in conjunction with DEV-C++ (`http://www.bloodshed.net/devcpp.html`) and the Mingw port of GCC (GNU Compiler Collection). The stand alone version of STABLE provides basic functions to estimate α-stable densities, distributions and to fit and test the fit of a series to an α-stable distribution. The stand alone version is available for download from `http://academic2.american.edu/~jpno` The library version is available from `http://www.robustanalysis.com/`. Thanks are due to John P. Nolan who made the library version available for this research.

Bob Rimmer's (Rimmer (2007)) package for MATHEMATICA adds α-stable functions to MATHEMATICA. This package allows these to be estimated directly (numerical integration of inverse Fourier transforms or using interpolation over a preestimated grid of values. The MATHEMATICA source code of the current version of these routines is available and these can be amended to give very high accuracy at the expense of processing time. The MATHEMATICA numerical integration routines can be used to estimate the density function but care must be taken to ensure that the integrals converge properly.

The Rmetrics routines Wuertz (2007) use the numerical integration routines in R to calculate the density function. We found that these routines were slower than the routines in MATHEMATICA and in the STABLE program.

A.7.2 Feasibility of Maximum Likelihood Estimation

The theory of maximum likelihood for estimation of α-stable parameters was set out in DuMouchel (1971, 1973, 1975). This requires a considerable amount of computation which was not fully feasible at the time. [9] Now both versions of the STABLE program provided maximum likelihood estimates of the parameters of an α-stable process. The calculations are fast and accurate for the range of parameter values encountered. It also provides standard errors of the parameter estimates. The method of calculating these standard errors is outlined in Nolan (2001) and is based on a precalculated grid. We used the Library density functions with C++ to estimate the α-stable GARCH processes in Chapter 5.

The MATHEMATICA routines do not provide standard errors of the parameters but these can be added by calculating and inverting the information matrix. An example of the MATHEMATICA density routines, used in the estimation of the regressions in Chapter 3, is given below.

Maximum likelihood estimation with the α-stable density functions in Rmetrics and the optimisation functions in R did not always converge. Sometimes convergence was only achieved after considerable fine-tuning. Convergence to an optimum was slow. It is possible that performance could be improved by resetting default parameters and varying the optimisation method. C++ programs based on Nolan's STABLE packages and the Rimmer MATHEMATICA routines performed better when used here. Appendix B.2 contains a copy of a C++ program used to calculate maximum likelihood estimates of the parameters of an α-stable GARCH process.

[9] To reduce the amount of calculations required DuMouchel based his maximum likelihood procedures on grouped data in the centre of the distribution.

APPENDIX B

Computer Listings

B.1 MATHEMATICA Program to Estimate Day of Week Effects

This section contains the output of the MATHEMATICA (Wolfram (2003)) program used to estimate the day of the week effects for returns on the ISEQ in Chapter 2. The original code was run in Version 5 but the version here has been updated to run in Version 6. There are some very minor changes in results between versions but these are not significant. The program output has been edited to allow long lines to flow to the next page. The output of some of the program has been suppressed.

Day of week analysis with stable residuals - Germany DAX30

Read and check data

```
Needs["StableM `"]
Directory[];
SetDirectory["C:\\WORK\\PHD\\Thesis\\thesis_1\\weekday\\MATHEMATICA_6"];
data = Import["returns.csv", "CSV"];
{nobs, vars} = Dimensions[data];
nobs = nobs − 2;
Table[data[[ii, jj]], {ii, 1, 4}, {jj, 1, vars}];
Table[data[[ii, jj]], {ii, nobs − 1, nobs + 2}, {jj, 1, vars}];
return = Table[data[[ii, 2]], {ii, 3, nobs + 2}];
```

Summary Statistics

Full Period

```
{"Mean = ", Mean[return],
"\nMedian = ", Median[return],
"\nVariance = ", Variance[return],
"\nStandard Deviation = ", StandardDeviation[return],
"\nSkewness = ", Skewness[return],
"\nKurtosis = ", Kurtosis[return]};
```

$\text{jb} = \text{nobs}\left(\frac{(\text{Kurtosis[return]}-3)^2}{24}+\frac{\text{Skewness[return]}^2}{6}\right);$

