The Petz Recovery Map

Alec Boyd

For Your Consideration

https://youtu.be/cJyGoGPXTj4?si=TcxUA4MaGbyw5V-A

https://youtu.be/cjIvu7e6Wq8?si=um9FYh3cSwDNh9ar

For Your Consideration

https://youtu.be/cJyGoGPXTj4?si=TcxUA4MaGbyw5V-A

https://youtu.be/cjIvu7e6Wq8?si=um9FYh3cSwDNh9ar

What does it mean to flip it and reverse it?

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Reverse-time evolution $\hat{U}(t)^R \equiv \hat{U}(-t) = e^{i\hat{H}t/\hbar}$

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Reverse-time evolution $\hat{U}(t)^R \equiv \hat{U}(-t) = e^{i\hat{H}t/\hbar}$

Reverse=Inverse: $\hat{U}(t)^R = \hat{U}(t)^{-1}$ since $\hat{U}(-t)\hat{U}(t) = e^{i\hat{H}t/\hbar - i\hat{H}t/\hbar}$

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Reverse-time evolution $\hat{U}(t)^R \equiv \hat{U}(-t) = e^{i\hat{H}t/\hbar}$

Reverse=Inverse: $\hat{U}(t)^R = \hat{U}(t)^{-1}$

$$\begin{aligned}\text{since } \hat{U}(-t)\hat{U}(t) &= e^{i\hat{H}t/\hbar - i\hat{H}t/\hbar} \\ &= e^0 \\ &= \mathbb{I} \equiv \sum_i |i\rangle\langle i|\end{aligned}$$

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Reverse-time evolution $\hat{U}(t)^R \equiv \hat{U}(-t) = e^{i\hat{H}t/\hbar}$

Reverse=Inverse: $\hat{U}(t)^R = \hat{U}(t)^{-1}$

$$\text{since } \hat{U}(-t)\hat{U}(t) = e^{i\hat{H}t/\hbar - i\hat{H}t/\hbar} = e^0 = \mathbb{I} \equiv \sum_i |i\rangle\langle i|$$

Reverse=adjoint: $\hat{U}(t)^R = \hat{U}(t)^\dagger$

$$\hat{U}(t)^\dagger = \left(e^{-i\hat{H}t/\hbar}\right)^\dagger$$

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Reverse-time evolution $\hat{U}(t)^R \equiv \hat{U}(-t) = e^{i\hat{H}t/\hbar}$

Reverse=Inverse: $\hat{U}(t)^R = \hat{U}(t)^{-1}$

$$\text{since } \hat{U}(-t)\hat{U}(t) = e^{i\hat{H}t/\hbar - i\hat{H}t/\hbar}$$
$$= e^0$$
$$= \mathbb{I} \equiv \sum_i |i\rangle\langle i|$$

Reverse=adjoint: $\hat{U}(t)^R = \hat{U}(t)^\dagger$

$$\hat{U}(t)^\dagger = \left(e^{-i\hat{H}t/\hbar}\right)^\dagger$$
$$= e^{-\left(i\hat{H}\right)^\dagger t/\hbar}$$

Unitary Time Reversal

Time evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Reverse-time evolution $\hat{U}(t)^R \equiv \hat{U}(-t) = e^{i\hat{H}t/\hbar}$

Reverse=Inverse: $\hat{U}(t)^R = \hat{U}(t)^{-1}$

$$\begin{aligned}\text{since } \hat{U}(-t)\hat{U}(t) &= e^{i\hat{H}t/\hbar - i\hat{H}t/\hbar} \\ &= e^0 \\ &= \mathbb{I} \equiv \sum_i |i\rangle\langle i|\end{aligned}$$

Reverse=adjoint: $\hat{U}(t)^R = \hat{U}(t)^\dagger$

$$\begin{aligned}\hat{U}(t)^\dagger &= \left(e^{-i\hat{H}t/\hbar}\right)^\dagger \\ &= e^{-\left(i\hat{H}\right)^\dagger t/\hbar} \\ &= e^{i\hat{H}t/\hbar} \qquad \text{because } \hat{H}^\dagger = \hat{H}\end{aligned}$$

Unitary as a Permutation?

Express unitary component-wise in a basis $$U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $U = \sum_j |\psi_j\rangle\langle j|$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $\quad U = \sum_j |\psi_j\rangle\langle j|$ $\quad U^\dagger = \sum_j |j\rangle\langle \psi_j|$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $U = \sum_j |\psi_j\rangle\langle j|$ $U^\dagger = \sum_j |j\rangle\langle\psi_j|$

Apply adjoint $U^\dagger U = \mathbb{I}$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $\quad U = \sum_j |\psi_j\rangle\langle j|$ $\quad U^\dagger = \sum_j |j\rangle\langle\psi_j|$

Apply adjoint $\quad U^\dagger U = \mathbb{I}$

$$\sum_{j',j} |j'\rangle\langle\psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $U = \sum_j |\psi_j\rangle\langle j|$ $U^\dagger = \sum_j |j\rangle\langle\psi_j|$

Apply adjoint $U^\dagger U = \mathbb{I}$

$$\sum_{j',j} |j'\rangle\langle\psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

Therefore: $\langle\psi_{j'}|\psi_j\rangle = \delta_{j',j}$

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $\quad U = \sum_j |\psi_j\rangle\langle j|$ $\quad U^\dagger = \sum_j |j\rangle\langle\psi_j|$

Apply adjoint $U^\dagger U = \mathbb{I}$

$$\sum_{j',j} |j'\rangle\langle\psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

Therefore: $\langle\psi_{j'}|\psi_j\rangle = \delta_{j',j}$ $\quad \{|\psi_j\rangle\}_j$ is a basis

Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle \quad U = \sum_j |\psi_j\rangle\langle j| \quad U^\dagger = \sum_j |j\rangle\langle\psi_j|$

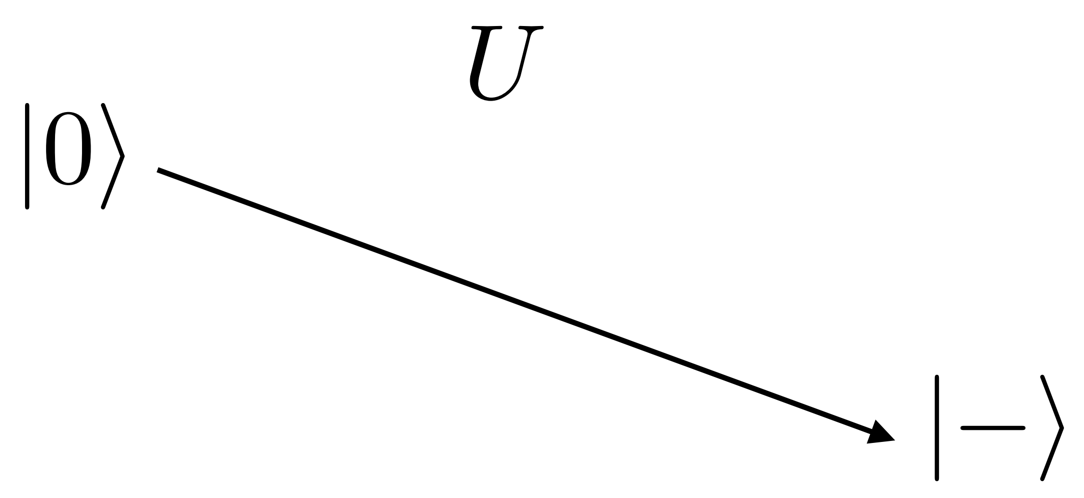
Apply adjoint $\quad U^\dagger U = \mathbb{I}$

$$\sum_{j',j} |j'\rangle\langle\psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

Therefore: $\langle\psi_{j'}|\psi_j\rangle = \delta_{j',j} \qquad \{|\psi_j\rangle\}_j$ is a basis

Unitaries permute between elements of different bases

$$|\psi_0\rangle = |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j} |i\rangle$ $\quad U = \sum_j |\psi_j\rangle\langle j|$ $\quad U^\dagger = \sum_j |j\rangle\langle \psi_j|$

Apply adjoint $\quad U^\dagger U = \mathbb{I}$

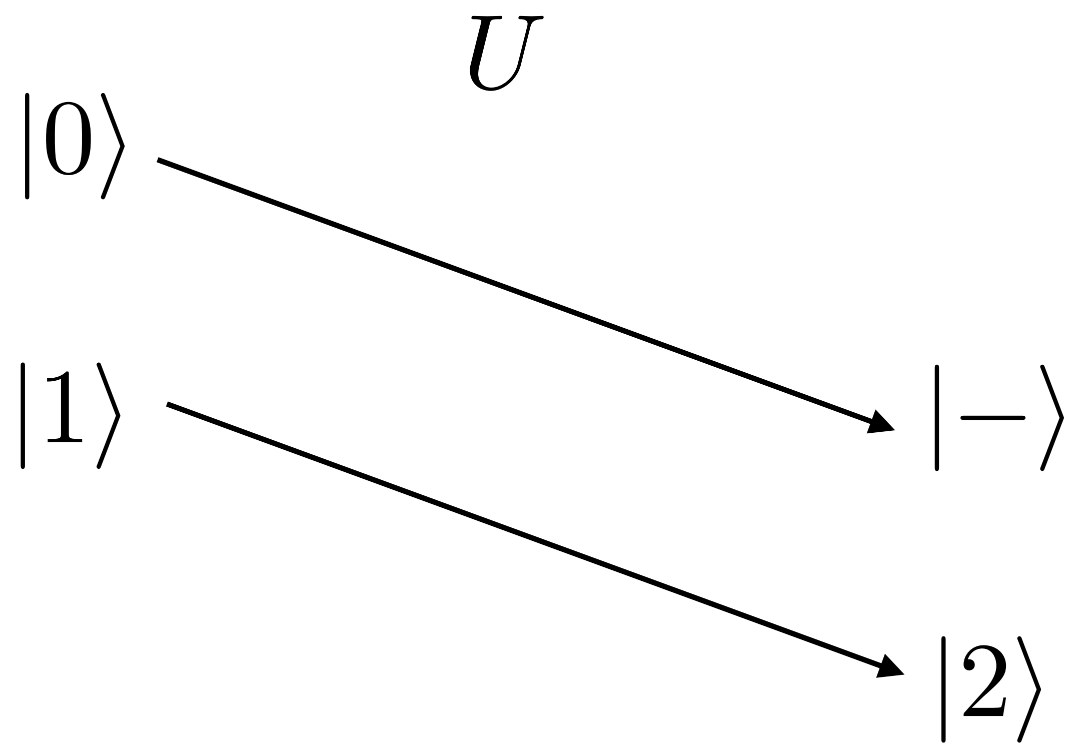
$$\sum_{j',j} |j'\rangle\langle \psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

Therefore: $\langle \psi_{j'}|\psi_j\rangle = \delta_{j',j}$ $\quad \{|\psi_j\rangle\}_j$ is a basis

Unitaries permute between elements of different bases

$$|\psi_0\rangle = |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_1\rangle = |2\rangle$$



Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j}|i\rangle \quad U = \sum_j |\psi_j\rangle\langle j| \quad U^\dagger = \sum_j |j\rangle\langle \psi_j|$

Apply adjoint $U^\dagger U = \mathbb{I}$

$$\sum_{j',j} |j'\rangle\langle\psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

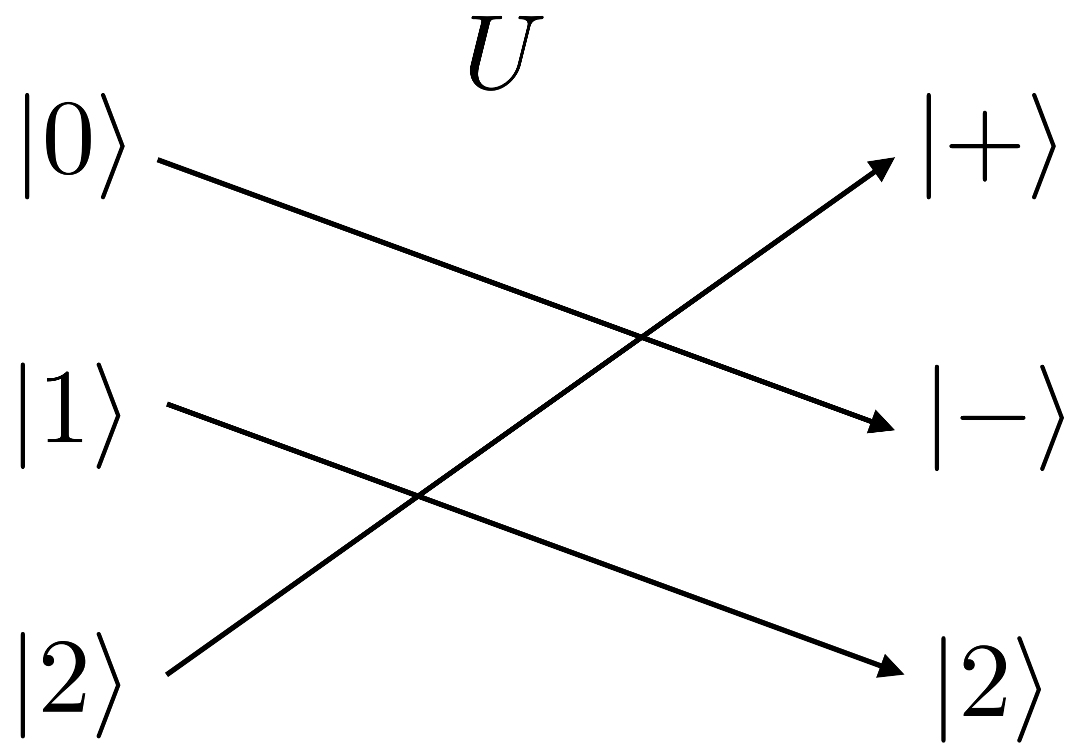
$\text{Therefore: } \langle\psi_{j'}|\psi_j\rangle = \delta_{j',j} \qquad \{|\psi_j\rangle\}_j \text{ is a basis}$

Unitaries permute between elements of different bases

$$|\psi_0\rangle = |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_1\rangle = |2\rangle$$

$$|\psi_2\rangle = |+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



Unitary as a Permutation?

