Properties of quantum states

partial trace

$M_{AB} \in L(A \otimes B)$

$M_A = tr_B[M_{AB}] := \sum_j (\mathbb{1}_A \otimes \langle B_j|) M_{AB} (\mathbb{1}_A \otimes |B_j\rangle)$

with $\{|B_j\rangle\}$ ONB

$|\psi_{AB}\rangle$: entangled iff $\nexists\ |\psi_A\rangle, |\psi_B\rangle$

$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$

↳ otherwise: product (seperable)

$\Leftrightarrow$ Schmidt rank $r = 1$

Purity of reduced states

ψ_{AB} product $\Rightarrow p(\varrho_A) = 1$ i.e. $\varrho_A = \psi_A = |\psi_A\rangle\langle\psi_A|$

ψ_{AB} entangled $\Rightarrow p(\varrho_A) < 1 \Rightarrow \varrho_A$: ensemble

$|\psi_{AB}\rangle = \sum_{k=1}^{r} \lambda_k |k\rangle_A \otimes |k\rangle_B \quad \lambda_k > 0 \quad \sum_k \lambda_k^2 = 1$

$\varrho_A = tr_B[\psi_{AB}] = \sum_{k=1}^{r} \lambda_k^2 |k\rangle\langle k|$

$\Rightarrow p(\varrho_A) = \sum_{k=1}^{r} \lambda_k^4 = 1$ iff $r = 1$

maximally-mixed state $\varrho_A = \sum_{k=1}^{d} \frac{1}{d} |k\rangle\langle k|$

maximally entangled states: $\lambda_k = \frac{1}{\sqrt{d}}$

canonically: $|\phi_d^+\rangle = \sum_{k=0}^{d-1} \frac{1}{\sqrt{d}} |k, k\rangle$

$$\propto \sum_{k=0}^{d-1} |k, k\rangle = |M\rangle$$

qubits: Bell states

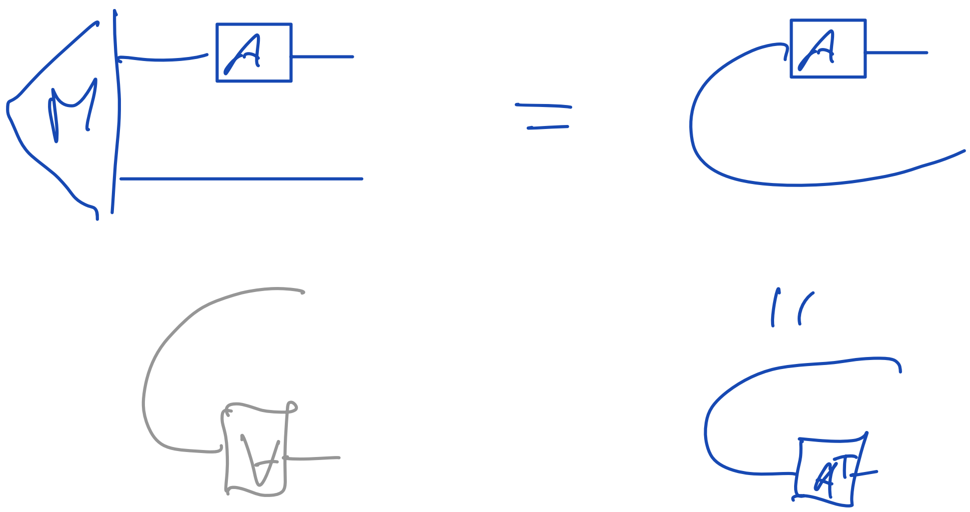
$$|\phi^\pm\rangle := \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\psi^\pm\rangle := \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

Write $|\phi^-\rangle$ in Schmidt decomposition

Lemma (Transpose trick)

$$A \otimes \mathbb{1} |M\rangle = \mathbb{1} \otimes A^T |M\rangle$$



Purification

Given ϱ_A can we understand it as $\operatorname{tr} \psi_{AB}$ for some $\psi_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$?

Purification of ϱ_A:

$\psi_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ with $\operatorname{tr}_B \psi_{AB} = \varrho_A$

requires $d_B \geq \operatorname{rank}(\varrho_A)$

$\varrho_A = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

generically: $|\psi_{AB}\rangle = \sum_j \sqrt{p_j}\, |\psi_j, j\rangle$

canonical: $|\psi_\varrho\rangle \equiv \sum_j \sqrt{p_j}\, |j, j\rangle$

$= \sqrt{\varrho} \otimes \mathbb{1}\, |M\rangle$

What other purifications?