α-stable fit of return series

```
parm1 = SFit[return, 1]//AbsoluteTiming;
parm1;
parm1 = parm1[[2]];
"Estimates of Stable Parameters of returns using FFT to estimate PDF"
Clear[maxlike];
maxlike[{(a_)?NumericQ, (b_)?NumericQ, (c_)?NumericQ, (d_)?NumericQ}]:=
−Dimensions[return]Log[c] + FLogLikelihood[(return − d)/c, {a, b, 1, 0}, 1];
```

```
i4 = IdentityMatrix[4];
h = 10^(-$MachinePrecision/4);
k = 10^(-$MachinePrecision/4);
u = parm1;
nhess =
Table[
1/(4hk) (maxlike[u + h i4[[ii]] + k i4[[jj]]] − maxlike[u − h i4[[ii]] + k i4[[jj]]]−
maxlike[u + h i4[[ii]] − k i4[[jj]]] + maxlike[u − h i4[[ii]] − k i4[[jj]]]),
{ii, 1, 4}, {jj, 1, 4}];
cov4 = Inverse[−Partition[Flatten[nhess], 4]];
cor4 = Table[cov4[[ii, jj]] / Sqrt[cov4[[ii, ii]]cov4[[jj, jj]]], {ii, 1, 4}, {jj, 1, 4}];
stdev4 = Table[Sqrt[cov4[[ii, ii]]], {ii, 1, 4}];
TableForm[Table[{parm1[[ii]], stdev4[[ii]], parm1[[ii]]/stdev4[[ii]]}, {ii, 1, 4}],
TableHeadings → {{"α", "β", "γ", "δ"}, {"Estimate", "St. error", "z-value"}}]
{"Maximum Likelihood", ml4 = FLogLikelihood[return, parm1, 1]}
"****************************************************"
```

Estimates of Stable Parameters of returns using FFT to estimate PDF

	Estimate	St. error	z-value
α	1.63252	0.0245188	66.5824
β	−0.0535249	0.0523431	−1.02258
γ	0.500529	0.00798075	62.717
δ	0.0548347	0.0156043	3.51408

{Maximum Likelihood, −5864.63}

****

```
SLogLikelihood[return, parm1, 1]//AbsoluteTiming;
FLogLikelihood[return, parm1, 1]//AbsoluteTiming;
```

Week Day effects - Full Sample

OLS Regression Estimates

```
<< LinearRegression`
Clear[monday, tuesday, wednesday, thursday, friday]
Clear[regdata]
```

regdata = Table[{data[[ii + 2, 3]], data[[ii + 2, 4]], data[[ii + 2, 5]], data[[ii + 2, 6]], data[[ii + 2, 7]], return[[ii]]}, {ii, 1, nobs}];

Regress[regdata, {monday, tuesday, wednesday, thursday, friday}, {monday, tuesday, wednesday, thursday, friday}, IncludeConstant → False]

DesignedRegress::tsos : Warning: the total sum of squares in the ANOVA Table is uncorrected (not centered on the response mean) when there is no constant term in the model; it is designated U Total.

DesignedRegress::rsqr : Warning: the RSquared and Adjusted

RSquared diagnostics are redefined when there is no constant term in the model.

ParameterTable →

	Estimate	SE	TStat	PValue
monday	0.0462835	0.0308487	1.50034	0.133595
tuesday	0.0471105	0.030832	1.52797	0.126587
wednesday	0.0450905	0.030832	1.46246	0.143685
thursday	0.0624206	0.0308487	2.02344	0.043085
friday	0.0566579	0.0308487	1.83664	0.0663274

RSquared → 0.00306445

AdjustedRSquared → 0.00198482

EstimatedVariance → 0.879318

ANOVA Table →

	DF	SumOfSq	MeanSq	FRatio	PValue
Model	5	12.4793	2.49587	2.83841	0.0145358
Error	4617	4059.81	0.879318		
U Total	4622	4072.29			

regrule = Regress[regdata, {monday, tuesday, wednesday, thursday, friday}, {monday, tuesday, wednesday, thursday, friday}, IncludeConstant → False, RegressionReport → {BestFitParameters, FitResiduals}];

beta0 = BestFitParameters/.regrule

{0.0462835, 0.0471105, 0.0450905, 0.0624206, 0.0566579}

resids = FitResiduals/.regrule;

Length[resids]