Express unitary component-wise in a basis $U = \sum_{i,j} U_{i,j} |i\rangle\langle j|$

Introduce vectors $|\psi_j\rangle \equiv \sum_i U_{i,j}|i\rangle \quad U = \sum_j |\psi_j\rangle\langle j| \quad U^\dagger = \sum_j |j\rangle\langle\psi_j|$

Apply adjoint $U^\dagger U = \mathbb{I}$

$$\sum_{j',j} |j'\rangle\langle\psi_{j'}|\psi_j\rangle\langle j| = \sum_j |j\rangle\langle j|$$

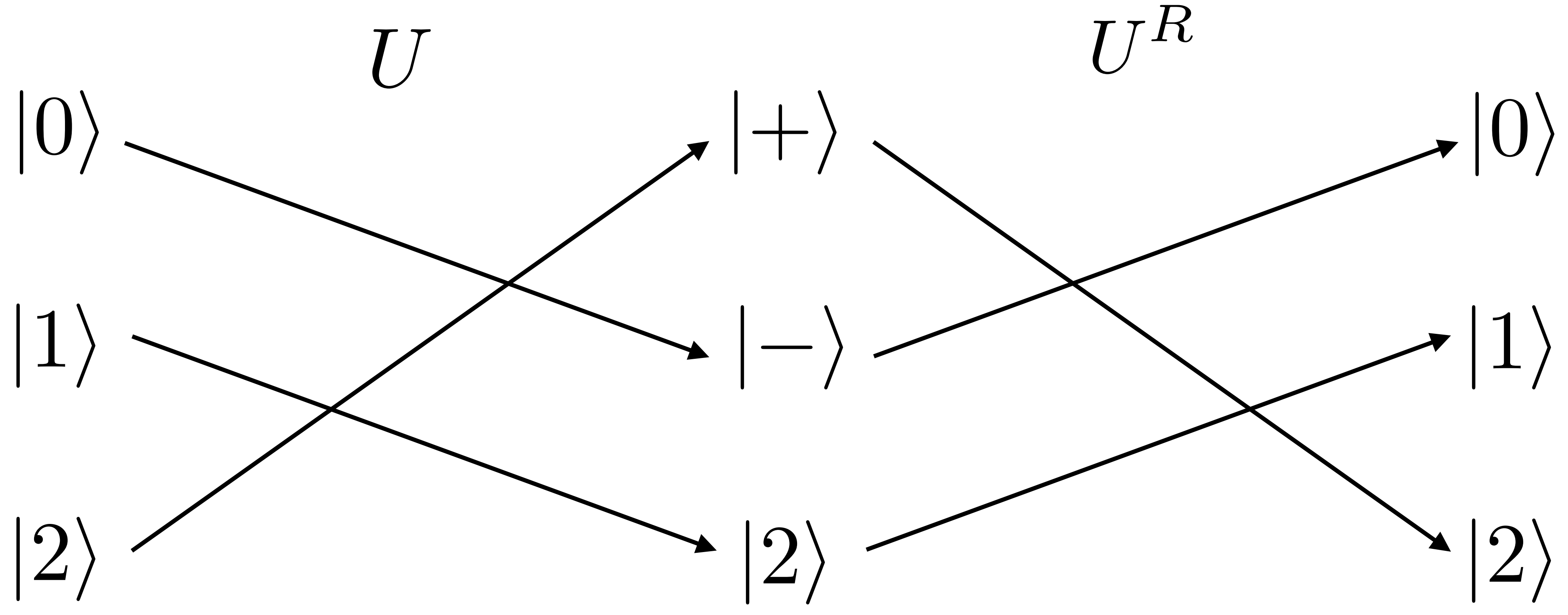
Therefore: $\langle\psi_{j'}|\psi_j\rangle = \delta_{j',j} \qquad \{|\psi_j\rangle\}_j$ is a basis

Unitaries permute between elements of different bases

$$|\psi_0\rangle = |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

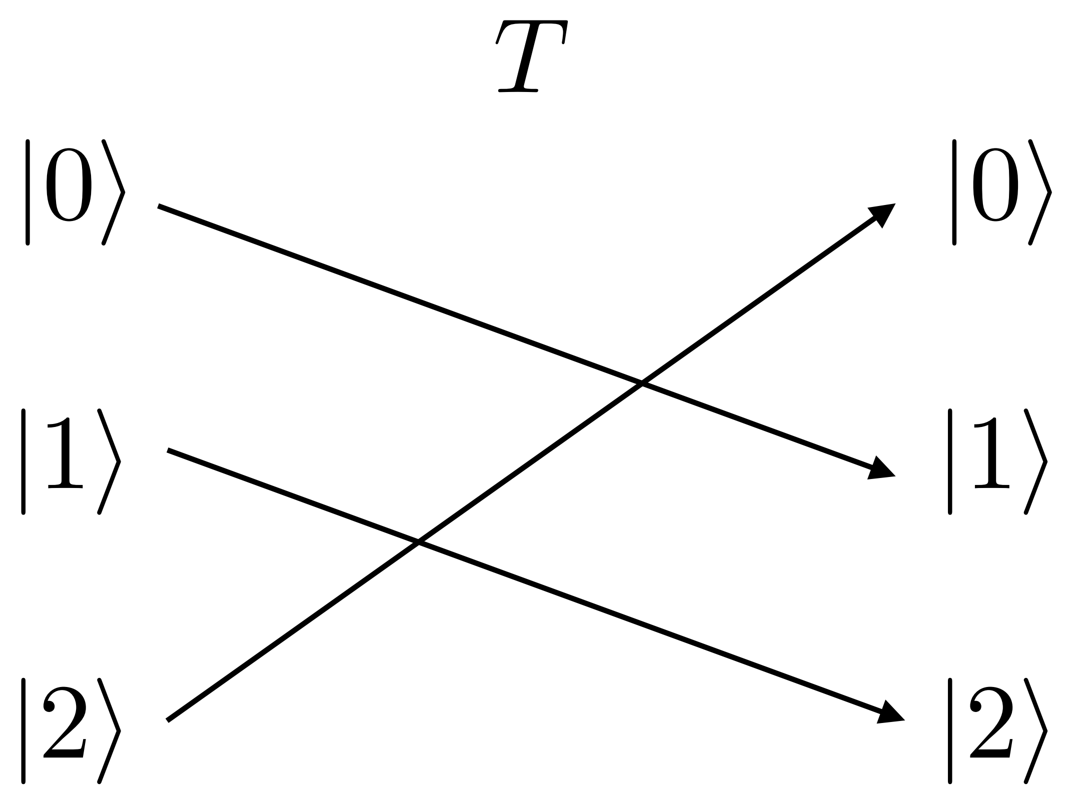
$$|\psi_1\rangle = |2\rangle$$

$$|\psi_2\rangle = |+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



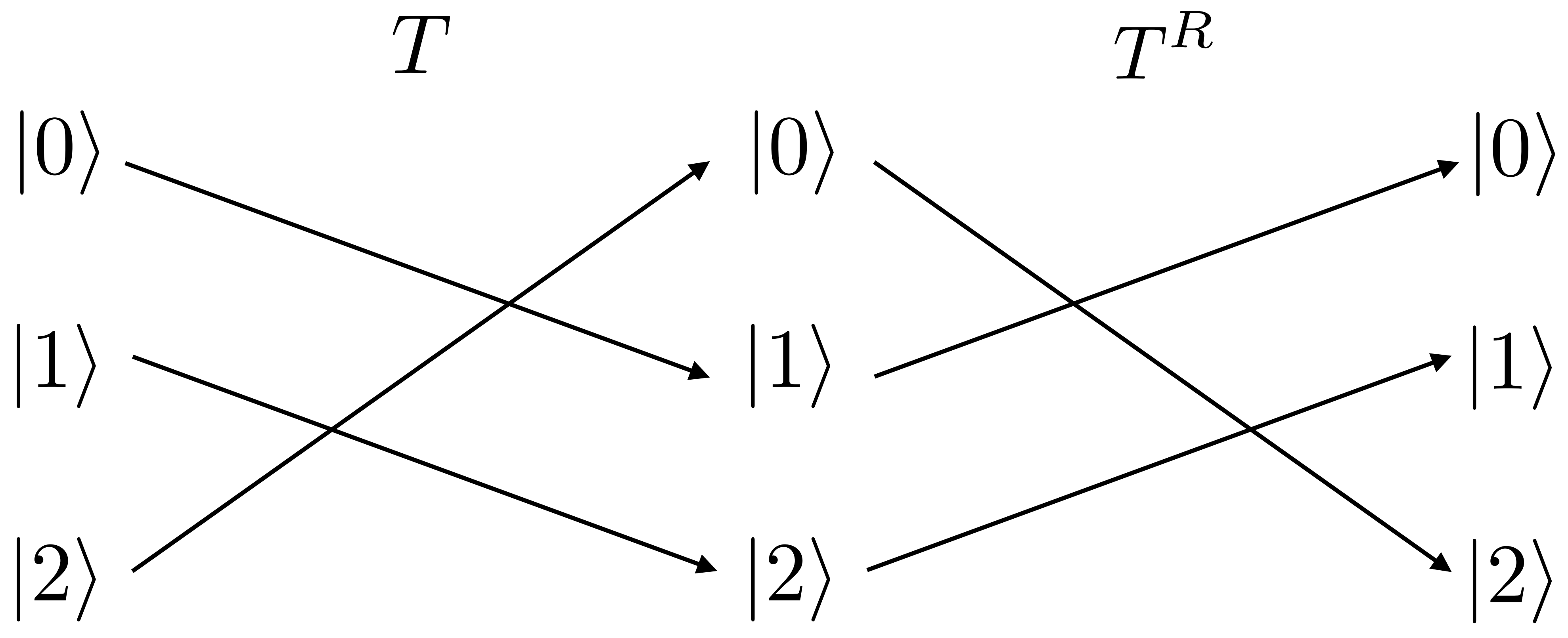
Classical Time Reversal

Classical permutation and time reversal:



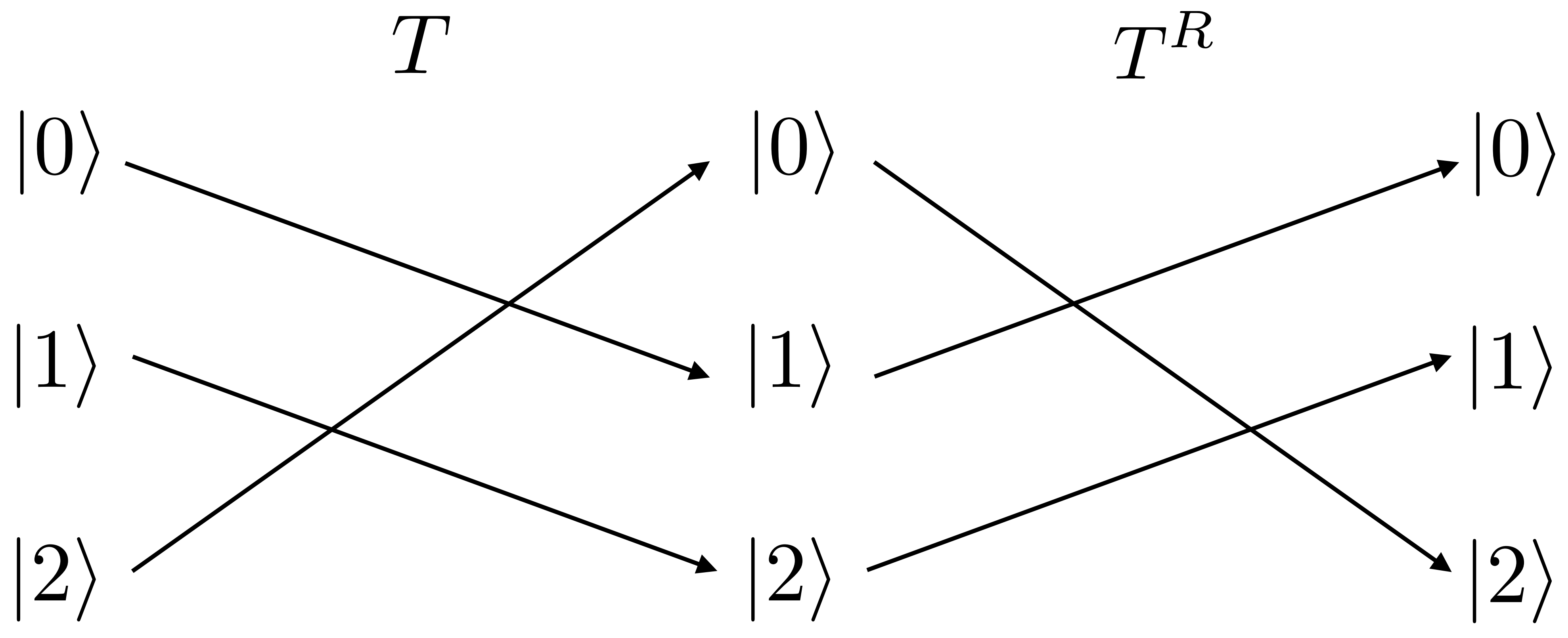
Classical Time Reversal

Classical permutation and time reversal:

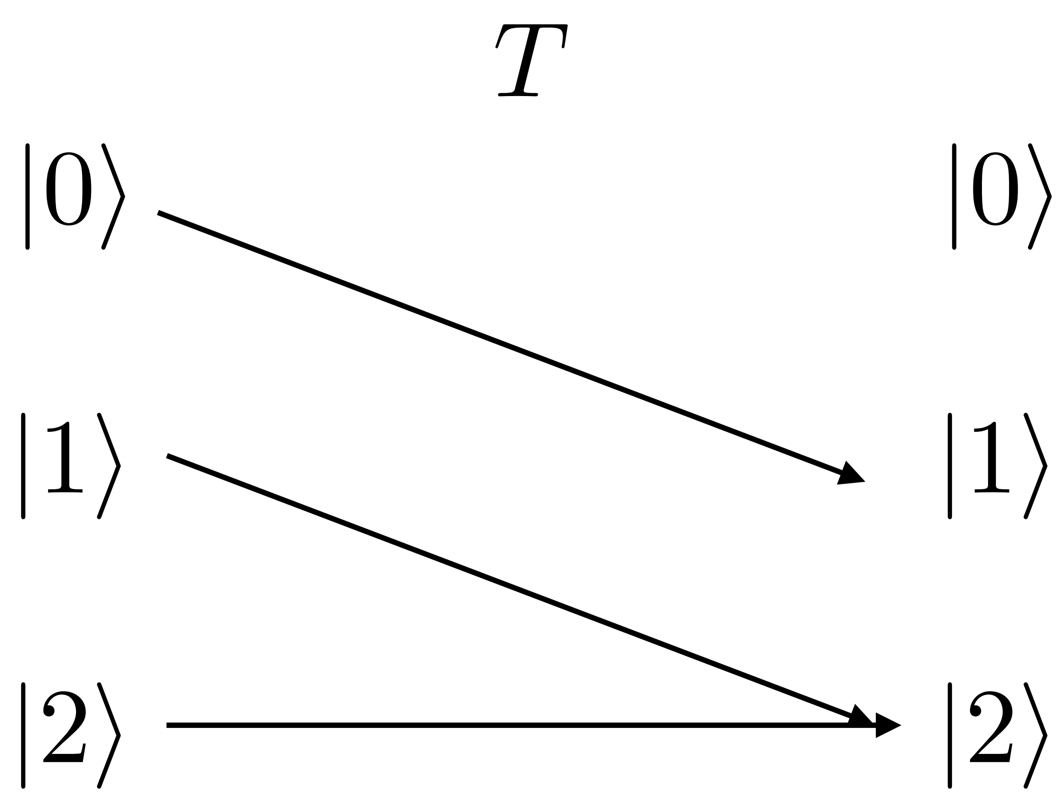


Classical Time Reversal

Classical permutation and time reversal:

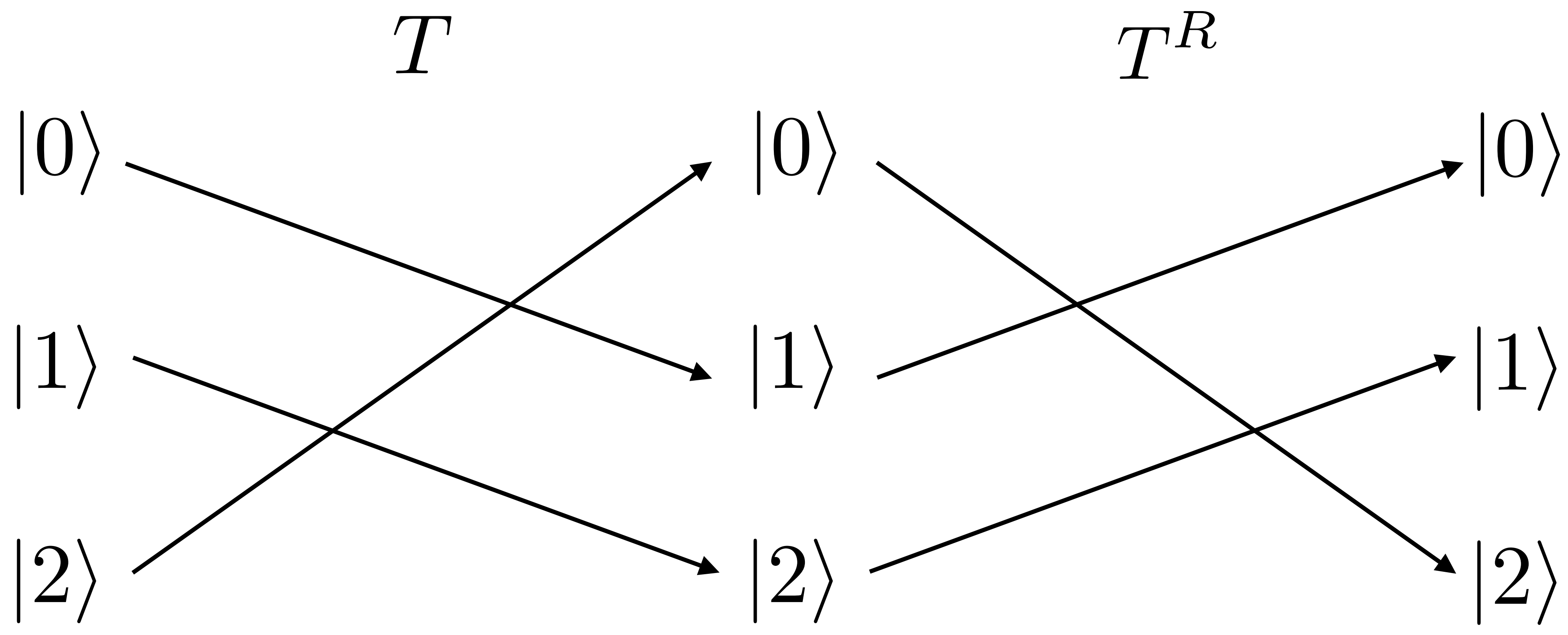


What can we do with classical permutations?
Is this everything?