Isometry

$V \in \mathcal{L}(X, Y)$ $\quad d_X \leq d_Y$

$V^\dagger V = \mathbb{1}_X$

$\Leftrightarrow \|V|v\rangle\| = \||v\rangle\| \quad \forall |v\rangle \in X$

$\langle v|V^\dagger V|v\rangle = \langle v|v\rangle \quad \forall v$

Unlike unitaries: $VV^\dagger \neq \mathbb{1}_Y$ (unless $d_X = d_Y$)

instead $VV^\dagger = \Pi_{\tilde{Y}}$

$(VV^\dagger)^2 = (VV^\dagger)(VV^\dagger) = V\mathbb{1}_X V^\dagger = VV^\dagger \equiv \Pi_{\tilde{Y}}$

Purifications are isometrically equivalent

ψ_{AB}, $\psi_{AB'}$ purifications of

$$\rho_A = \sum p_j |\psi_j\rangle\langle\psi_j|$$

w.l.o.g. $d_{B'} \geq d_B$

$\Rightarrow \exists$ isometry $V \in \mathcal{L}(\mathcal{H}_B, \mathcal{H}_{B'})$

$$|\psi_{AB'}\rangle = \mathbb{1}_A \otimes V |\psi_{AB}\rangle$$

For any purification $\exists \{|j\rangle_X\}$ ONB s.t.:

$$|\psi_{AX}\rangle = \sum_j \sqrt{p_j}\, |j\rangle_A |j\rangle_X$$

For B, B': $\quad V = \sum_k^{d_B} |k\rangle_{B'} \langle k|_B \qquad VV^\dagger = \Pi_{B'}$

$$V^\dagger V = \mathbb{1}_B \quad \square$$

Purifications of ρ in general:

$$\{\psi_\rho(V) : |\psi_\rho(V)\rangle \equiv \sqrt{\rho} \otimes V |M\rangle\}_V$$

What about ensembles $\{(p_j, |\psi_j\rangle)\}$?

with $\rho = \sum p_j |\psi_j\rangle\langle\psi_j|$

not necessarily ON

Schrödinger-HJW theorem

$\{(p_j, |\psi_j\rangle)\}$ vs. $\{(q_i, |\phi_i\rangle)\}$

$\mathbb{1} \otimes V |\psi_\rho\rangle$ $\qquad$ $\mathbb{1} \otimes U |\psi_\rho\rangle$

$\Rightarrow$ measure in $\{V|j\rangle\}$ or $\{U|i\rangle\}$ basis of ancillary space.

Sidenote:

$\rho = \sum r_j |j\rangle\langle j|$

Schur-Horn Thm

$\Rightarrow \exists \{(p_j, \psi_j)\} : \rho = \sum p_j |\psi_j\rangle\langle\psi_j|$ iff $r \succ p$

majorises

Hilbert-Schmidt inner product

(Frobenius inner product)

$A, B \in L(X, Y)$

$\langle A, B \rangle = \mathrm{tr}[A^\dagger B]$

$\Rightarrow$ norm $\|A\|_2 = \sqrt{\langle A, A \rangle}$

Frobenius norm

Schatten p-norms:

$\|A\|_p = \left(\mathrm{tr}\left[(A^\dagger A)^{p/2}\right]\right)^{1/p}$

define $\|A\|_\infty = \max\{\|Au\| : u \in X \ \|u\| \leq 1\}$

$\hookrightarrow$ spectral norm

$p = 1$: trace norm

singular values

$\|A\|_1 = \mathrm{tr}\left[\sqrt{A^\dagger A}\right] = \sum_k \lambda_k$

$= \sum_j |a_j|$

for A Hermitian with eigenvalues a_j

How similar/different are two states ϱ & σ?

distance: $\|\varrho-\sigma\|_1 = \mathrm{tr}\left[\sqrt{(\varrho-\sigma)^\dagger(\varrho-\sigma)}\right]$

define **trace distance** $D_{tr}(\varrho,\sigma) = \frac{1}{2}\|\varrho-\sigma\|_1$

$0 \leq D_{tr}(\varrho,\sigma) \leq 1$

example $\varrho=\sigma$ example: $\varrho = |0\rangle\langle 0|$, $\sigma = |1\rangle\langle 1|$

POVMs

$M_j \geq 0$ with $\sum_j M_j = \mathbb{1}$

$p_j = \mathrm{tr}[M_j \varrho]$

$$D_{tr}(\varrho,\sigma) = \max_{0\leq M\leq \mathbb{1}} \mathrm{tr}[\underbrace{M(\varrho-\sigma)}_{}]$$

$= \mathrm{tr}[M\varrho] - \mathrm{tr}[M\sigma] = \Delta p_M$

Proof: homework or literature

Properties:

- non-negativity: $D_{tr}(\varrho,\sigma) \geq 0$
- symmetry: $D_{tr}(\varrho,\sigma) = D_{tr}(\sigma,\varrho)$
- triangle inequality $D_{tr}(\varrho,\gamma) \leq D(\varrho,\sigma) + D_{tr}(\sigma,\gamma)$