4622

ListPlot[resids]

```
Length[resids];
Mean[resids];
Median[resids];
Commonest[resids];
Skewness[resids];
Kurtosis[resids];
jb = nobs (Skewness[resids]^2/6 + Kurtosis[resids]^2/24);
regrule2 = Regress[regdata, {monday, tuesday, wednesday, thursday, friday},
{monday, tuesday, wednesday, thursday, friday}, IncludeConstant → False,
RegressionReport → {CovarianceMatrix, CorrelationMatrix}];
regrule2;
parm0 = SFit[resids]; (*Initial values for Stable estimates*)
FLogLikelihood[resids, parm0, 1];
SLogLikelihood[resids, parm0, 1];
"Estimates of Stable Parameters of Residuals of OLS Regression - Initial values"
Clear[maxlike];
maxlike[{(a_)?NumericQ, (b_)?NumericQ, (c_)?NumericQ, (d_)?NumericQ}]:=
−Dimensions[return]Log[c] + FLogLikelihood[(return − d)/c, {a, b, 1, 0}, 1];
i4 = IdentityMatrix[4];
h = 10^(−$MachinePrecision/4);
k = 10^(−$MachinePrecision/4);
u = parm1;
nhess =
Table[
1/(4hk) (maxlike[u + hi4[[ii]] + ki4[[jj]]] − maxlike[u − hi4[[ii]] + ki4[[jj]]]−
maxlike[u + hi4[[ii]] − ki4[[jj]]] + maxlike[u − hi4[[ii]] − ki4[[jj]]]),
{ii, 1, 4}, {jj, 1, 4}];
cov4 = Inverse[−Partition[Flatten[nhess], 4]];
cor4 = Table[cov4[[ii, jj]] / Sqrt[cov4[[ii, ii]]cov4[[jj, jj]]], {ii, 1, 4}, {jj, 1, 4}];
stdev4 = Table[Sqrt[cov4[[ii, ii]]], {ii, 1, 4}];
TableForm[Table[{parm0[[ii]], stdev4[[ii]], parm0[[ii]]/stdev4[[ii]]}, {ii, 1, 4}],
TableHeadings → {{"α", "β", "γ", "δ"}, {"Estimate", "St. error", "z-value"}}]
{"Maximum Likelihood", ml4 = FLogLikelihood[return, parm0, 1]}
```

```
"*****************************************************"
Null
```

Estimates of Stable Parameters of Residuals of OLS Regression - Initial values

	Estimate	St. error	z-value
α	1.63206	0.0245188	66.5637
β	−0.0524173	0.0523431	−1.00142
γ	0.500401	0.00798075	62.701
δ	0.00340179	0.0156043	0.218004