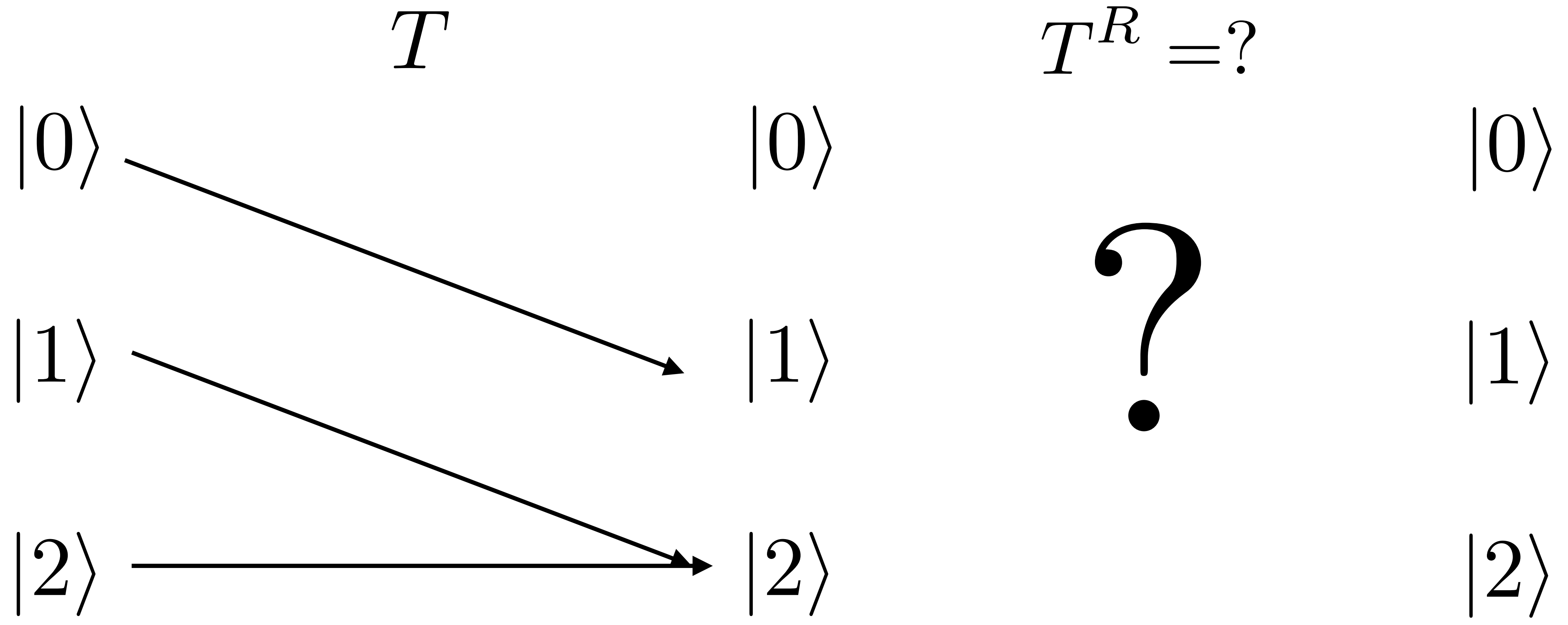


Classical Time Reversal

Classical permutation and time reversal:

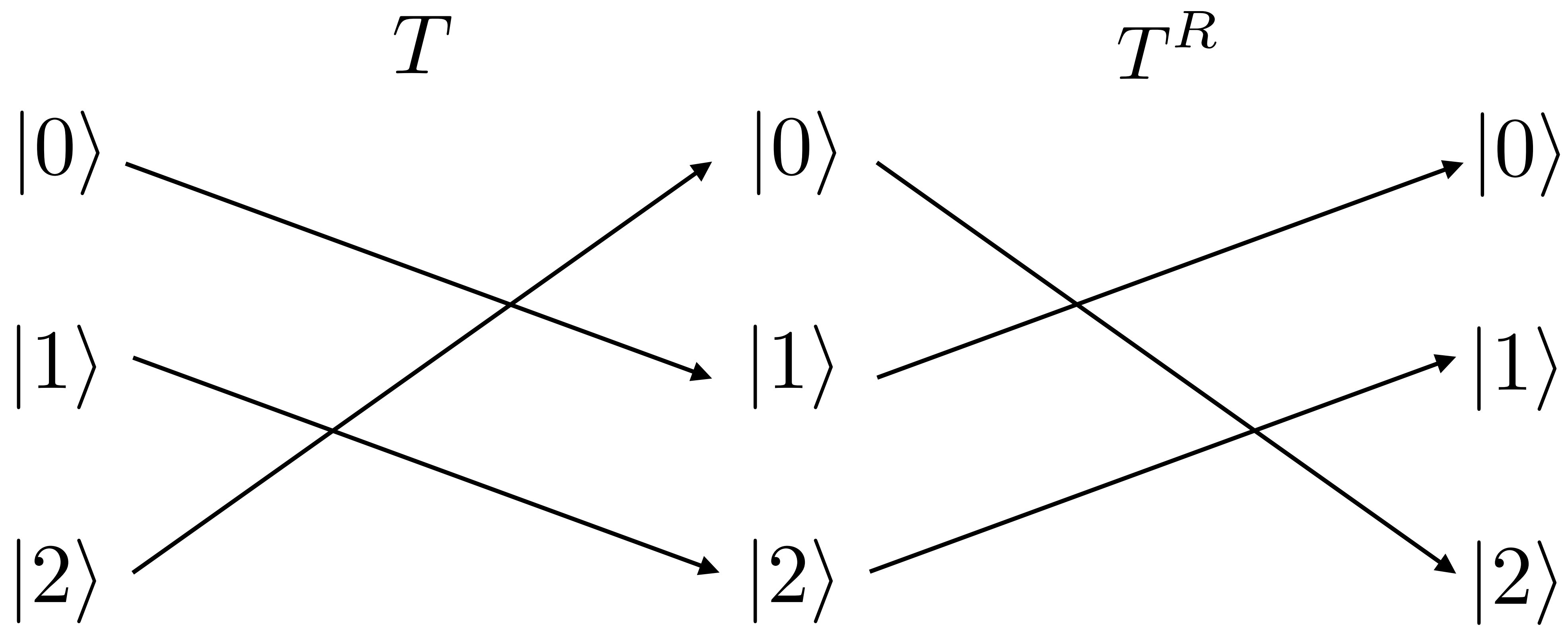


What can we do with classical permutations?
Is this everything?

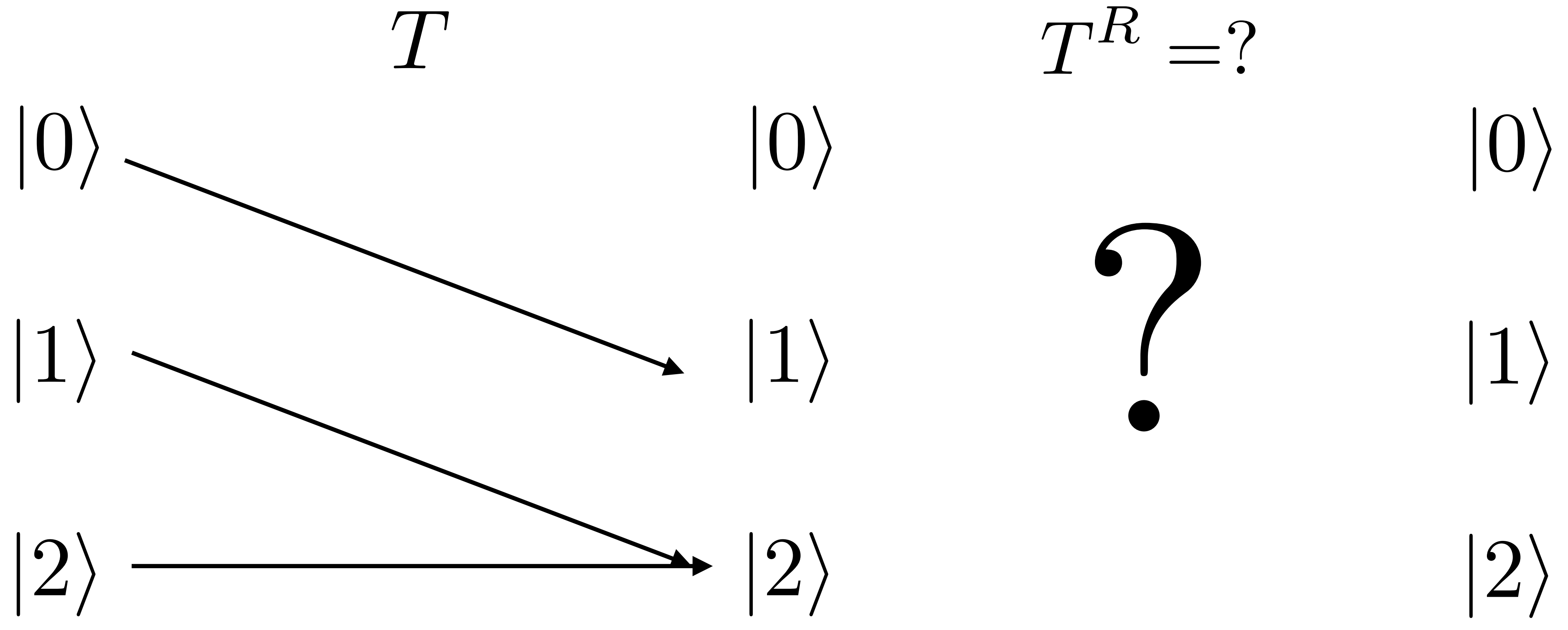


Classical Time Reversal

Classical permutation and time reversal:



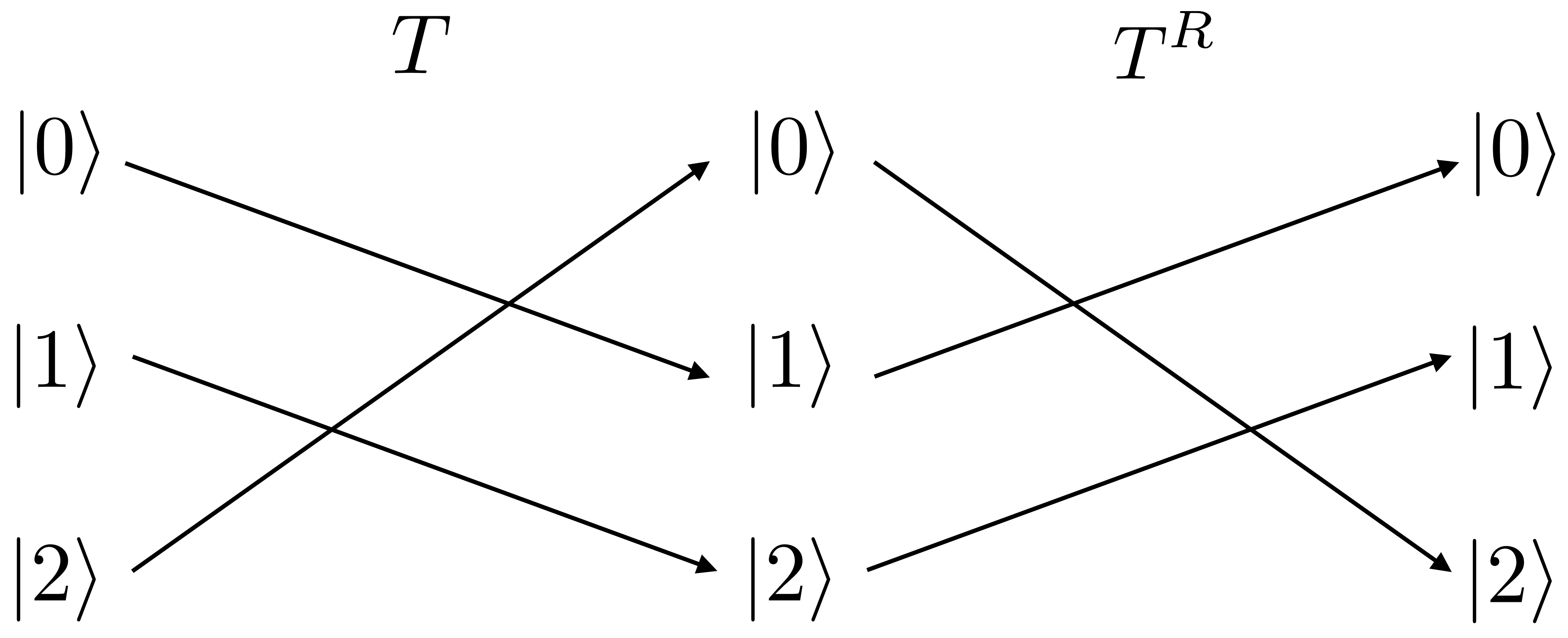
What can we do with classical permutations?
Is this everything?



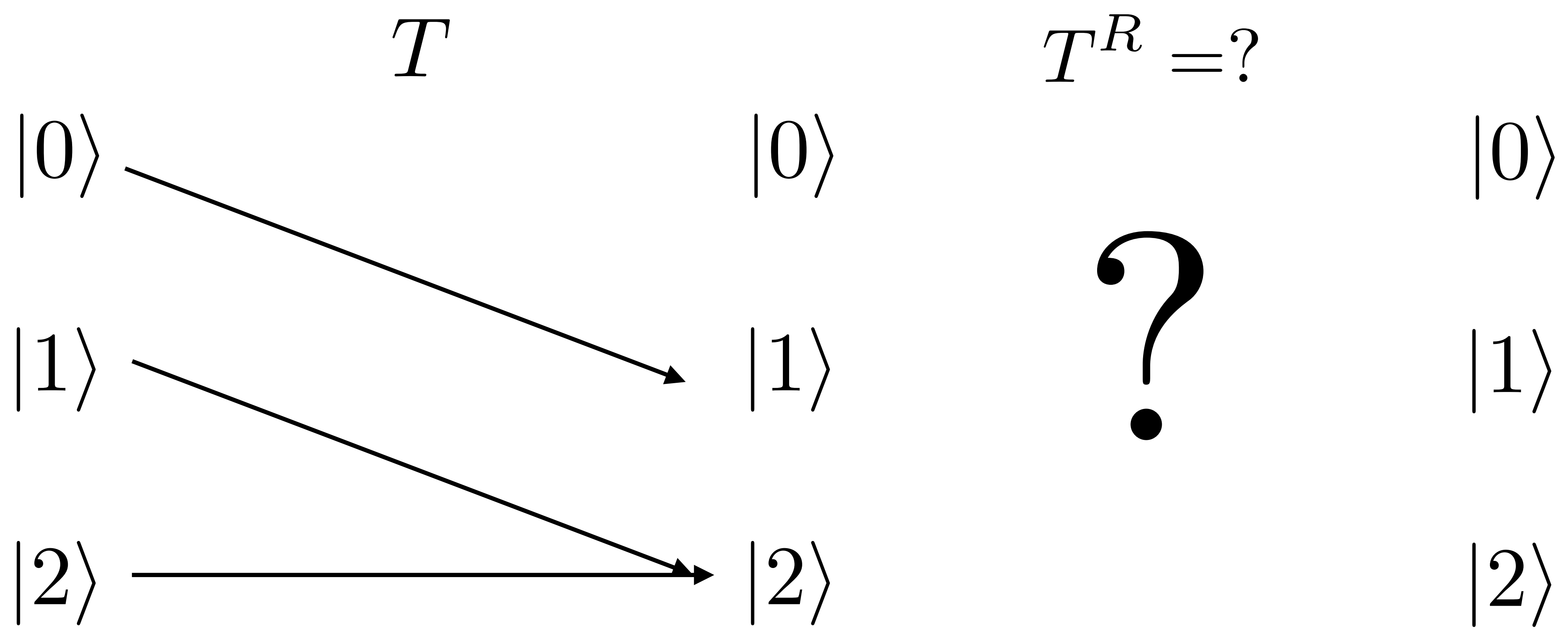
IRREVERSIBLE TRANSFORMATION
Exact previous configuration cannot be determined

Classical Time Reversal

Classical permutation and time reversal:



What can we do with classical permutations?
Is this everything?

