When Stackelberg and Cournot Equilibria Coincide*

Luca Colombo†
Trinity College Dublin and University of Bologna

Paola Labrecciosa‡
Trinity College Dublin and University of Bologna

February 25, 2004

Abstract

We compare two-stage Stackelberg with Cournot equilibrium under the assumption of quantity competition and homogeneous good. We show that, when the curvature of the inverse market demand equals the total number of firms in the industry, the outcome of the two games coincides.

JEL Classification: C73, D43.

Key Words: Stackelberg equilibrium, Cournot equilibrium, oligopoly.

In this note, we investigate the relationship between two-stage Stackelberg and Cournot equilibria under the assumption of quantity competition and homogeneous good. We show that, when the inverse market demand is convex, the outcome of the two games may coincide. This occurs when the curvature of the inverse market demand equals the total number of firms in the industry.

Consider a scenario where multiple leaders choose outputs simultaneously and independently at first and multiple followers choose outputs simultaneously and independently later, given the leaders' total output. The set of players (firms) is

*We would like to thank Eric Maskin and Jim Markusen for insightful comments and suggestions. The usual disclaimer applies.

†Department of Economics, Trinity College Dublin, Foster Place, Dublin 2, Ireland; email: colombol@tcd.ie; tel. +353-1-6082228; fax. +353-1-6772503. Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy, email: colombo@spbo.unibo.it; tel. +39-051-2092629; fax. +39-051-2092664.

‡Department of Economics, Trinity College Dublin, Dublin 2, Ireland; email: labreccp@tcd.ie; tel. +353-1-6082325; fax. +353-1-6772503. Department of Economics, University of Bologna, Strada Maggiore 45, 40125 Bologna, Italy, email: labrecciosa@spbo.unibo.it; tel. +39-051-2092629; fax. +39-051-2092664.

$\mathcal{P} = \mathcal{F} \cup \mathcal{S}$, where $\mathcal{F}$ is the set of first movers and $\mathcal{S}$ is the set of second movers. There are $N_F \geq 1$ leaders and $N_S \geq 1$ followers, with $N = N_F + N_S$ denoting the total number of firms in the industry. The inverse demand for a homogeneous product is defined by $p : R_+ \rightarrow R_+$. For every industry output $Q \in R_+$ this function specifies the market clearing price $p(Q)$, with $p^{\prime} < 0$ and $p^{\prime\prime} > 0$. Industry output is $Q = \sum_{i=1}^{N_F} q_i + \sum_{j=1}^{N_S} q_j$, with $i \in \mathcal{F}$ and $j \in \mathcal{S}$, where q_i and q_j are the quantities produced by the generic leader and the generic follower, respectively. Each firm has the same cost function, defined by $c : R_+ \rightarrow R_+$, satisfying $c^{\prime} > 0$.

We solve the game by backward induction, considering first the follower j's problem. This, under the Cournot-Nash assumption, consists in setting the profit maximizing quantity taking as given the quantities of the rivals. First order condition writes (assuming an inner solution exists):

$$p^{\prime} q_j + p - c^{\prime} = 0, \ \ \forall j \in \mathcal{S} \tag{1}$$

which implicitly defines the follower j's reaction function. Second order condition is satisfied if:

$$p^{\prime\prime} q_j + 2p^{\prime} - c^{\prime\prime} \leq 0, \ \ \forall j \in \mathcal{S} \tag{2}$$

which can be rewritten:

$$\gamma \leq \frac{2}{s_j}, \ \ \forall j \in \mathcal{S} \tag{3}$$

where $\gamma = -Q \left(p^{\prime\prime} / p^{\prime} \right)$ is the curvature of the inverse market demand and $s_j = q_j / Q$ is the market share of a generic follower.

The leader i's program consists in maximizing its profits, π_i, under the constraint given by (1). The Lagrangian function writes:

$$\mathcal{L}_i = \pi_i - \lambda \left(p^{\prime} q_j + p - c^{\prime} \right), \ \ \forall i \in \mathcal{F} \tag{4}$$

First order conditions are:

$$\begin{cases} \dfrac{\partial \mathcal{L}_i}{\partial q_i} = 0 \Rightarrow p^{\prime} q_i + p - c^{\prime} - \lambda \left(p^{\prime\prime} q_j + p^{\prime} \right) = 0 \\ \dfrac{\partial \mathcal{L}_1}{\partial q_j} = 0 \Rightarrow p^{\prime} q_i - \lambda \left(p^{\prime\prime} q_j + 2p^{\prime} - c^{\prime\prime} \right) = 0 \\ \dfrac{\partial \mathcal{L}_1}{\partial \lambda} = 0 \Rightarrow p^{\prime} q_j + p - c^{\prime} = 0 \end{cases} \tag{5}$$

which, given the concavity assumptions, are also sufficient. By applying the condition of symmetry among followers and among leaders, and the definition of industry output, we get:

$$\begin{cases} p^{\prime} q_i + p - c^{\prime} - \lambda \left(p^{\prime\prime} \dfrac{Q - N_F q_i}{N_S} + p^{\prime} \right) = 0 \\ p^{\prime} q_i - \lambda \left(p^{\prime\prime} \dfrac{Q - N_F q_i}{N_S} + 2p^{\prime} - c^{\prime\prime} \right) = 0 \\ p^{\prime} \dfrac{Q - N_F q_i}{N_S} + p - c^{\prime} = 0 \end{cases} \tag{6}$$

which can be solved for q_i:

$$q_i = \left[\frac{\gamma}{\varepsilon}\left(1 - \frac{2c^{'}}{p} + \frac{\left(c^{'}\right)^2}{p^2}\right) - 2\left(1 - \frac{c^{'}}{p}\right) + \frac{c^{''}}{p^{'}}\left(1 - \frac{c^{'}}{p}\right)\right]\left(\frac{p}{p^{'} - c^{''}}\right) \tag{7}$$

where $\varepsilon = -Q\left(p^{'}/p\right)$ is the elasticity of the inverse market demand w.r.t. industry output. Once solved (1), evaluated at the equilibrium, for q_j:

$$q_j = -\frac{p - c^{'}}{p^{'}} \tag{8}$$

it is immediate to verify that $q_i = q_j$ when:

$$\gamma = \frac{\varepsilon}{L} \tag{9}$$

where $L = (p - c)/p$ is the Lerner Index (or price-cost margin). Since in the symmetric Cournot game $\varepsilon = N\,L$, we can write:

Proposition 1 *When $\gamma = N$ Stackelberg and Cournot equilibria coincide.*

Example In order to illustrate our result, let us provide a simple example with constant elasticity and curvature of the inverse market demand and constant returns to scale. It is worth noting that these assumptions are not required for our proposition to hold.

Consider the following inverse market demand:

$$p = \frac{1}{Q} \tag{10}$$

with $p^{'} = -1/Q^2 < 0$; $p^{''} = 2/Q^3 > 0$, $\varepsilon = 1$, $\gamma = 2$. Assume $N = 2$ and $c^{'} = 1$. The follower's reaction function turns out to be:

$$q_j\left(q_i\right) = \sqrt{q_i} - q_i \tag{11}$$

Accordingly, the leader has to maximize the following Lagrangian:

$$\mathcal{L}_i = \pi_i - \lambda\left(\sqrt{q_i} - q_i - q_j\right) \tag{12}$$

First order conditions write:

$$\begin{cases} \dfrac{\partial \mathcal{L}_i}{\partial q_i} = 0 \Rightarrow \dfrac{\partial\left(\left(\dfrac{1}{q_i + q_j} - 1\right)q_i - \lambda\left(\sqrt{q_i} - q_i - q_j\right)\right)}{\partial q_i} = 0 \\ \dfrac{\partial \mathcal{L}_i}{\partial q_j} = 0 \Rightarrow \dfrac{\partial\left(\left(\dfrac{1}{q_i + q_j} - 1\right)q_i - \lambda\left(\sqrt{q_i} - q_i - q_j\right)\right)}{\partial q_j} = 0 \\ \dfrac{\partial \mathcal{L}_i}{\partial \lambda} = 0 \Rightarrow -\sqrt{q_i} + q_i + q_j = 0 \end{cases} \tag{13}$$

from which we can obtain the Stackelberg equilibrium quantities:

$$q_i^* = q_j^* = \frac{1}{4} \tag{14}$$

It is straightforward to verify that Stackelberg and Cournot equilibria coincide. Graphically:

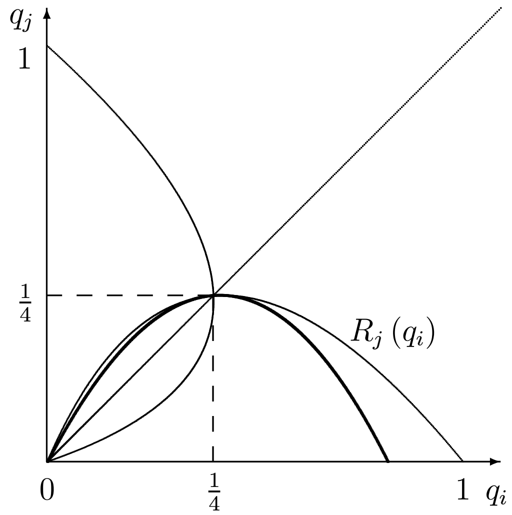

Figure 1 : Equilibria Comparison

The equivalence between Stackelberg and Cournot equilibria is due to the fact that the point where the isoprofit curve of firm i (the thick line) is tangent to the reaction function of firm j, $R_j(q_i)$, exactly coincides with the point where the reaction function of firm j intersects the 45 degree line.

References

[1] Cournot, A. (1838), Recherches sur les principes mathématiques de la théorie des richesses, Hachette, Paris.

[2] Stackelberg, H. Von (1934), Marktform und Gleichgewicht, Springer-Verlag, Berlin-Wien.