{Maximum Likelihood, −5875.31}

****

Stable Estimates

function Definitions

```
Clear[sregfit, fregfit]
fregfit[s_]:=
Module[{w, za0, zb0, zc0, a0, b0, c0, d0, za, zb, b10, b20, b30, b40, b50, nobs2},
nobs2 = Length[s];
{a0, b0, c0, d0} = parm0;
{b10, b20, b30, b40, b50} = beta0;
za0 = N[Sqrt[Log[2/a0]]];
maxlike[(a_)?NumericQ, (b_)?NumericQ, (c_)?NumericQ, (b1_)?NumericQ,
(b2_)?NumericQ, (b3_)?NumericQ, (b4_)?NumericQ, (b5_)?NumericQ]:=
−nobs2Log[c]+
FLogLikelihood[(s − (b1monday + b2tuesday + b3wednesday + b4thursday + b5friday))/c,
{a, b, 1, 0}, 1];
w = FindMaximum [a = 2Exp [−za^2];
maxlike[a, If[Abs[b] > 1, Sign[b], b], Abs[c], b1, b2, b3, b4, b5],
{{za, za0, −10.0, 10.0}, {b, b0, −1, 1}, {c, c0, −∞, ∞}, {b1, b10}, {b2, b20},
{b3, b30}, {b4, b40}, {b5, b50}}];
{w[[1]], 2 * Exp[−za^2], b, c, b1, b2, b3, b4, b5}/.w[[2]]];
sregfit[s_]:=
Module[{w, za0, zb0, zc0, a0, b0, c0, d0, za, zb, b10, b20, b30, b40, b50, nobs2},
```

```
nobs2 = Length[s];
{a0, b0, c0, d0} = parm0;
{b10, b20, b30, b40, b50} = beta0;
za0 = N[Sqrt[Log[2/a0]]];
maxlike[(a_)?NumericQ, (b_)?NumericQ, (c_)?NumericQ, (b1_)?NumericQ,
(b2_)?NumericQ, (b3_)?NumericQ, (b4_)?NumericQ, (b5_)?NumericQ]:=
−nobs2Log[c]+
SLogLikelihood[(s − (b1monday + b2tuesday + b3wednesday + b4thursday + b5friday))/c,
{a, b, 1, 0}, 1];
w = FindMaximum [a = 2Exp [−za^2] ;
maxlike[a, If[Abs[b] > 1, Sign[b], b], Abs[c], b1, b2, b3, b4, b5],
{{za, za0, −10, 10}, {b, b0, −1, 1}, {c, c0, −∞, ∞}, {b1, b10}, {b2, b20},
{b3, b30}, {b4, b40}, {b5, b50}}];
{w[[1]], 2 ∗ Exp[−za^2], b, c, b1, b2, b3, b4, b5}/.w[[2]]];
Clear[monday, tuesday, wednesday, thursday, friday]
monday = Table[data[[ii, 3]], {ii, 2, nobs + 1}];
tuesday = Table[data[[ii, 4]], {ii, 2, nobs + 1}];
wednesday = Table[data[[ii, 5]], {ii, 2, nobs + 1}];
thursday = Table[data[[ii, 6]], {ii, 2, nobs + 1}];
friday = Table[data[[ii, 7]], {ii, 2, nobs + 1}];
parmtotal = fregfit[return];
{−"5864.32", "1.63218", −"0.0536357",
"0.", "0.0491851", "0.0693213",
"0.0516465", "0.0594664"}
parm8total = Table[parmtotal[[ii]], {ii, 2, 9}]
{1.63218, −0.0536357, 0.500413, 0.0444915, 0.0491851, 0.0693213, 0.0516465, 0.0594664}
"Estimates of Standard Errors of Stable Parameters and Regression coefficients
with Stable Errors (using FFT to estimate PDF)"
Clear[maxlike, i8, h, k, u, nhess, cov8, cor8];
maxlike[{(a_)?NumericQ, (b_)?NumericQ, (c_)?NumericQ, (b1_)?NumericQ, (b2_)?NumericQ,
(b3_)?NumericQ, (b4_)?NumericQ, (b5_)?NumericQ}]:=
−Dimensions[return]Log[c]+
FLogLikelihood[(return − (b1monday + b2tuesday + b3wednesday + b4thursday + b5friday))/c,
```

```
{a, b, 1, 0}, 1];
i8 = IdentityMatrix[8];
h = 10^(-$MachinePrecision/5);
k = 10^(-$MachinePrecision/5);
u = parm8total;
nhess =
Table[
1/(4hk) (maxlike[u + h i8[[ii]] + k i8[[jj]]] - maxlike[u - h i8[[ii]] + k i8[[jj]]]-
maxlike[u + h i8[[ii]] - k i8[[jj]]] + maxlike[u - h i8[[ii]] - k i8[[jj]]]),
{ii, 1, 8}, {jj, 1, 8}];
cov8 = Inverse[-Partition[Flatten[nhess], 8]];
cor8 = Table[cov8[[ii, jj]] / Sqrt[cov8[[ii, ii]] cov8[[jj, jj]]], {ii, 1, 8}, {jj, 1, 8}];
stdevtotal = Table[Sqrt[cov8[[ii, ii]]], {ii, 1, 8}];
TableForm[Table[{parm8total[[ii]], stdevtotal[[ii]], parm8total[[ii]]/stdevtotal[[ii]]},
{ii, 1, 8}],
TableHeadings → {{"α", "β", "γ", "b1", "b2", "b3", "b4", "b5"},
{"Estimate", "St. error", "z-value"}}]

{"Maximum Likelihood", mltotal = maxlike[parm8total]}
```

Estimates of Standard Errors of Stable Parameters and Regression
coefficients with Stable Errors (using FFT to estimate PDF)

	Estimate	St. error	z-value
α	1.63218	0.024538	66.5167
β	−0.0536357	0.0522964	−1.02561
γ	0.500413	0.00797772	62.7263
b1	0.0444915	0.027191	1.63626
b2	0.0491851	0.027377	1.79658
b3	0.0693213	0.027202	2.54839
b4	0.0516465	0.0271538	1.902
b5	0.0594664	0.0274676	2.16497