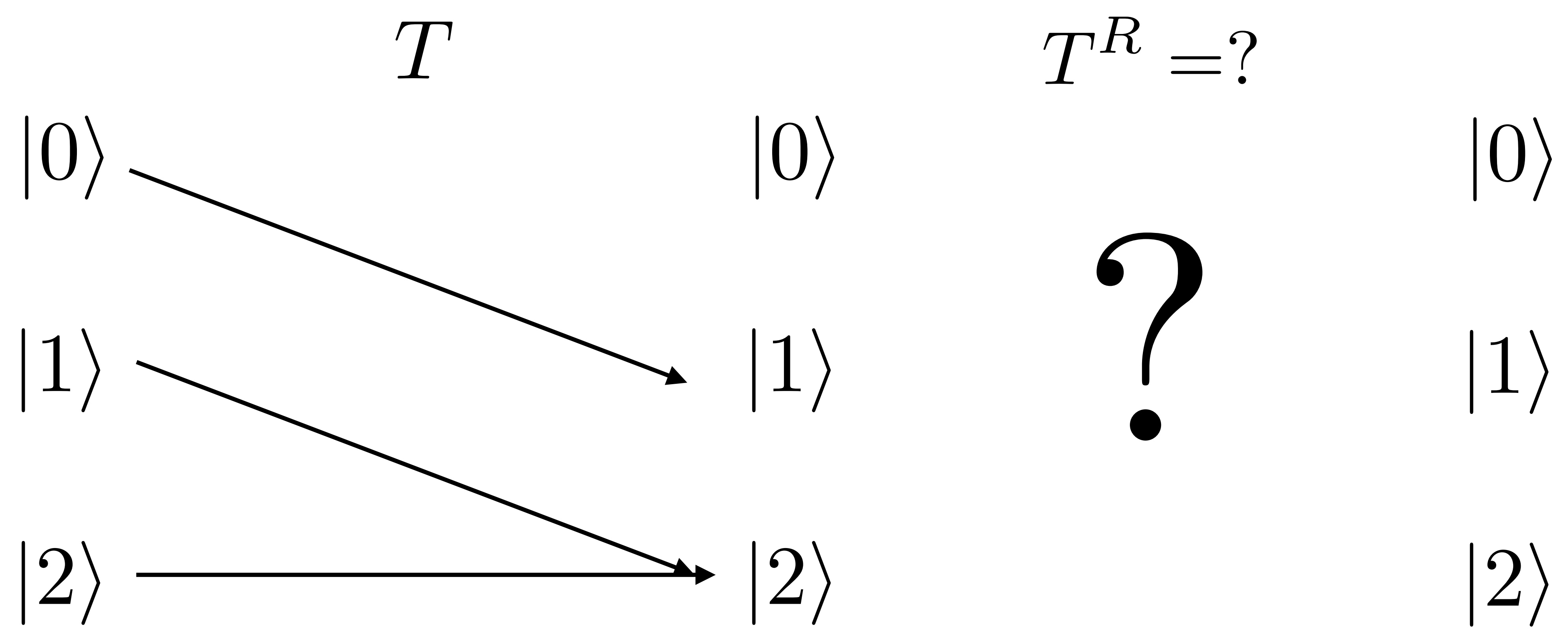


IRREVERSIBLE TRANSFORMATION
Exact previous configuration cannot be determined

https://www.youtube.com/watch?v=v30b5IAgwQw

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation

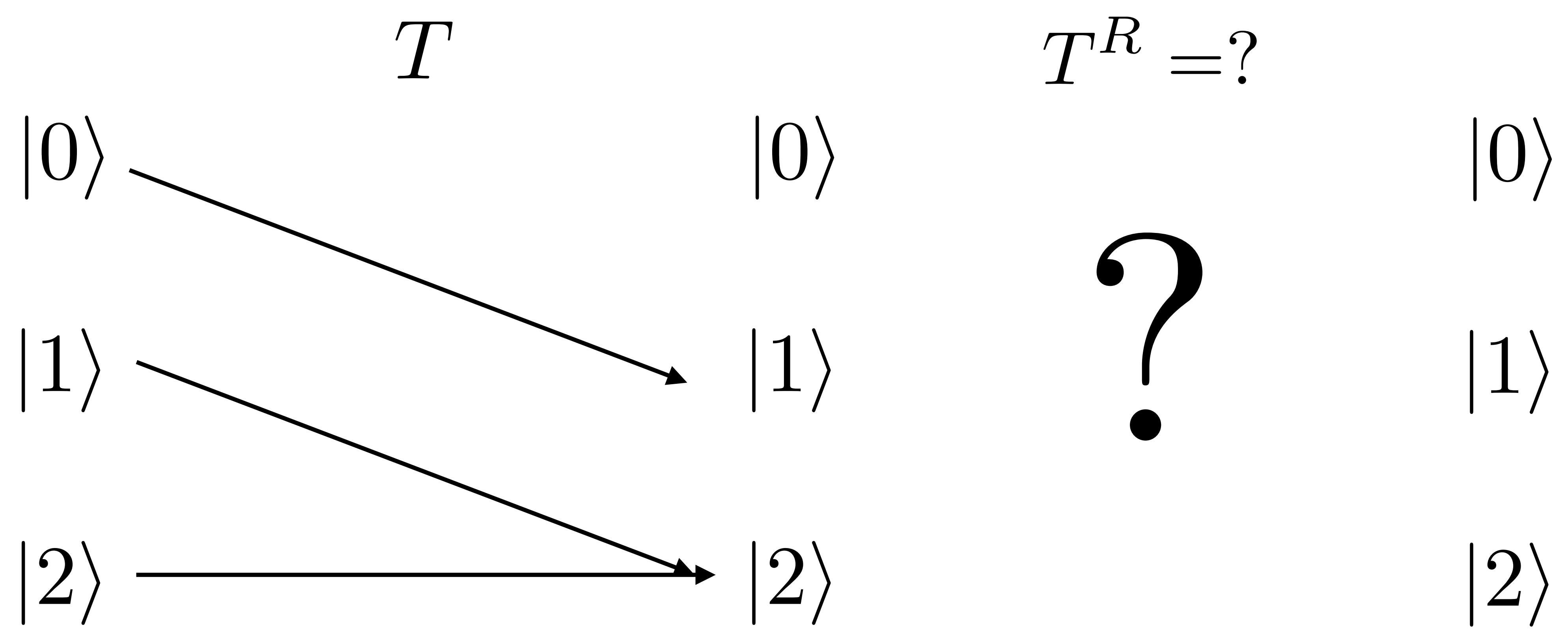


conditional probability:

$$T = \sum_{i,j} T_{i \to j} |j\rangle\langle i|$$

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation

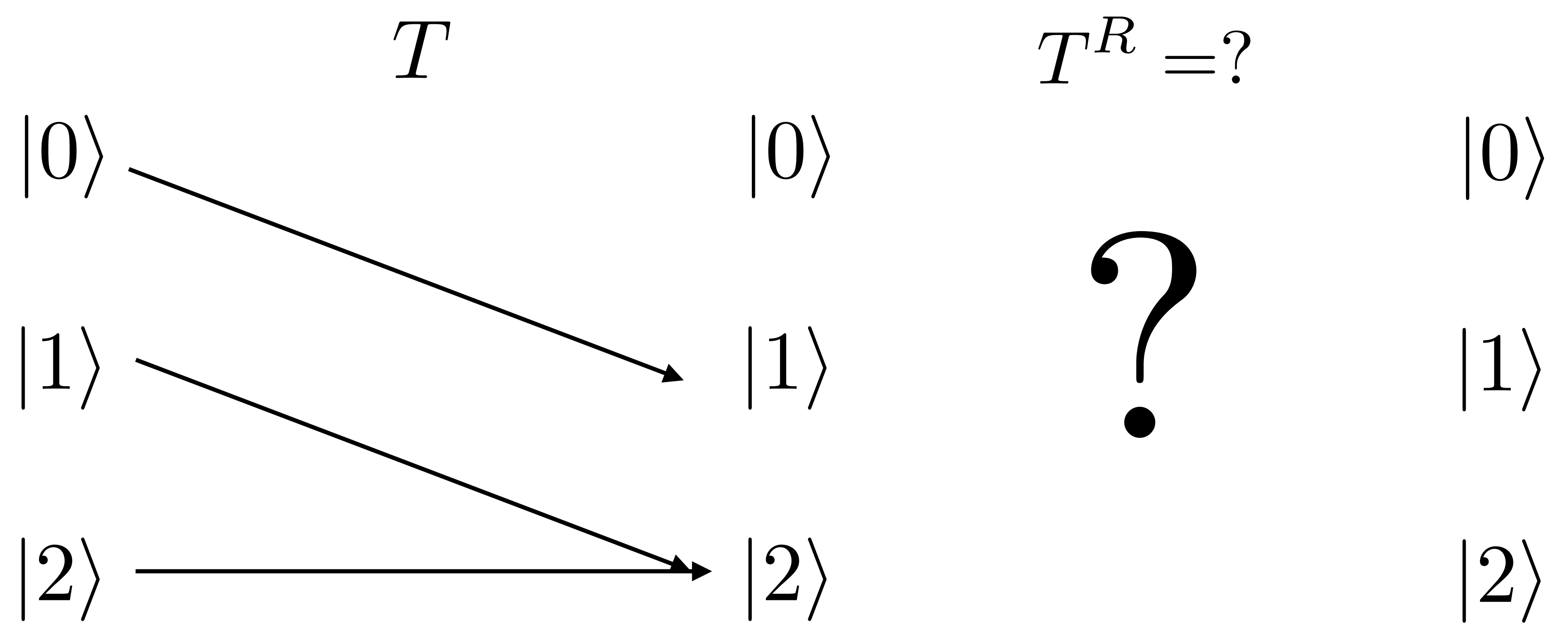


conditional probability:

$$T = \sum_{i,j} T_{i\to j} |j\rangle\langle i| \qquad T_{i\to j} \equiv \Pr(S_t = j | S_0 = i)$$

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



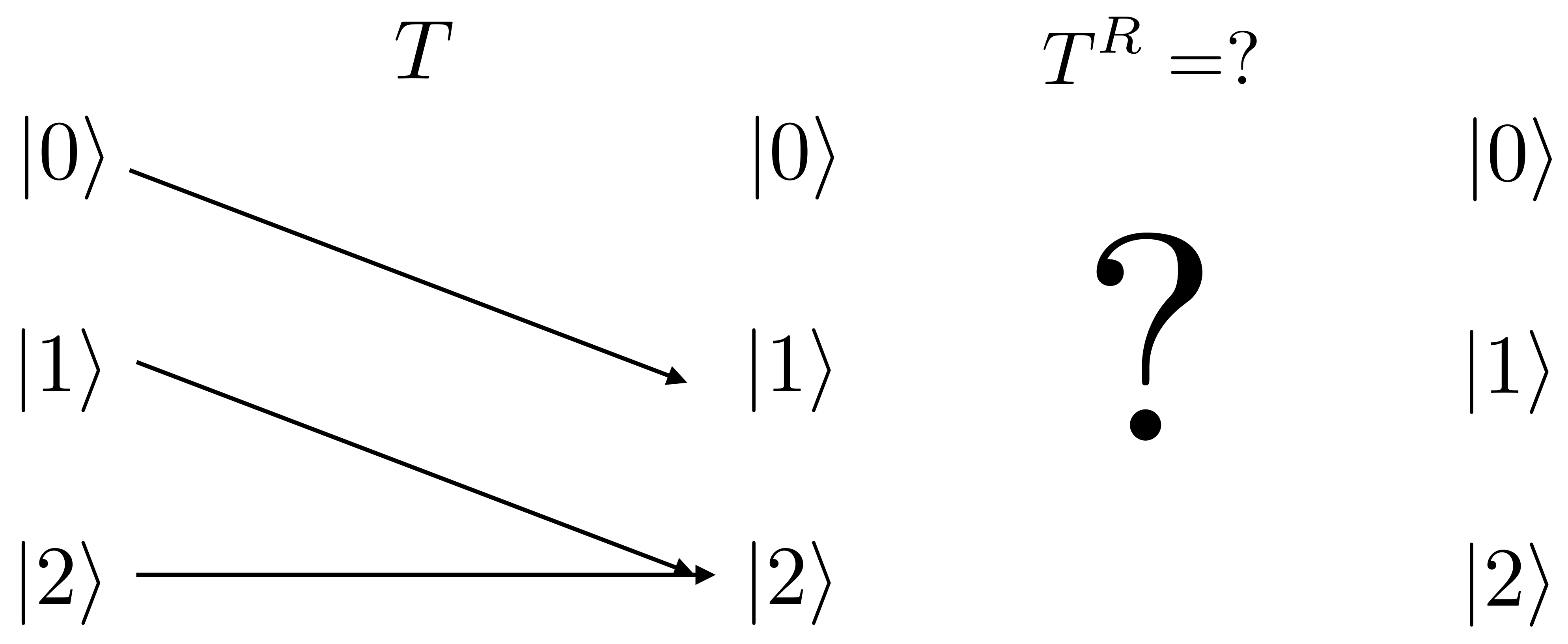
conditional probability:

$$T = \sum_{i,j} T_{i\to j} |j\rangle\langle i| \qquad T_{i\to j} \equiv \Pr(S_t = j | S_0 = i)$$

S_t realizes state at time t

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



conditional probability:

$$T = \sum_{i,j} T_{i\to j} |j\rangle\langle i| \qquad T_{i\to j} \equiv \Pr(S_t = j | S_0 = i)$$

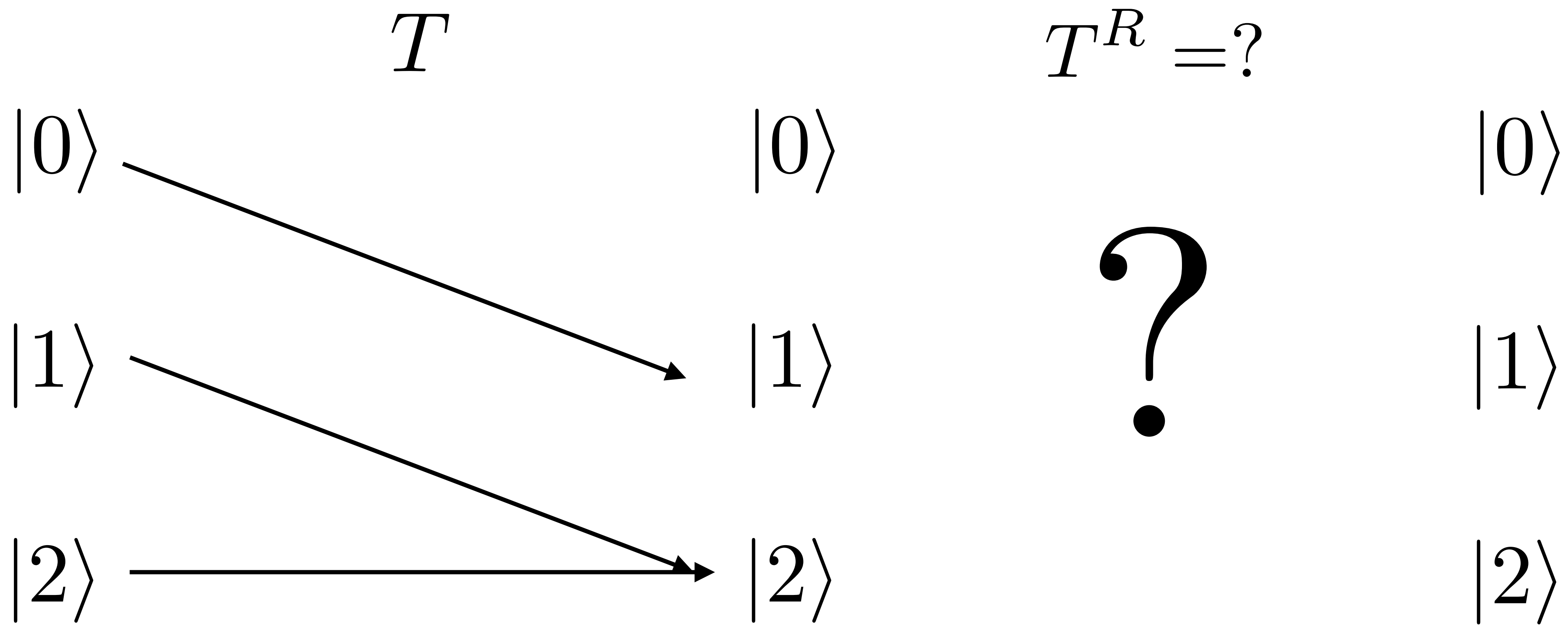
Deterministic permutation is a subcase:

$$T_{i\to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



conditional probability:

$$T = \sum_{i,j} T_{i \to j} |j\rangle\langle i| \qquad T_{i \to j} \equiv \Pr(S_t = j | S_0 = i)$$