Pick M: $D_{tr}(\varrho,\gamma) = \mathrm{tr}[M(\varrho-\gamma)]$

$$= \mathrm{tr}[M(\varrho-\sigma)] + \mathrm{tr}[M(\sigma-\gamma)]$$
$$\leq \mathrm{tr}[\tilde{M}_1(\varrho-\sigma)] + \mathrm{tr}[\tilde{M}_2(\sigma-\gamma)]$$
$$= D_{tr}(\varrho,\sigma) + D_{tr}(\sigma,\gamma) \quad \square$$

• unitary invariance (actually: isometric)

$$\|\rho-\sigma\|_1 = \|U\rho U^\dagger - U\sigma U^\dagger\|$$

• monotonicity (in general: also under qu. channels)

$$D_{tr}(\rho_A, \sigma_A) \le D_{tr}(\rho_{AB}, \sigma_{AB})$$

$$\mathrm{tr}[M_A(\rho_A - \sigma_A)] = \mathrm{tr}[M_A \otimes \mathbb{1}_B (\rho_{AB} - \sigma_{AB})]$$

$$\le \max_{M_{AB}} \mathrm{tr}[M_{AB}(\rho_{AB} - \sigma_{AB})]$$

$$= D_{tr}(\rho_{AB}, \sigma_{AB}) \quad \square$$

Fidelity:

$$F(\rho,\sigma) = \|\sqrt{\sigma}\sqrt{\rho}\|_1$$

$$= \mathrm{tr}\left(\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)$$

For pure states: $F(\psi,\phi) = \mathrm{tr}\sqrt{\psi\phi\psi} = |\langle\psi|\phi\rangle|$

Polar decomposition

$\forall A \in L(X)$

$\exists$ W unitary, P pos. semidefinite:

$A = WP$ (or $A = \tilde{P}\tilde{W}$)

with $P = |A| = \sqrt{A^\dagger A}$

SVD: $A = U\Lambda V$

$W = UV$ ← unitary

$P = V^\dagger \Lambda V$ ← pos. semi-def. because

$\langle\psi|\Lambda|\psi\rangle \ge 0 \quad \forall |\psi\rangle$

equally all $|\tilde{\psi}\rangle = V|\psi\rangle$

$A^\dagger A = PW^\dagger WP = P^2 \Rightarrow P = \sqrt{A^\dagger A}$

Uhlmann's Theorem

$$F(\rho,\sigma) = \max_{W,V} |\langle \psi_\rho(V) | \psi_\sigma(W)\rangle| :$$

$$\mathrm{tr}_{B_\rho} \psi_\rho(V) = \rho \,, \quad \mathrm{tr}_{B_\sigma} \psi_\sigma(V) = \sigma$$

Proof:

$$|\psi_\rho(V)\rangle = \sqrt{\rho} \otimes V |M\rangle$$

$$|\psi_\sigma(W)\rangle = \sqrt{\sigma} \otimes W |M\rangle$$

$$\langle \psi_\sigma(W) | \psi_\rho(V)\rangle = \langle M | \sqrt{\sigma}\sqrt{\rho} \otimes W^\dagger V |M\rangle$$

$$= \langle M | \sqrt{\sigma}\sqrt{\rho} (\mathbb{1} \otimes W^\dagger V) |M\rangle$$

$$\mathrm{tr}[A] = \langle M | A \otimes \mathbb{1} | M\rangle$$

$$= \langle M | \sqrt{\sigma}\sqrt{\rho} \underbrace{(W^\dagger V)^T}_{U} \otimes \mathbb{1} |M\rangle$$

$$= \mathrm{tr}[\sqrt{\sigma}\sqrt{\rho}\, U]$$

$$\sqrt{\sigma}\sqrt{\rho} = X \overbrace{\sqrt{(\sqrt{\sigma}\sqrt{\rho})^\dagger (\sqrt{\sigma}\sqrt{\rho})}}^{} \qquad (\sqrt{\sigma}\sqrt{\rho})^\dagger (\sqrt{\sigma}\sqrt{\rho}) = \sqrt{\rho}\,\sigma\sqrt{\rho}$$

X: unitary

$$= \mathrm{tr}\left(U X \sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)$$

$$\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}} = \sum_j \lambda_j |j\rangle\langle j|$$

$$= \sum_j \lambda_j \langle j | U X | j\rangle$$

maximal for $U = X^\dagger$

$$\max_{U,V} |\langle \psi_\sigma(W) | \psi_\rho(V)\rangle| = \mathrm{tr}\left(\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)$$

$$= F(\rho,\sigma) \quad \square$$

Properties

symmetry: $F(\varrho,\sigma) = F(\sigma,\varrho)$

monotonicity: $F(\varrho_{AB},\sigma_{AB}) \leq F(\varrho_A,\varrho_B)$

extends to qu. channels

Proof: at home

Bounds: $0 \leq F(\varrho,\sigma) \leq 1$

Bures angle & Bures distance

$$\cos\left[D_{BA}(\varrho,\sigma)\right] = F(\varrho,\sigma)$$

$$D_{BD}^2(\varrho,\sigma) = 2(1-F(\varrho,\sigma))$$