{Maximum Likelihood, {−5864.32}}

****

B.2 C++ Program to Estimate α-stable GARCH Process

The program below was used to estimate the α-stable processes which were used in the VaR estimates in tables 5.25 and 5.26. The program uses the STABLE library functions (Nolan (2005a)) which have been described in Subsections A.7.1 and A.7.2 of Appendix A. The program also calls routines from Press et al. (2007). Their file `quasinewton.h` was amended to ensure convergence. Some error checking that was included in the original program has been deleted in the version below. The program outputs data for analysis in R (R Development Core Team (2008)).

```
//
// stablegarch.cpp
//
// Program to estimate Stable-garch process (TS-Garch with
// alpha-stable innovations
// John Frain
// Version of 16 August 2008
// (derived from Version of 2 May 2008)


#include <iostream>
#include <cmath>
#include <fstream>

// Include files from Numerical Recipes

#include "C:\WORK\CPP_general\nr_301\include\nr3.h"
#include "C:\WORK\CPP_general\nr_301\include\moment.h"
#include "C:\WORK\CPP_general\nr_301\include\erf.h"
#include "C:\WORK\CPP_general\nr_301\include\amoeba.h"
#include "C:\WORK\CPP_general\nr_301\include\ludcmp.h"
#include "C:\WORK\CPP_general\nr_301\include\qrdcmp.h"
#include "C:\WORK\CPP_general\nr_301\include\roots_multidim.h"
#include "C:\WORK\CPP_general\nr_301\include\quasinewton2.h"
#include "C:\WORK\CPP_general\nr_301\include\gaussj.h"

// Include header files from John Nolan's STABLE Library

#include "stable.hpp"
```

```
//own header file

#include "stablegarch.h"

using namespace std;


Int main()
{
    // read data and load file
    Int nrow,ncol;
    string txt;
    ifstream fp("loss.dat");
    fp >> nrow >> ncol;
    cout << "matrix is " << nrow << " by " << ncol << "\n" ;
    if (fp.fail()){
                   cout << "Data file loss.dat not found" << endl;
                   return 1;
                   }
        //NR::nrerror("Data file loss.dat not found");

    NRmatrix<Doub> inputmat(nrow,ncol); // matrix to hold input data

// Read file
    for (int ii=0;ii<nrow;ii++){
        getline(fp,txt);
        for (int jj=0;jj<ncol;jj++) {
        fp >> inputmat[ii][jj];
        }
}

// Extract returns

   NRvector<Doub> returns(nrow);
   for (Int ii=0;ii<nrow;ii++){
       returns[ii] = inputmat[ii][3];
       }

// Estimate gamma[0] to pass to revised likelihood function
double theta[4];
double * xtemp;
xtemp= new double[nrow];
for (int i = 0; i<nrow ;i++){
     xtemp[i]=inputmat[i][3];
     }
Int param = 1;
Int ierr = 1;
STABLEFITMLE(&nrow,xtemp,theta,&param,&ierr); // we need theta[2]
Doub gamma0 = theta[2];
```

```
// Run Nedler Mead
//
cout << "\nInitial Estimate of Parameters using Nedler Mead"<<endl;

    param_ll loglik1(returns, nrow, gamma0);

    VecDoub point(6);
    point[0] =  1.80135; // Starting Values for Parameters - alpha
    point[1] = 0.199358; // beta
    point[2] = -0.00857765; // delta
    point[3] = 0.026424; // omega
    point[4] = 0.091525; // alpha1
    point[5] = 0.904622; // beta1

    Doub del = 0.0001;

    const Doub ftol = 1.0e-6;
    Amoeba am(ftol);
    VecDoub pmin(6);
    pmin=am.minimize(point, del, loglik1);

    Doub ave, adev, sdev, var, skew, curt;
    moment(returns, ave, adev, sdev, var, skew, curt);

    cout << "Loglikelihood           " <<setprecision(12) <<am.fmin<< endl;
    cout << "Number of function evaluations was " << am.nfunc <<endl;
    cout << "alpha   = "<< pmin[0] << endl;
    cout << "beta    = "<< pmin[1] << endl;
    cout << "delta   = "<< pmin[2] << "\n" << endl;
    cout << "omega   = "<< pmin[3] << endl;
    cout << "alpha1  = "<< pmin[4] << endl;
    cout << "beta1   = "<< pmin[5] << endl;