Deterministic permutation is a subcase:

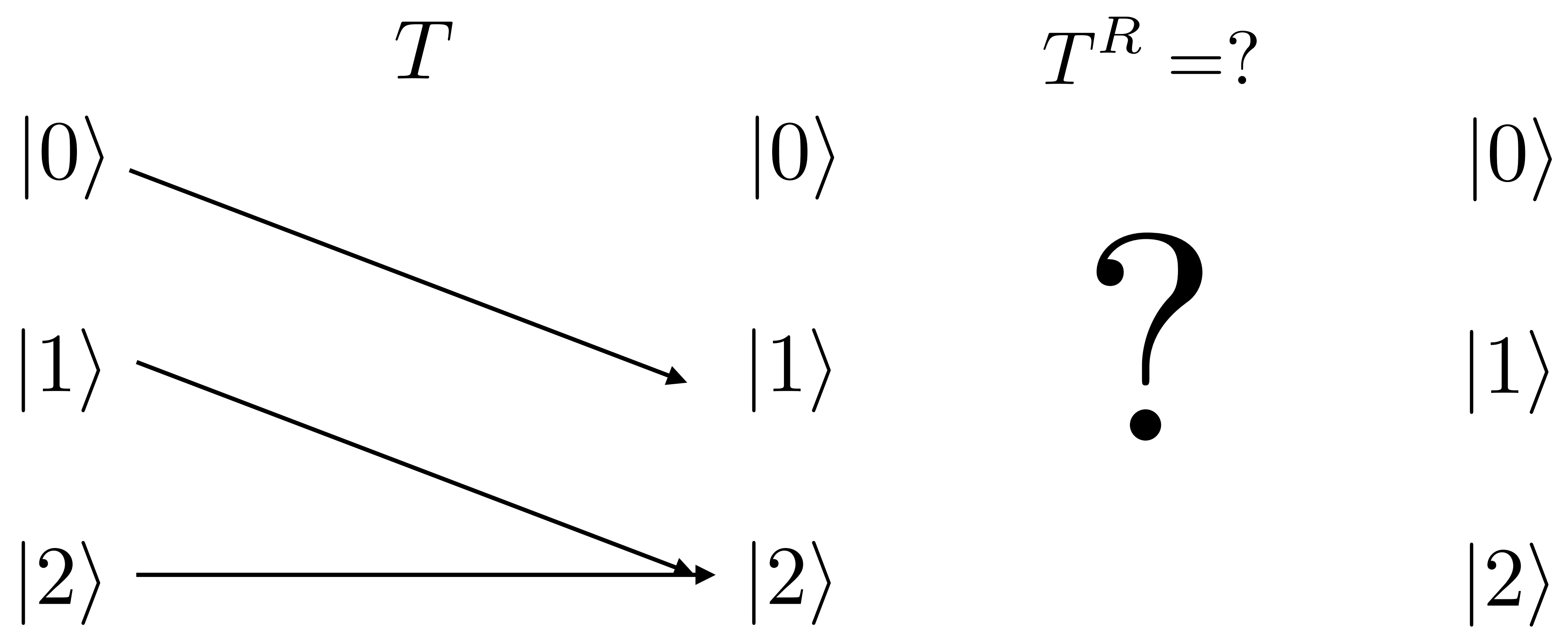
$$T_{i \to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Given an initial probability $\Pr(S_0 = i)$

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



Given an initial probability $\Pr(S_0 = i)$

1) Final probability:

$$\Pr(S_t = j) = \sum_i \Pr(S_0 = i) T_{i \to j}$$

conditional probability:

$$T = \sum_{i,j} T_{i \to j} |j\rangle\langle i| \qquad T_{i \to j} \equiv \Pr(S_t = j | S_0 = i)$$

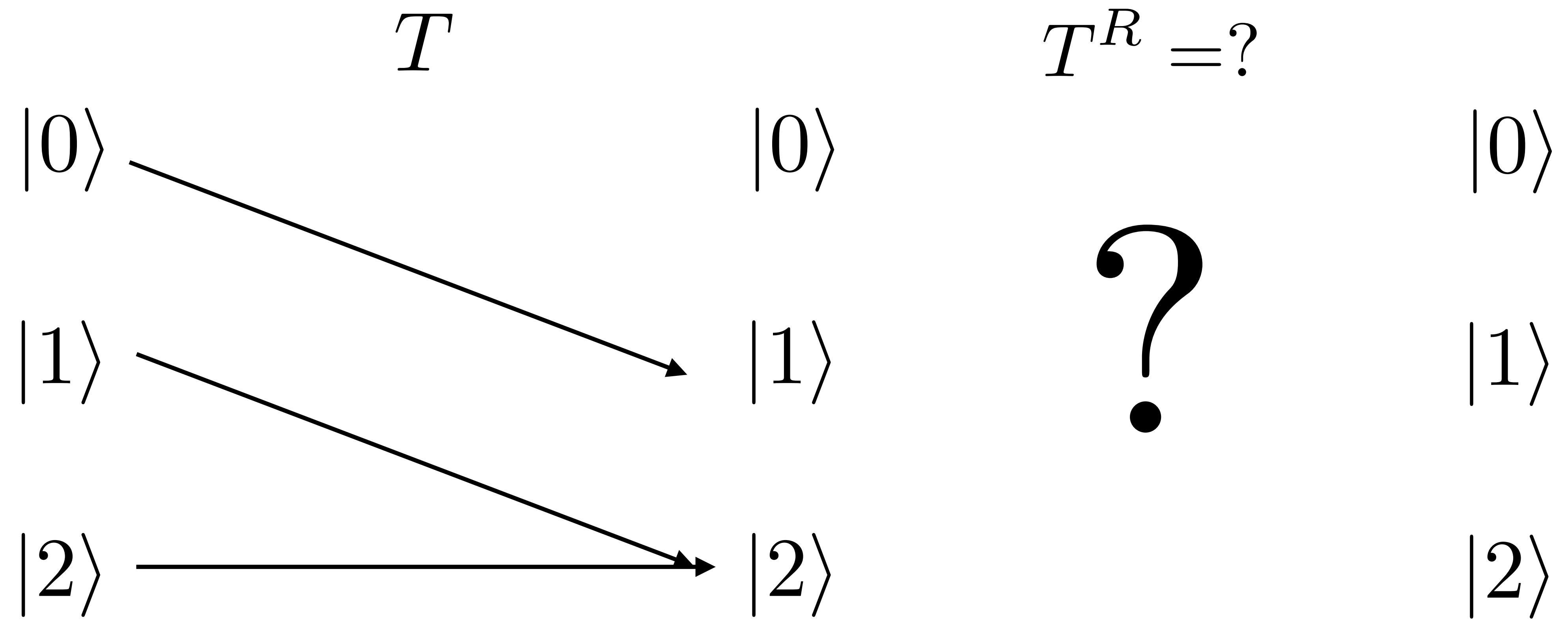
Deterministic permutation is a subcase:

$$T_{i \to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



conditional probability:

$$T = \sum_{i,j} T_{i\to j} |j\rangle\langle i| \qquad T_{i\to j} \equiv \Pr(S_t = j | S_0 = i)$$

Deterministic permutation is a subcase:

$$T_{i\to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Given an initial probability $\Pr(S_0 = i)$

1) Final probability:

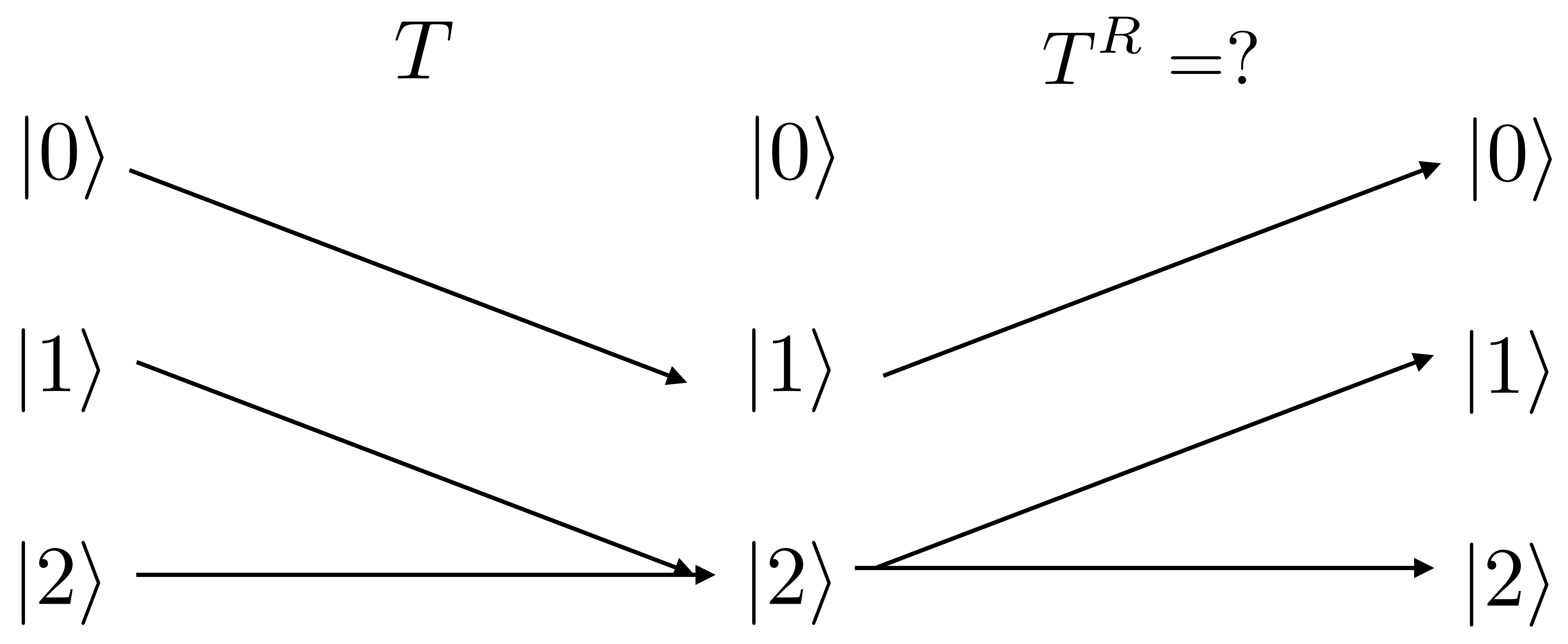
$$\Pr(S_t = j) = \sum_i \Pr(S_0 = i) T_{i\to j}$$

2) Joint probability over initial and final state:

$$\Pr(S_t = j, S_0 = i) = \Pr(S_0 = i) T_{i\to j}$$

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



conditional probability:

$$T = \sum_{i,j} T_{i\to j} |j\rangle\langle i| \qquad T_{i\to j} \equiv \Pr(S_t = j | S_0 = i)$$

Deterministic permutation is a subcase:

$$T_{i\to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Given an initial probability $\Pr(S_0 = i)$

1) Final probability:

$$\Pr(S_t = j) = \sum_i \Pr(S_0 = i) T_{i\to j}$$

2) Joint probability over initial and final state:

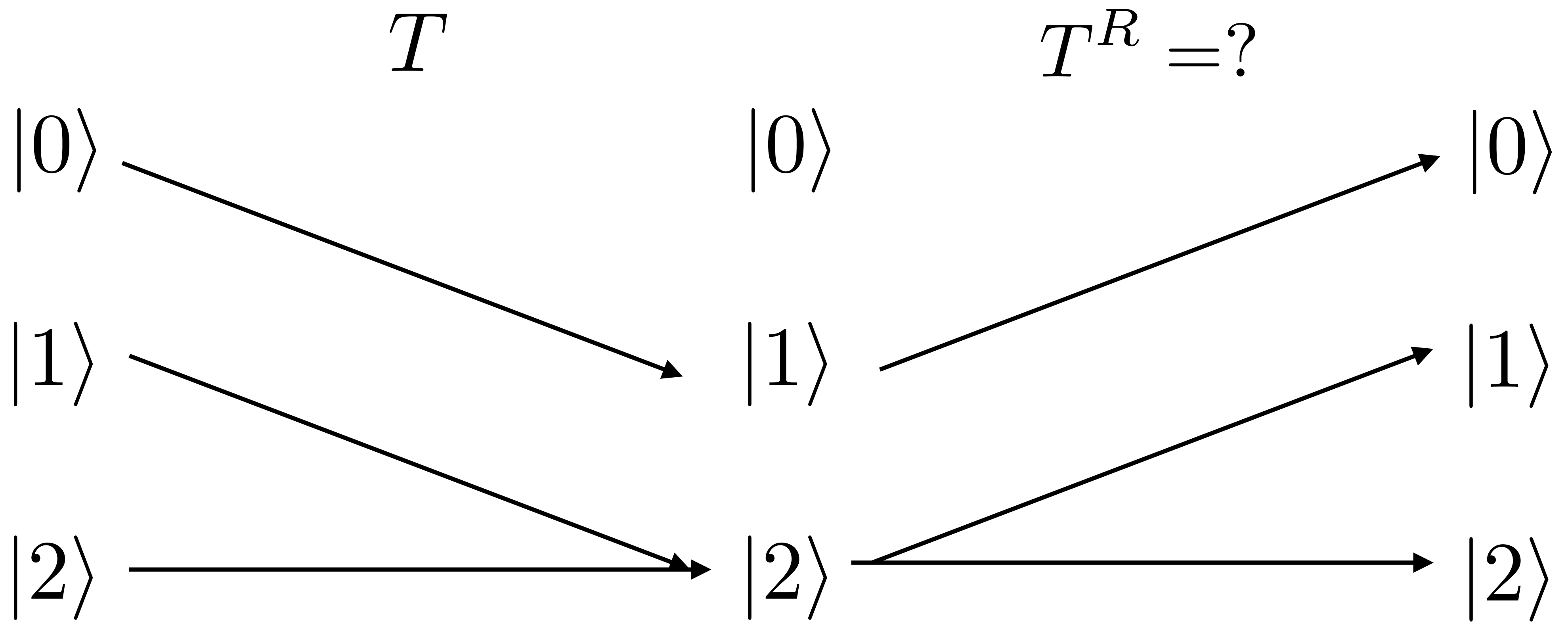
$$\Pr(S_t = j, S_0 = i) = \Pr(S_0 = i) T_{i\to j}$$

3) Probability of initial state given final state:

$$\Pr(S_0 = i | S_t = j) = \frac{\Pr(S_0 = i) T_{i\to j}}{\Pr(S_t = j)} \equiv T^R_{j\to i}$$

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



conditional probability:

$$T = \sum_{i,j} T_{i \to j} |j\rangle\langle i| \qquad T_{i \to j} \equiv \Pr(S_t = j | S_0 = i)$$

Deterministic permutation is a subcase:

$$T_{i \to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Given an initial probability $\Pr(S_0 = i)$

