Derivation of the consumption-CAPM

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In the standard one-period CAPM, investors maximize their expected utility in the mean-variance world. That is, their behaviour is based on the one-period mean and variance of returns. In contrast, the consumption-CAPM is an intertemporal model within which investors maximize their expected lifetime utility. In this model, financial assets are used to smooth the path of consumption over time, selling assets when times are bad and investing in assets when times are good. Assets whose returns have a high negative covariance with consumption will be held with a low risk premium. Conversely, assets whose returns have a high positive covariance with consumption are not so useful when times are bad, and they will command a high risk premium to convince investors to hold them. This model therefore associates an asset's systematic risk with the state of the economy (consumption).

1 Setup of the model

A representative investor maximizes lifetime utility that depends only on current consumption and future consumption. The expected utility from future consumption is discounted; the discount factor is equal to θ. In a two-period model, lifetime utility is given by

$$U(C_t, C_{t+1}) = u(C_t) + \theta E_t(u(C_{t+1}))$$

At time t, the investor receives an exogenous endowment of resources, denoted as e_t, which is allocated between consumption and assets. The number of assets that he will buy is denoted as ς and the price for each unit of the assets is p_t. At time $t+1$ assets pay a dividend d_{t+1} that can be used, along with a new exogenous endowment of resources, for consumption purposes in period $t+1$. The problem of the investor is written as

$$\max_{\varsigma} u(C_t) + \theta E_t(u(C_{t+1})) \tag{1}$$

subject to

$$C_t + p_t\varsigma = e_t \tag{2}$$

$$C_{t+1} = e_{t+1} + \varsigma(p_{t+1} + d_{t+1}) \tag{3}$$

2 Solution

Substituting the two constraints into the objective function, we have

$$U(C_t, C_{t+1}) = u(e_t - p_t\varsigma) + \theta E_t u(e_{t+1} + \varsigma(p_{t+1} + d_{t+1}))$$

Differentiating with respect to ς yields a first-order condition:

$$u'(C_t)(-p_t) + \theta E_t u'(C_{t+1})(p_{t+1} + d_{t+1}) = 0$$

Dividing all terms by $u'(C_t)$ and adding p_t to both sides of the equation, we get

$$p_t = \theta E_t \left[\frac{u'(C_{t+1})}{u'(c_t)}((p_{t+1} + d_{t+1}) \right]$$

Defining the ratio of marginal utilities as $m_{t,t+1} = \frac{u'(C_{t+1})}{u'(c_t)}$, using the fact that the gross rate of return on a risky asset is defined as $1 + r_{j,t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$, we obtain

$$1 = \theta E_t \left[m_{t,t+1} \frac{p_{t+1} + d_{t+1}}{p_t} \right]$$

$$1 = \theta E_t \left[m_{t,t+1}(1 + r_{j,t+1}) \right]$$

This relationship shows clearly that returns on risky assets and consumption are related (through the ratio of marginal utilities).

For any two random variables x and y, $E(xy) = E(x)E(y) + cov(x, y)$, so

$$1 = \theta \left[E_t(m_{t,t+1}) E_t(1 + r_{j,t+1}) + cov_t(m_{t,t+1}, 1 + r_{j,t+1}) \right] \tag{4}$$

A risk-free asset always gives a certain return, denoted as $r_{f,t+1}$. In the case of a risk-free asset, $cov_t(m_{t,t+1}, 1 + r_{f,t+1}) = 0$ and equation (4) simplifies to

$$1 = \theta E_t(m_{t,t+1}) E_t(1 + r_{f,t+1})$$

$$1 + r_{f,t+1} = \frac{1}{\theta E_t(m_{t,t+1})}$$

We are now ready to derive an expression for the expected excess return on a risky security j. Starting with equation (4),

$$\frac{1 - \theta cov_t(m_{t,t+1}, 1 + r_{j,t+1})}{\theta E_t(m_{t,t+1})} = E_t(1 + r_{j,t+1})$$

Using the expression for the risk-free security,

$$(1 - \theta cov_t(m_{t,t+1}, 1 + r_{j,t+1}))(1 + r_{f,t+1}) = E_t(1 + r_{j,t+1})$$

$$E_t(1 + r_{j,t+1} - (1 + r_{f,t+1})) = -(1 + r_{f,t+1})\theta cov_t(m_{t,t+1}, 1 + r_{j,t+1})$$

and therefore,

$$E_t(r_{j,t+1}) - r_{f,t+1} = -\frac{1}{E_t(m_{t,t+1})} cov_t(m_{t,t+1}, 1 + r_{j,t+1}) \tag{5}$$

Reintroducing the definition of $m_{t,t+1} = \frac{u'(C_{t+1})}{u'(C_t)}$, we have

$$E_t(r_{j,t+1}) - r_{f,t+1} = -\frac{1}{E_t(u'(C_{t+1}))} cov_t(u'(C_{t+1}), 1 + r_{j,t+1}) \tag{6}$$

3 Discussion

Equation (6) shows that the expected excess returns on two assets i and j will differ only because the covariance of their respective returns with (the marginal utility of) consumption differs. An asset whose return has a negative covariance with the marginal utility of consumption and hence, a positive covariance with consumption (thanks to diminshing marginal utility), must offer a high expected rate of return in order for investors to hold this asset. Since this asset covaries positively with consumption, it is not very useful in so far as it does not allow for consumption smoothing. In other words, it does not offer a high return when times are bad. Conversely, an asset whose return covaries negatively with consumption provide insurance against bad times and are useful for consumption smoothing. Thus, they will be held even if the expected rate of return is relatively low.