    cout << "\n" <<endl;
    cout << "\nMean of original data " << ave << endl;
    cout << "Standard Deviation      " << sdev << endl;
    cout << "Loglikelihood           " << am.fmin<< endl;
    cout << "Number of function evaluations was " << am.nfunc <<endl;


Funcd<param_ll> ll(loglik1);
//cout<< "functor likelihood " << ll (point)<<endl;

const Doub gtol = 1.0e-8;
Int iter;
Doub fret;
MatDoub hessout(nrow,nrow);
dfpmin(pmin,gtol,iter,fret,ll,hessout);

cout << "\nFinal Parameter Estimates using DFP " <<endl;
```

```
cout << "Loglikelihood " <<setprecision(12) <<fret<< endl;
cout << "Number of iterations was " << iter << "\n"<<endl;
    cout << "alpha   = "<< pmin[0] << endl;
    cout << "beta    = "<< pmin[1] << endl;
    cout << "delta   = "<< pmin[2] << endl;
    cout << "omega   = "<< pmin[3] << endl;
    cout << "alpha1  = "<< pmin[4] << endl;
    cout << "beta1   = "<< pmin[5] << endl;

cout << "\nVariance-covariance matrix derived from inverse of Hessian from ML " <<endl;

cout << hessout[0][0] <<", "<<hessout[0][1]<<", "<<hessout[0][2]<<", "<<hessout[0][3]
     <<", "<<hessout[0][4]<<", "<<hessout[0][5] <<endl;
cout << hessout[1][0] <<", "<<hessout[1][1]<<", "<<hessout[1][2]<<", "<<hessout[1][3]
     <<", "<<hessout[1][4]<<", "<<hessout[1][5] <<endl;
cout << hessout[2][0] <<", "<<hessout[2][1]<<", "<<hessout[2][2]<<", "<<hessout[2][3]
     <<", "<<hessout[2][4]<<", "<<hessout[2][5] <<endl;
cout << hessout[3][0] <<", "<<hessout[3][1]<<", "<<hessout[3][2]<<", "<<hessout[3][3]
     <<", "<<hessout[3][4]<<", "<<hessout[3][5] <<endl;
cout << hessout[4][0] <<", "<<hessout[4][1]<<", "<<hessout[4][2]<<", "<<hessout[4][3]
     <<", "<<hessout[4][4]<<", "<<hessout[4][5] <<endl;
cout << hessout[5][0] <<", "<<hessout[5][1]<<", "<<hessout[5][2]<<", "<<hessout[5][3]
     <<", "<<hessout[5][4]<<", "<<hessout[5][5] <<endl;

cout << "\nStandard Errors derived from inverse of Hessian from ML " <<endl;
// Hessout is inverse of hessian

cout << sqrt(hessout[0][0]) <<endl;
cout << sqrt(hessout[1][1]) <<endl;
cout << sqrt(hessout[2][2]) <<endl;
cout << sqrt(hessout[3][3]) <<endl;
cout << sqrt(hessout[4][4]) <<endl;
cout << sqrt(hessout[5][5]) <<endl;

cout<<"\n"<<endl;

//Alternative estimate of variances



Int nparm;
nparm=point.size(); // Number of parameters estimates
MatDoub altcovar(nparm,nparm); // Covariance Matrix Matrix

ncovar(point, returns, nrow, gamma0, altcovar);

cout << "\nAlternative estimate of variance-covariance matrix" <<endl;
cout << "Numerical derivatives of Loglikelihood function\n" << endl;
```

```
cout << altcovar[0][0] <<", "<<altcovar[0][1]<<", "<<altcovar[0][2]<<", "<<altcovar[0][3]
     <<", "<<altcovar[0][4]<<", "<<altcovar[0][5] <<endl;
cout << altcovar[1][0] <<", "<<altcovar[1][1]<<", "<<altcovar[1][2]<<", "<<altcovar[1][3]
     <<", "<<altcovar[1][4]<<", "<<altcovar[1][5] <<endl;
cout << altcovar[2][0] <<", "<<altcovar[2][1]<<", "<<altcovar[2][2]<<", "<<altcovar[2][3]
     <<", "<<altcovar[2][4]<<", "<<altcovar[2][5] <<endl;
cout << altcovar[3][0] <<", "<<altcovar[3][1]<<", "<<altcovar[3][2]<<", "<<altcovar[3][3]
     <<", "<<altcovar[3][4]<<", "<<altcovar[3][5] <<endl;
cout << altcovar[4][0] <<", "<<altcovar[4][1]<<", "<<altcovar[4][2]<<", "<<altcovar[4][3]
     <<", "<<altcovar[4][4]<<", "<<altcovar[4][5] <<endl;
cout << altcovar[5][0] <<", "<<altcovar[5][1]<<", "<<altcovar[5][2]<<", "<<altcovar[5][3]
     <<", "<<altcovar[5][4]<<", "<<altcovar[5][5] <<endl;


cout << "\nAlternative estimate of standard errors" <<endl;
cout << sqrt(altcovar[0][0]) << endl;
cout << sqrt(altcovar[1][1]) << endl;
cout << sqrt(altcovar[2][2]) << endl;
cout << sqrt(altcovar[3][3]) << endl;
cout << sqrt(altcovar[4][4]) << endl;
cout << sqrt(altcovar[5][5]) << endl;