1) Final probability:

$$\Pr(S_t = j) = \sum_i \Pr(S_0 = i) T_{i \to j}$$

2) Joint probability over initial and final state:

$$\Pr(S_t = j, S_0 = i) = \Pr(S_0 = i) T_{i \to j}$$

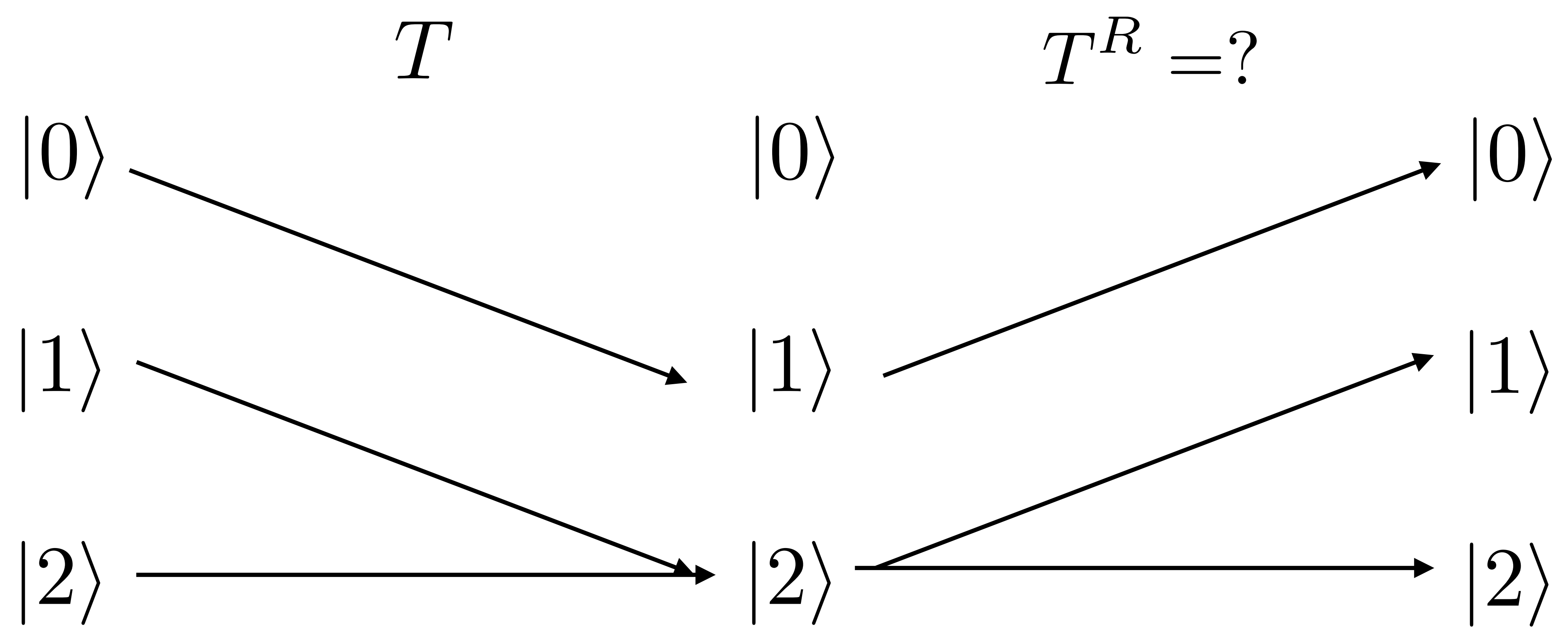
3) Probability of initial state given final state:

$$\Pr(S_0 = i | S_t = j) = \frac{\Pr(S_0 = i) T_{i \to j}}{\Pr(S_t = j)} \equiv T^R_{j \to i}$$

Bayes Rule

Probabilistic Time Reversal (Bayes Rule)

Consider probabilistic transformation



conditional probability:

$$T = \sum_{i,j} T_{i\to j} |j\rangle\langle i| \qquad T_{i\to j} \equiv \Pr(S_t = j | S_0 = i)$$

Deterministic permutation is a subcase:

$$T_{i\to j} = \delta_{j,p(j)}$$

S_t realizes state at time t

Given an initial probability $\Pr(S_0 = i)$

1) Final probability:

$$\Pr(S_t = j) = \sum_i \Pr(S_0 = i) T_{i\to j}$$

2) Joint probability over initial and final state:

$$\Pr(S_t = j, S_0 = i) = \Pr(S_0 = i) T_{i\to j}$$

3) Probability of initial state given final state:

$$\Pr(S_0 = i | S_t = j) = \frac{\Pr(S_0 = i) T_{i\to j}}{\Pr(S_t = j)} \equiv T^R_{j\to i}$$

Bayes Rule (comes from preservation of joint state over time)

$$\Pr(S_t = j) T^R_{j\to i} = \Pr(S_0 = i) T_{i\to j}$$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i \to j} |j\rangle\langle i|$$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i\to j}|j\rangle\langle i|$$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger$$

CPTP map implement with Kraus operators

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i\to j}|j\rangle\langle i|$$

$$|\rho_t\rangle = T|\rho_0\rangle$$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger$$

$$\rho_t = \Lambda(\rho_0)$$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i\to j}|j\rangle\langle i|$$

$$|\rho_t\rangle = T|\rho_0\rangle$$

constraints:

Normalization $\sum_j \langle j|\rho_t\rangle = 1$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger$$

$$\rho_t = \Lambda(\rho_0)$$

constraints:

Normalization $\sum_j \langle j|\rho_t|j\rangle = 1$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i\to j}|j\rangle\langle i|$$

$$|\rho_t\rangle = T|\rho_0\rangle$$

constraints:

Normalization $\sum_j \langle j|\rho_t\rangle = 1$

Normalization $\sum_j \langle j|T|i\rangle = 1 \text{ for all } i$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger$$

$$\rho_t = \Lambda(\rho_0)$$

constraints:

Normalization $\sum_j \langle j|\rho_t|j\rangle = 1$

Kraus condition $\langle i|\sum_m K_m^\dagger K_m|j\rangle = \delta_{i,j}$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i\to j}|j\rangle\langle i|$$

$$|\rho_t\rangle = T|\rho_0\rangle$$

constraints:

Normalization $\sum_j \langle j|\rho_t\rangle = 1$

Normalization $\sum_j \langle j|T|i\rangle = 1 \text{ for all } i$

state over time:

$$|\rho_{0,t}\rangle = \sum_{i,j} \Pr(S_0 = i) T_{i\to j}|i\rangle \otimes |j\rangle$$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger$$

$$\rho_t = \Lambda(\rho_0)$$

constraints:

Normalization $\sum_j \langle j|\rho_t|j\rangle = 1$

Kraus condition $\langle i|\sum_m K_m^\dagger K_m|j\rangle = \delta_{i,j}$

General Quantum Maps CPTP Operators

Classical

$$|\rho_0\rangle = \sum_i \Pr(S_0 = i)|i\rangle$$

$$T = \sum_{i,j} T_{i\to j}|j\rangle\langle i|$$

$$|\rho_t\rangle = T|\rho_0\rangle$$

constraints:

Normalization $\sum_j \langle j|\rho_t\rangle = 1$

Normalization $\sum_j \langle j|T|i\rangle = 1 \text{ for all } i$

state over time:

$$|\rho_{0,t}\rangle = \sum_{i,j} \Pr(S_0 = i) T_{i\to j}|i\rangle \otimes |j\rangle$$

Quantum

$$\rho_0 = \sum_i \Pr(S_0 = i)|s_i\rangle\langle s_i|$$

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger$$

$$\rho_t = \Lambda(\rho_0)$$

constraints:

Normalization $\sum_j \langle j|\rho_t|j\rangle = 1$

Kraus condition $\langle i|\sum_m K_m^\dagger K_m|j\rangle = \delta_{i,j}$

state over time:

?

Quantum State Over Time

One candidate: use the Jamiolkowski channel state “conditional quantum state”

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \ \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

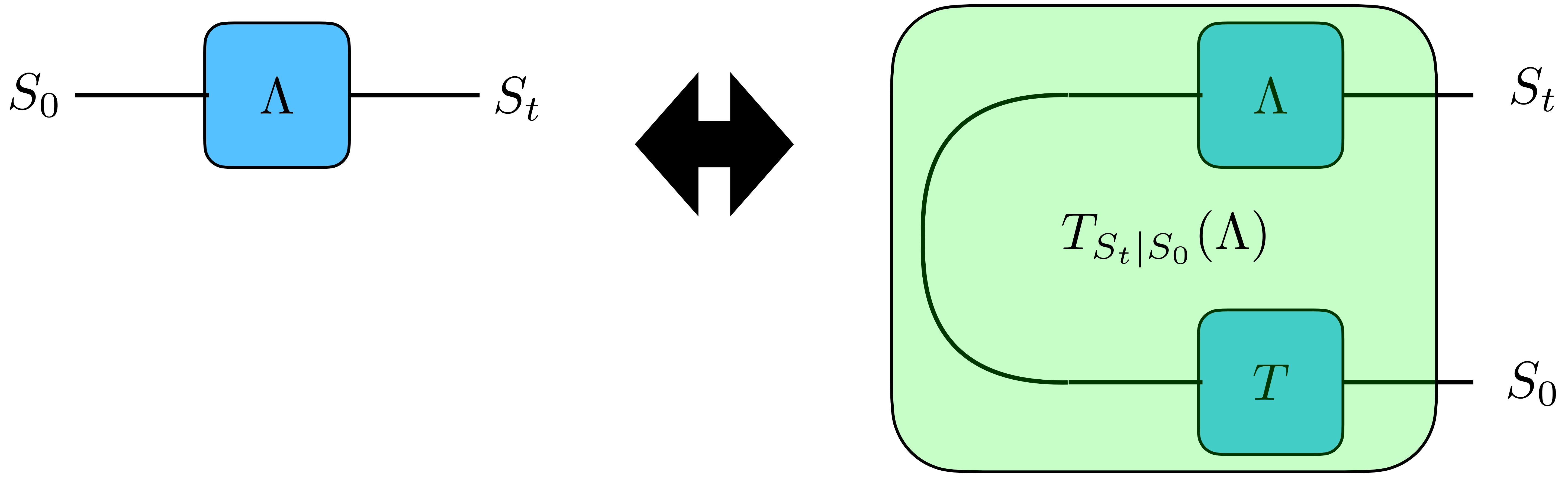
[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes’ rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state “conditional quantum state”

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive



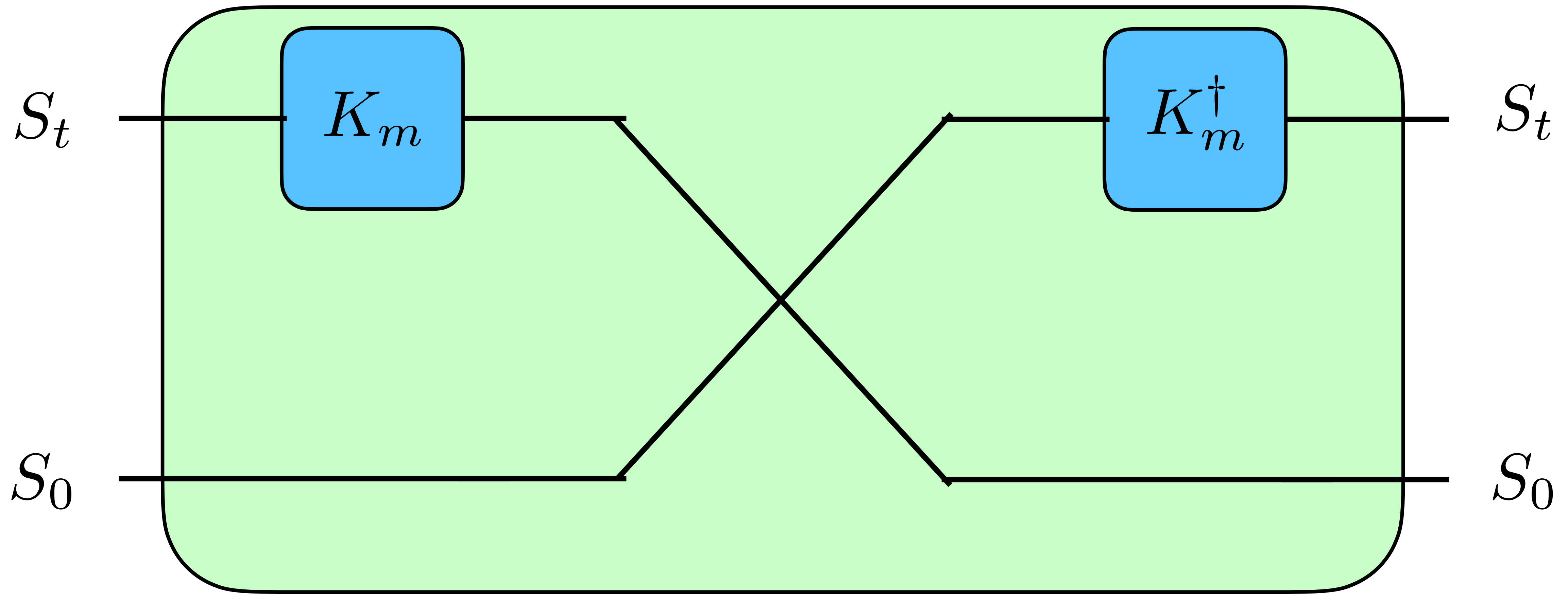
[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes’ rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive



[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \ \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

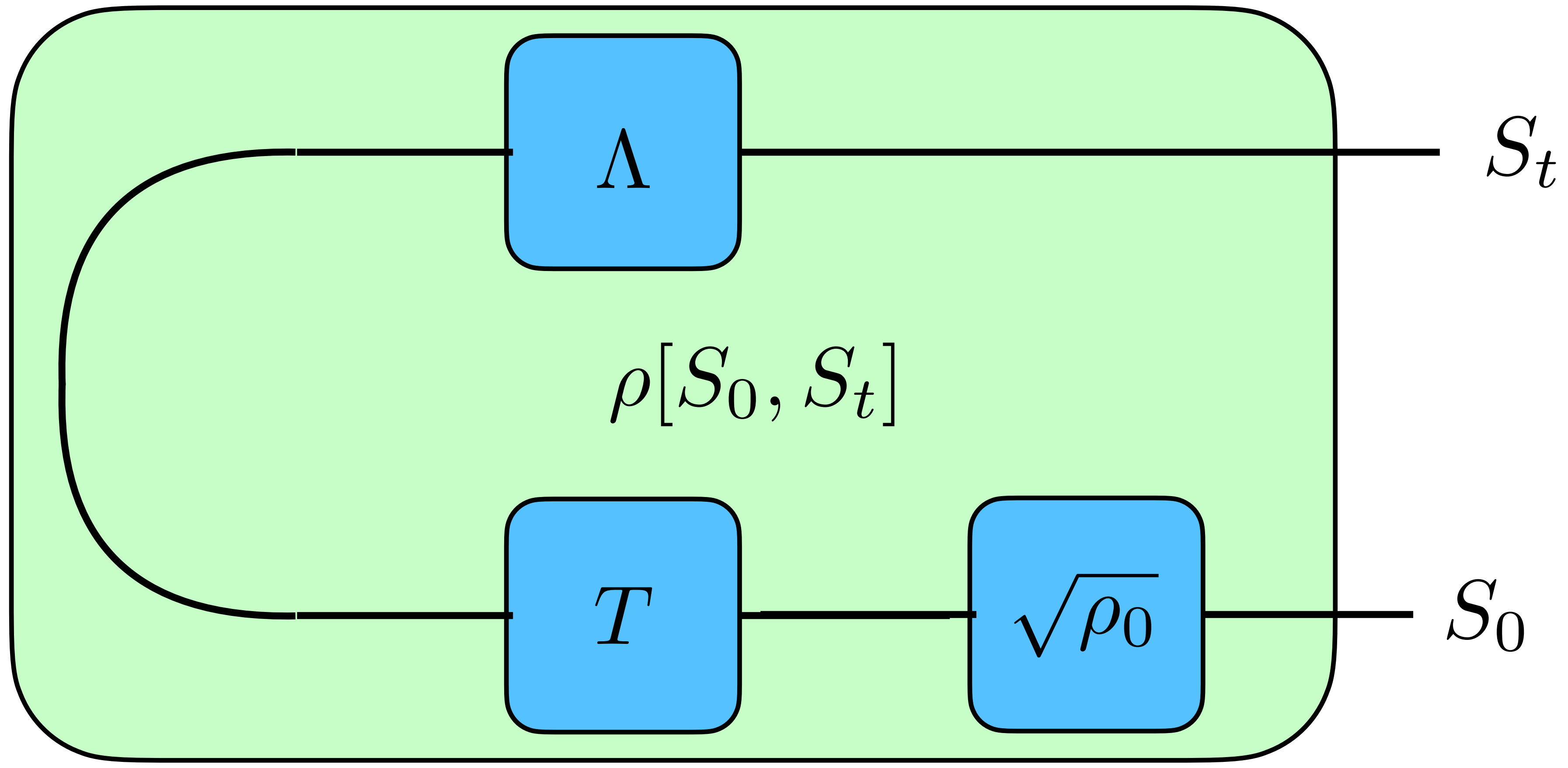
One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \ \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$