// Gather results and output to a file;
// Columns of matrix include
//
// Column 0 : year
// Column 1 : month
// Column 2 : day
// Column 3 : data
// Column 4 : fit (mu in this case)
// Column 5 : residual(t) ( x(t) - mu )
// Column 6 : sigmasq(t)
// Column 7 : standardised residual ( residual(t)/sqrt(sigmasq(t)))


MatDoub results(nrow,13);

double alpha = pmin[0];
double beta = pmin[1];
double delta =pmin[2];
double omega  = pmin[3];
double alpha1 = pmin[4];
double beta1  = pmin[5];


results[0][0]=inputmat[0][0];// Year
results[0][1]=inputmat[0][1];// Month
results[0][2]=inputmat[0][2];//Day
results[0][3]=inputmat[0][3];// loss
results[0][4]=delta;//
```

```
results[0][5]=inputmat[0][3]-delta;// unstandardised residual
results[0][6]=gamma0;//
results[0][7]=results[0][5]/results[0][6];// standardised residual
int no_quantile = 5 ;
double p[5] = {.900, .950, .990, .995, .999}; // quantile levels
double quantile[no_quantile];
int iparam = 1;
STABLEQUANT(&no_quantile,p,quantile,&alpha,&beta,&gamma0,&delta,&iparam,&ierr);

// STABLEQUANT(&nrow, p, q, &alpha, &beta, &gamma0, &delta,&param, &ierr);
errchk("STABLEQUANT",&ierr);
results[0][8]= quantile[0]; //10% quantile
results[0][9]= quantile[1]; //5% quantile
results[0][10]=quantile[2]; //1% quantile
results[0][11]=quantile[3]; //0.5% quantile
results[0][12]=quantile[4]; //0.1% quantile
for ( int i = 1; i< nrow; i++){
    results[i][0]=inputmat[i][0];
    results[i][1]=inputmat[i][1];
    results[i][2]=inputmat[i][2];
    results[i][3]=inputmat[i][3];
    results[i][4]=delta;
    results[i][5]=inputmat[i][3]-delta;
    results[i][6]=omega + alpha1 * abs(results[i-1][5]) +
                      beta1 * results[i - 1][6];
    results[i][7]=results[i][5]/results[i][6];

    STABLEQUANT(&no_quantile, p, quantile, &alpha, &beta, &results[i][6], &delta,&param, &ierr);
    errchk("STABLEQUANT",&ierr);
    results[i][8]= quantile[0]; //10% quantile
    results[i][9]= quantile[1]; //5% quantile
    results[i][10]=quantile[2]; //1% quantile
    results[i][11]=quantile[3]; //0.5% quantile
    results[i][12]=quantile[4]; //0.1% quantile
    }

ofstream out;

out.open("results.csv");

if (!out){
    cout<<"Unable to open output file: " <<endl;
    cerr<<"Unable to open output file: " <<out <<endl;
    return -1;
}

out << "Year, Month, Day, Loss, Fit, Residual, Sigmasq, zt, VAR100, VAR050, VAR010, VAR005, VAR001" << endl;
for ( int i = 0; i< nrow; i++){
    out      << setprecision(4)<< results[i][0] << ","
             << setprecision(2)<< results[i][1] << ","
```

```
            << setprecision(2)<< results[i][2] << ","
            << setprecision(12)<< results[i][3] << ","
            << setprecision(12)<< results[i][4] << ","
            << setprecision(12)<< results[i][5] << ","
            << setprecision(12)<< results[i][6] << ","
            << setprecision(12)<< results[i][7] << ","
            << setprecision(12)<< results[i][8] << ","
            << setprecision(12)<< results[i][9] << ","
            << setprecision(12)<< results[i][10] << ","
            << setprecision(12)<< results[i][11] << ","
            << setprecision(12)<< results[i][12] << endl;
            }



    return 0;
}
```

```
//
// stablegarch.h
//
// Header file for stablegardh.cpp


#ifndef GARCHFIT_H__
#define GARCHFIT_H__

// Various probability density functions
Doub pdf1(Doub x, Doub mean, Doub std);
Doub pdf2(Doub x, Doub mean, Doub std);
Doub pdf3(Doub x, Doub alpha, Doub beta, Doub gamma, Doub delta);

// Error check for Stable routines
int errchk(char *str, int *ierr);

int ncovar(const VecDoub_I &x, const VecDoub &returns,const Int nrow, MatDoub_O &var);
int ncovar2(const VecDoub_I &x, const VecDoub &returns, const Int nrow, MatDoub_O &var);



Doub pdf1(Doub x, Doub mean, Doub std)
{
       return (1/(sqrt(2.0*M_PI)*std))*( exp(-pow((x-mean)/std,2)/2))    ;
       }
```

```
Doub pdf2(Doub x, Doub mean, Doub std)
{
       Normaldist gauss(mean,std);
       return gauss.p(x);
       }

Doub pdf3(Doub x, Doub alpha, Doub beta, Doub gamma, Doub delta){
     Doub y;
     Int ierr;
     Int iparam = 1;
     Int n=1;
     STABLEPDF(&n,&x,&y,&alpha,&beta,&gamma,&delta,&iparam,&ierr);
     return y;
     }


struct param_ll {
    VecDoub x;
    Int nobs;
    Doub gamma0;
    param_ll(VecDoub_I xx, Int nnobs,Doub ggamma0):x(xx), nobs(nnobs) , gamma0(ggamma0){ }

    Doub operator () (VecDoub_I params)

     {
        Doub alpha = params[0];
        Doub beta = params[1];
        Doub delta = params[2];
        Doub omega  = params[3];
        Doub alpha1 = params[4];
        Doub beta1  = params[5];
        if ( alpha > 2 | alpha <1 | beta <-1 | beta >1 |
             omega <= 0.0 | alpha1 < 0.0 | beta1<0.0 ) {
                   cout<< "problem ? "<<endl;
                   cout<<alpha<<", "<<beta<<", "<<delta<<", "<<omega<<", "<<alpha1<<", "<<beta1<< "\n"<<endl
                   return 1.0e8;
                   }

        VecDoub gamma(nobs, 0.0);
        gamma[0] = gamma0;
        Doub ll = log(pdf3(x[0], alpha, beta, gamma[0] ,delta));

        for (Int ii = 1; ii < nobs; ii++) {
            gamma[ii] = omega + alpha1 * abs(x[ii - 1]-delta) +
                          beta1 * gamma[ii - 1];
            ll += log(pdf3(x[ii], alpha, beta, gamma[ii], delta));
            }
        return - ll;
    }
};
```

```
//
int ncovar(const VecDoub_I &x, const VecDoub &returns, const Int nrow, const Doub gamma0, MatDoub_O &var)
        {
        param_ll loglik(returns,nrow,gamma0);
                Int n=x.size();
                Doub h1 = 1.0e-4; // multipliers to calculate finite differences
                Doub k1 = 1.0e-4;
                Doub h, k; // Finite differences
        VecDoub_IO x1;
        VecDoub_IO x2;
        VecDoub_IO x3;
        VecDoub_IO x5;
        VecDoub_IO x6;
        VecDoub_IO x7;
        MatDoub hessian1(n,n);
        for (Int i=0;i<n;i++) {
                    x1=x;
                    x6=x;
                    h=h1 * abs(x[i]);
                    x1[i]=x[i]+h;
                    x6[i]=x[i]-h;
                    hessian1[i][i]=( (loglik(x1)-loglik(x))-(loglik(x)-loglik(x6)) )/(h*h);
                                for (Int j=0;j<i;j++){
                    x1=x;
                    x2=x;
                    x3=x;
                    x5=x;
                    x6=x;
                    x7=x;
                    h=h1 * abs(x[i]);
                    k=k1 * abs(x[j]);
                    x1[i]=x[i]+h;
                    x1[j]=x[j]+k;
                    x2[i]=x[i]+h;
                    x3[j]=x[j]+k;
                    x5[i]=x[i]-h;
                    x5[j]=x[j]-k;
                    x6[i]=x[i]-h;
                    x7[j]=x[j]-k;

                    hessian1[i][j]= (
                      ( (loglik(x1)-loglik(x2)) - (loglik(x3)-loglik(x)) )+
                      ( (loglik(x5)-loglik(x6)) - (loglik(x7)-loglik(x)) ) )/(2.0*h*k);
                    hessian1[j][i]=hessian1[i][j] ;
                                }
                        }

        LUdcmp alu(hessian1);
        alu.inverse(var);
```

```
                  return 0;
        }

// Error Check for stable routines

int errchk(char *str, int *ierr) {
    if (*ierr != 0) {
              cout << "\n Stable error " << ierr<< " in " << str <<endl;
              return -1;
              }
    return 0;
}


#endif // GARCHFIT_H__
```

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