[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

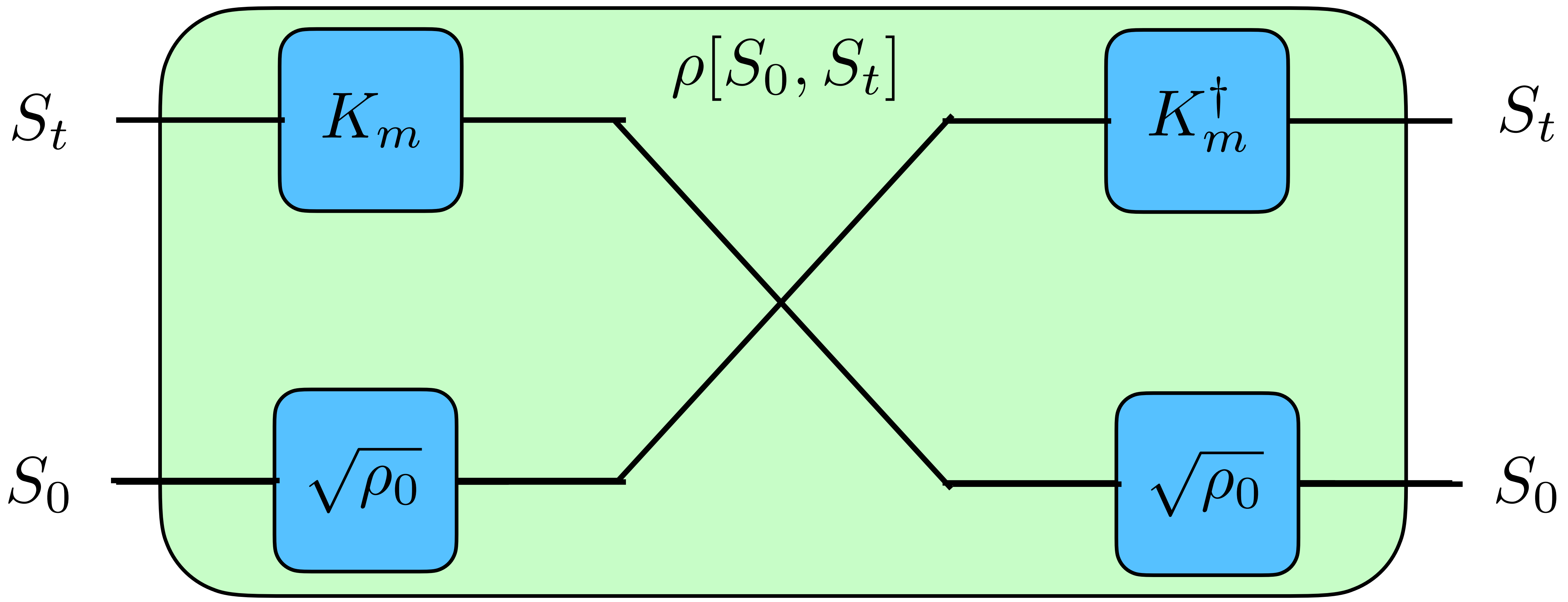
One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \;\in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$



[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\quad \mathrm{Tr}_{S_0}\left(\rho[S_0, S_t]\right) = \rho_t$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \ \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\quad \mathrm{Tr}_{S_0}\left(\rho[S_0, S_t]\right) = \rho_t \quad \mathrm{Tr}_{S_t}\left(\rho[S_0, S_t]\right) = \rho_0$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\quad \mathrm{Tr}_{S_0}\left(\rho[S_0, S_t]\right) = \rho_t \qquad \mathrm{Tr}_{S_t}\left(\rho[S_0, S_t]\right) = \rho_0$

A condition for quantum Bayes rule

state over time $\rho[S_0, S_\tau]$ is the same whether evaluated from past to future or from future to past

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\mathrm{Tr}_{S_0}(\rho[S_0, S_t]) = \rho_t$ $\mathrm{Tr}_{S_t}(\rho[S_0, S_t]) = \rho_0$

A condition for quantum Bayes rule

state over time $\rho[S_0, S_\tau]$ is the same whether evaluated from past to future or from future to past

$$T_{S_0|S_t}(\Lambda^R) \star (\mathbb{I}_{S_0} \otimes \rho_t) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t})$$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \;\in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\qquad \mathrm{Tr}_{S_0}\left(\rho[S_0, S_t]\right) = \rho_t \qquad \mathrm{Tr}_{S_t}\left(\rho[S_0, S_t]\right) = \rho_0$

A condition for quantum Bayes rule

state over time $\rho[S_0, S_\tau]$ is the same whether evaluated from past to future or from future to past

$$T_{S_0|S_t}(\Lambda^R) \star (\mathbb{I}_{S_0} \otimes \rho_t) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \qquad \text{classical: } T^R_{j \to i} \Pr(S_t = j) = T_{i \to j} \Pr(S_0 = i)$$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t}) \qquad \mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\quad \mathrm{Tr}_{S_0}\left(\rho[S_0, S_t]\right) = \rho_t \quad \mathrm{Tr}_{S_t}\left(\rho[S_0, S_t]\right) = \rho_0$

A condition for quantum Bayes rule

state over time $\rho[S_0, S_\tau]$ is the same whether evaluated from past to future or from future to past

$$T_{S_0|S_t}(\Lambda^R) \star (\mathbb{I}_{S_0} \otimes \rho_t) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \qquad \text{classical: } T^R_{j\to i}\,\Pr(S_t = j) = T_{i\to j}\,\Pr(S_0 = i)$$

Rewritten:

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Quantum State Over Time

One candidate: use the Jamiolkowski channel state "conditional quantum state"

$$T_{S_t|S_0}(\Lambda) \equiv \sum_{i,j} |i_{S_0}\rangle\langle j_{S_0}| \otimes \Lambda(|j_{S_t}\rangle\langle i_{S_t}|) \in \mathcal{B}(\mathcal{H}_{S_0} \otimes \mathcal{H}_{S_t})$$

$$\mathrm{Tr}\left[T_{S_t|S_0}(\Lambda)(\rho_0 \otimes \mathbb{I}_{S_t})\right] = \Lambda(\rho_0)$$

Like Choi state, but not generally positive

Leiffer-Spekkens state over time:

$$\rho[S_0, S_t] \equiv T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t}) \quad A \star B \equiv B^{1/2} A B^{1/2}$$

Returns desired marginal probabilities $\quad \mathrm{Tr}_{S_0}(\rho[S_0, S_t]) = \rho_t \quad \mathrm{Tr}_{S_t}(\rho[S_0, S_t]) = \rho_0$

A condition for quantum Bayes rule

state over time $\rho[S_0, S_\tau]$ is the same whether evaluated from past to future or from future to past

$$T_{S_0|S_t}(\Lambda^R) \star (\mathbb{I}_{S_0} \otimes \rho_t) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \mathbb{I}_{S_t})$$

classical: $T^R_{j\to i} \Pr(S_t = j) = T_{i\to j} \Pr(S_0 = i)$

Rewritten:

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$

classical: $T^R_{j\to i} = T_{i\to j} \Pr(S_0 = i) \Pr(S_t = j)^{-1}$

[Parzygnat, Arthur J., and James Fullwood. "From time-reversal symmetry to quantum Bayes' rules." *PRX Quantum* 4.2 (2023): 020334.]
[Bera, Mohit Lal, and Manabendra Nath Bera. "Quantum Bayes' rule affirming consistency in measurement inferences in quantum mechanics." *Physical Review A* 108.1 (2023): 012224.]

Petz Recovery Map

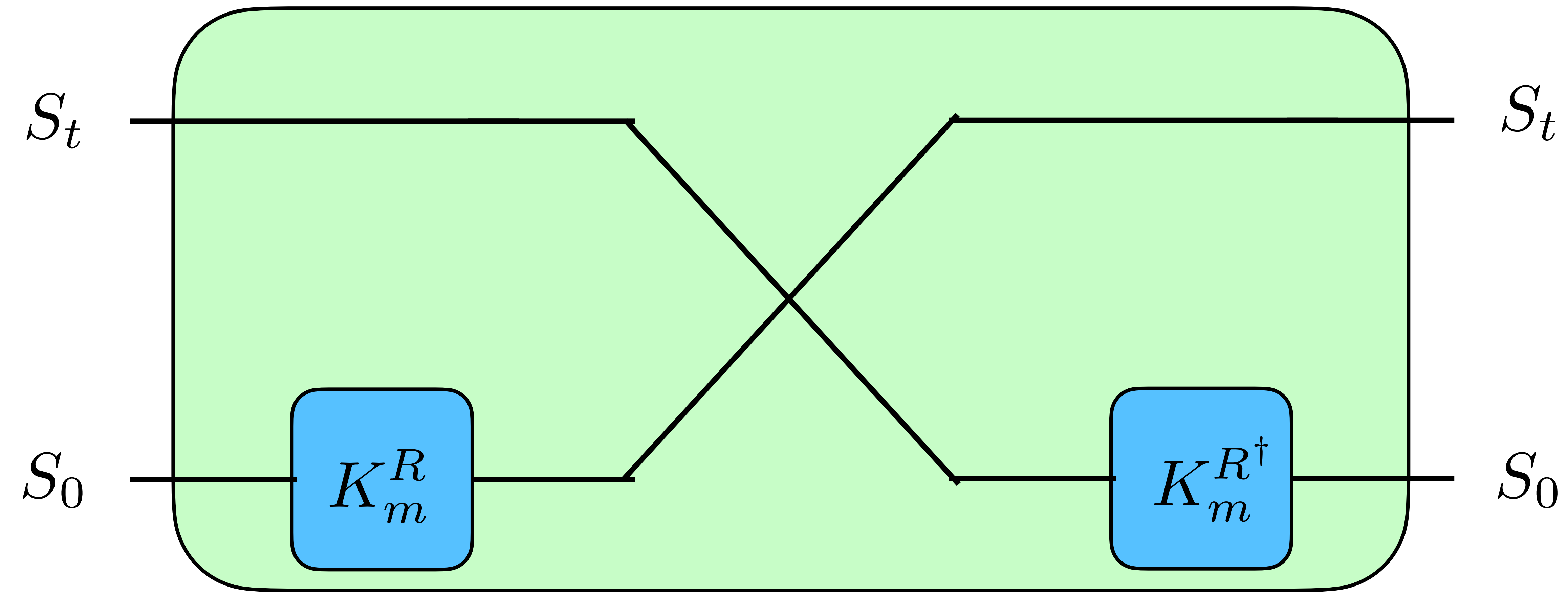
From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

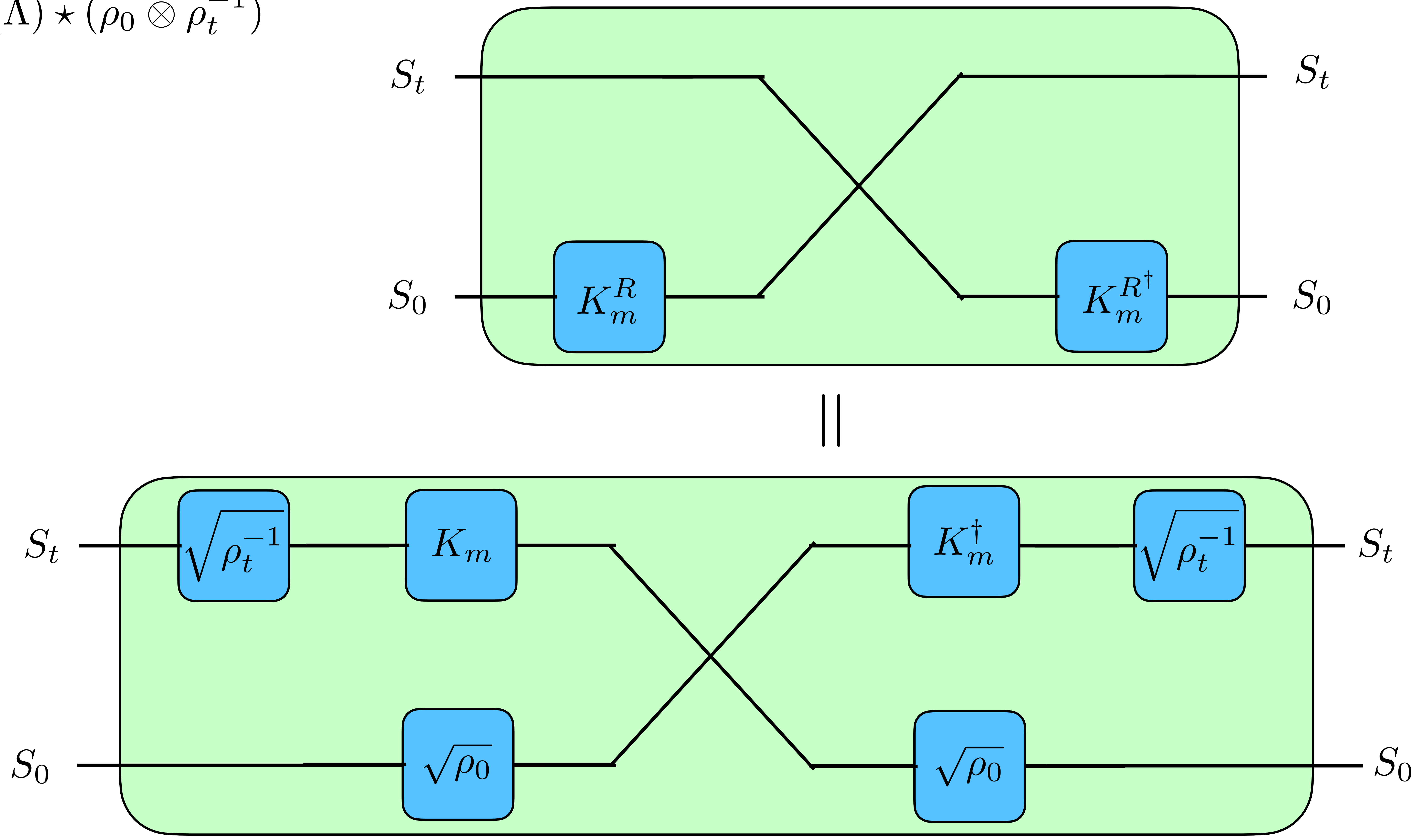
$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$



Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$

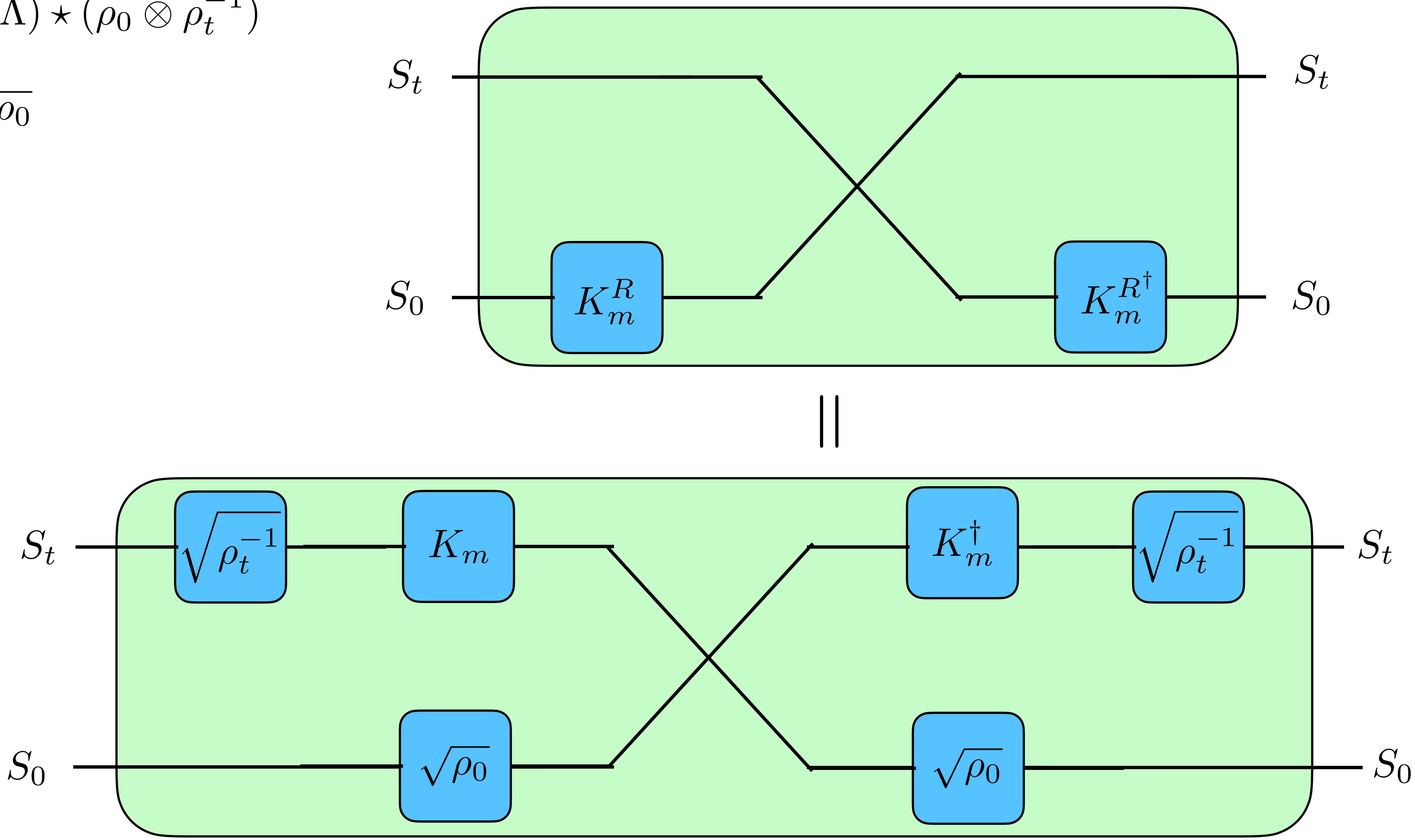


Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$

$$K_m^R = \sqrt{\rho_t^{-1}} K_m^\dagger \sqrt{\rho_0}$$



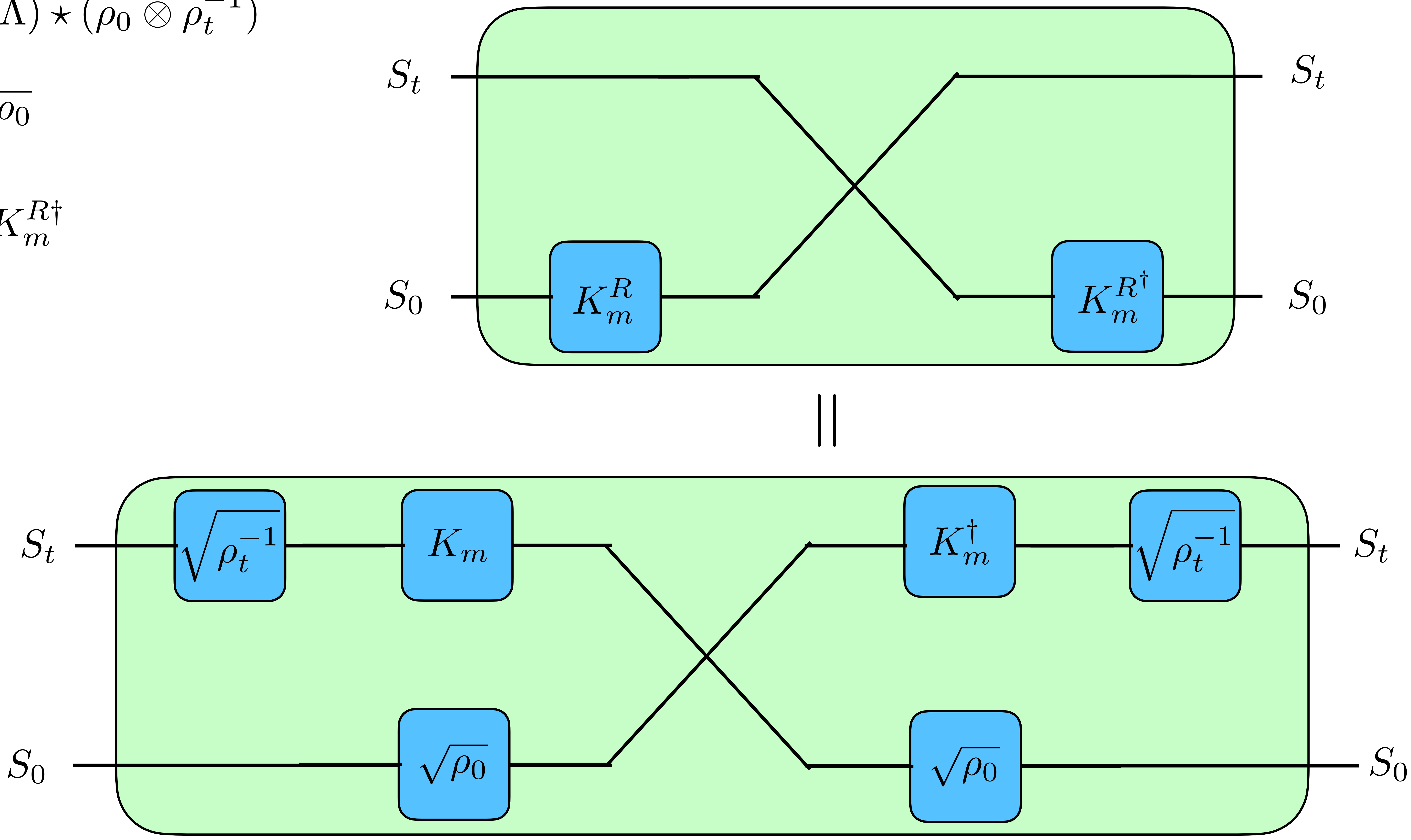
Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1})$$

$$K_m^R = \sqrt{\rho_t^{-1}} K_m^\dagger \sqrt{\rho_0}$$

$$\Lambda^R(\cdot) = \sum_m K_m^R \cdot K_m^{R\dagger}$$



Petz Recovery Map

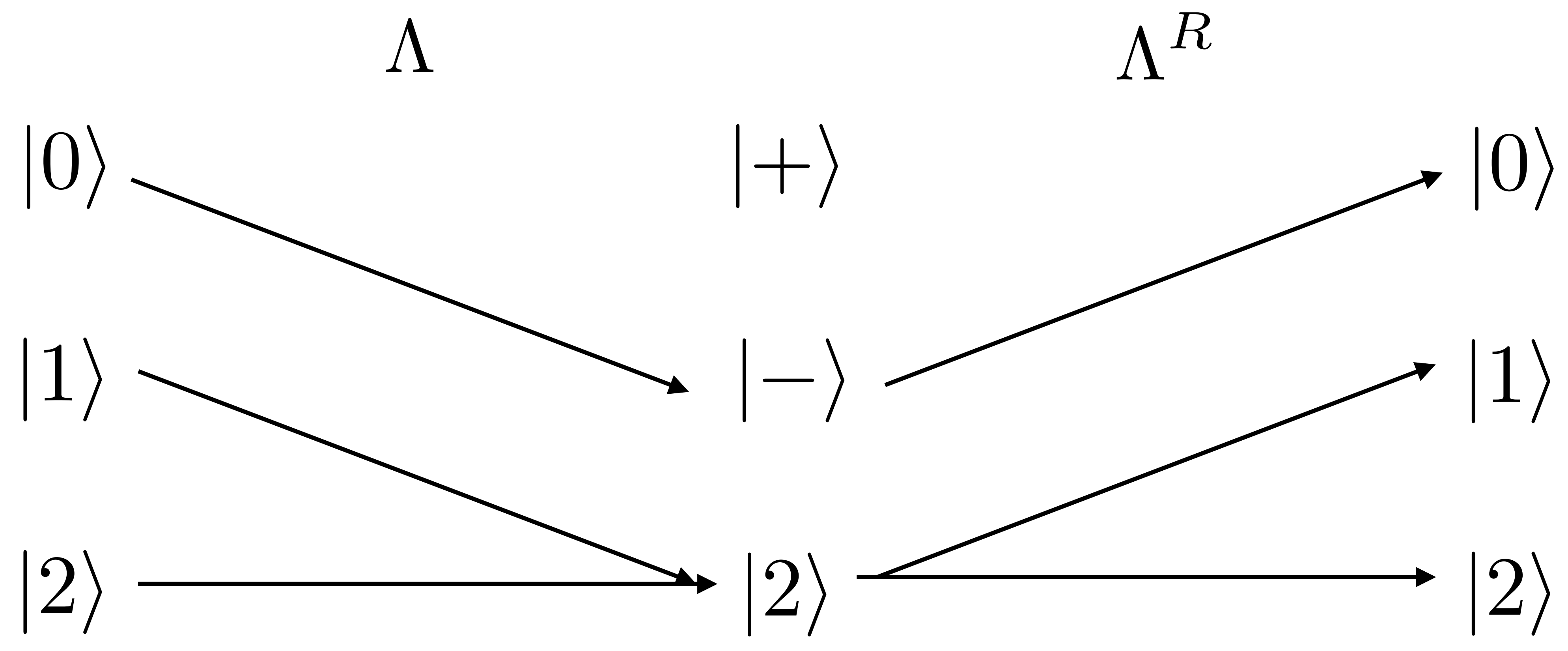
From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}}$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

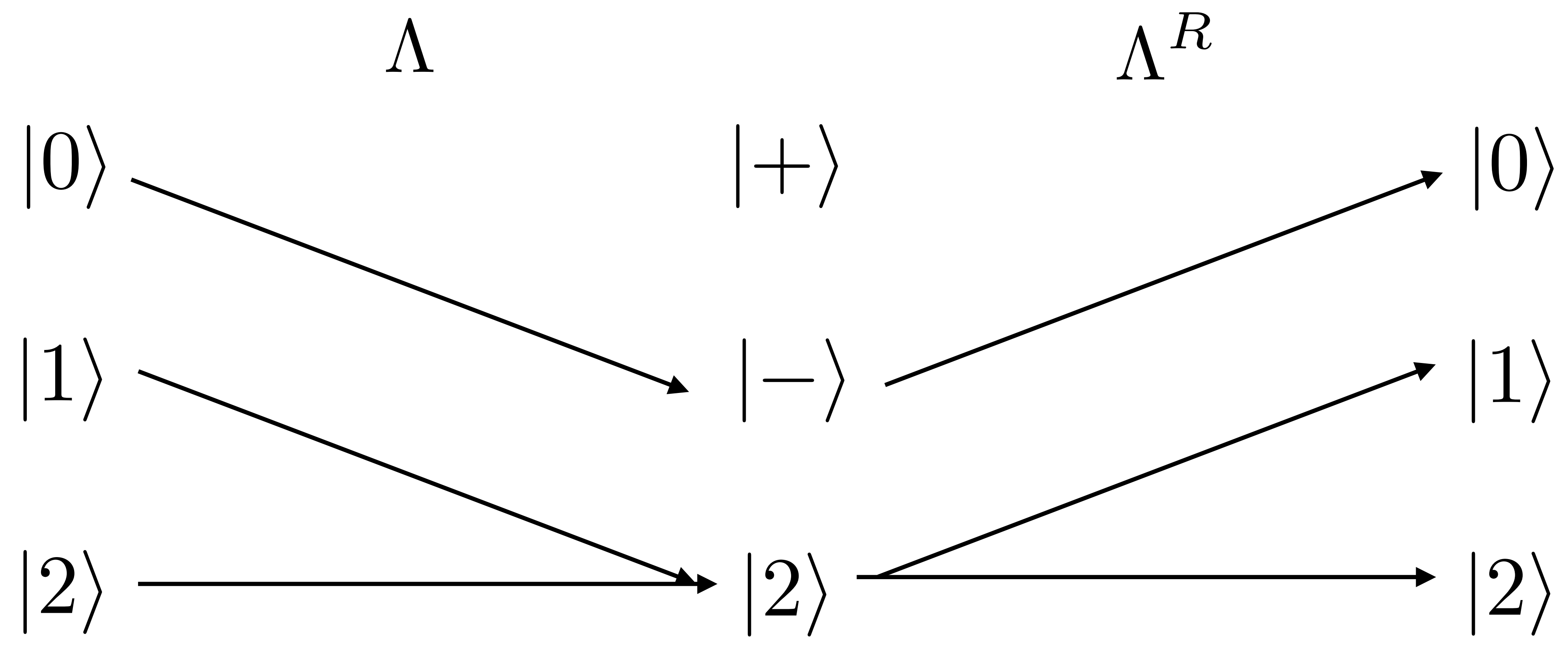
$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}}$$



Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}} \qquad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m \right)$$

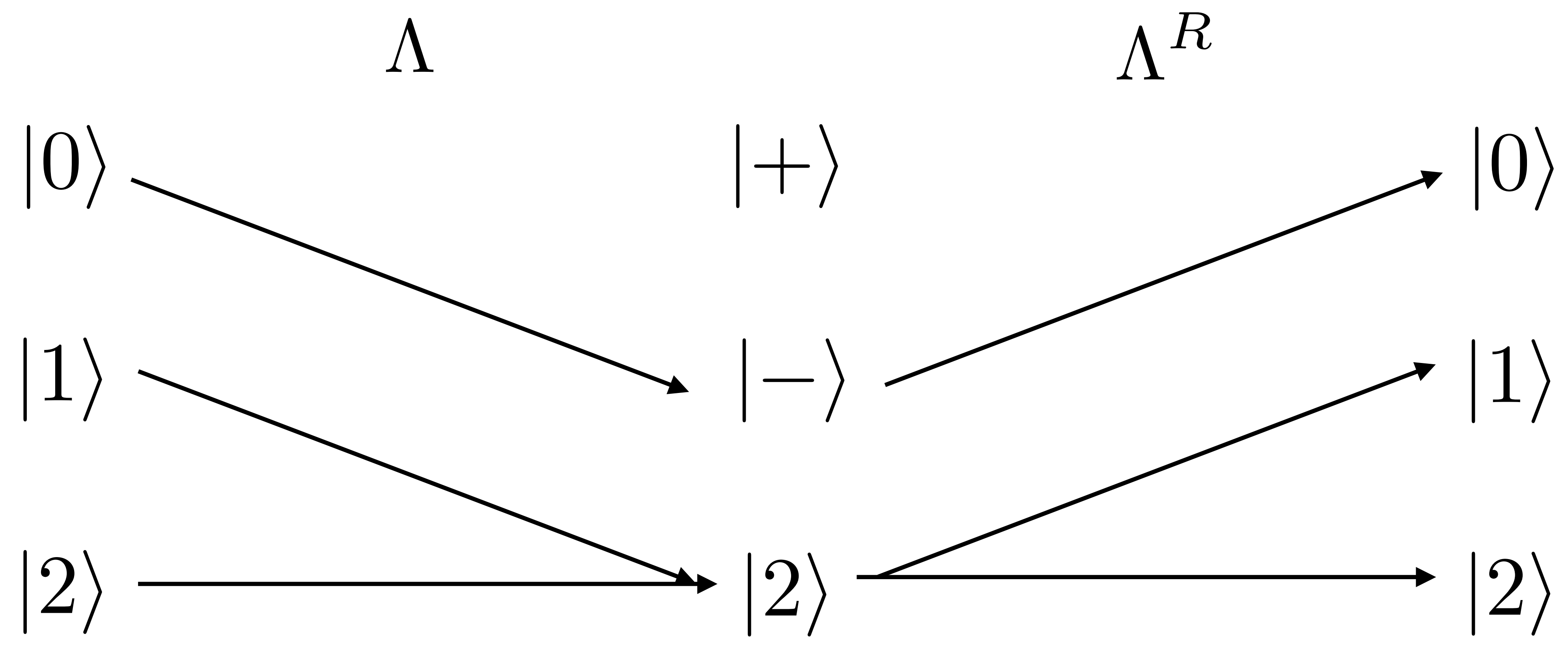


Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}} \qquad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m \right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$

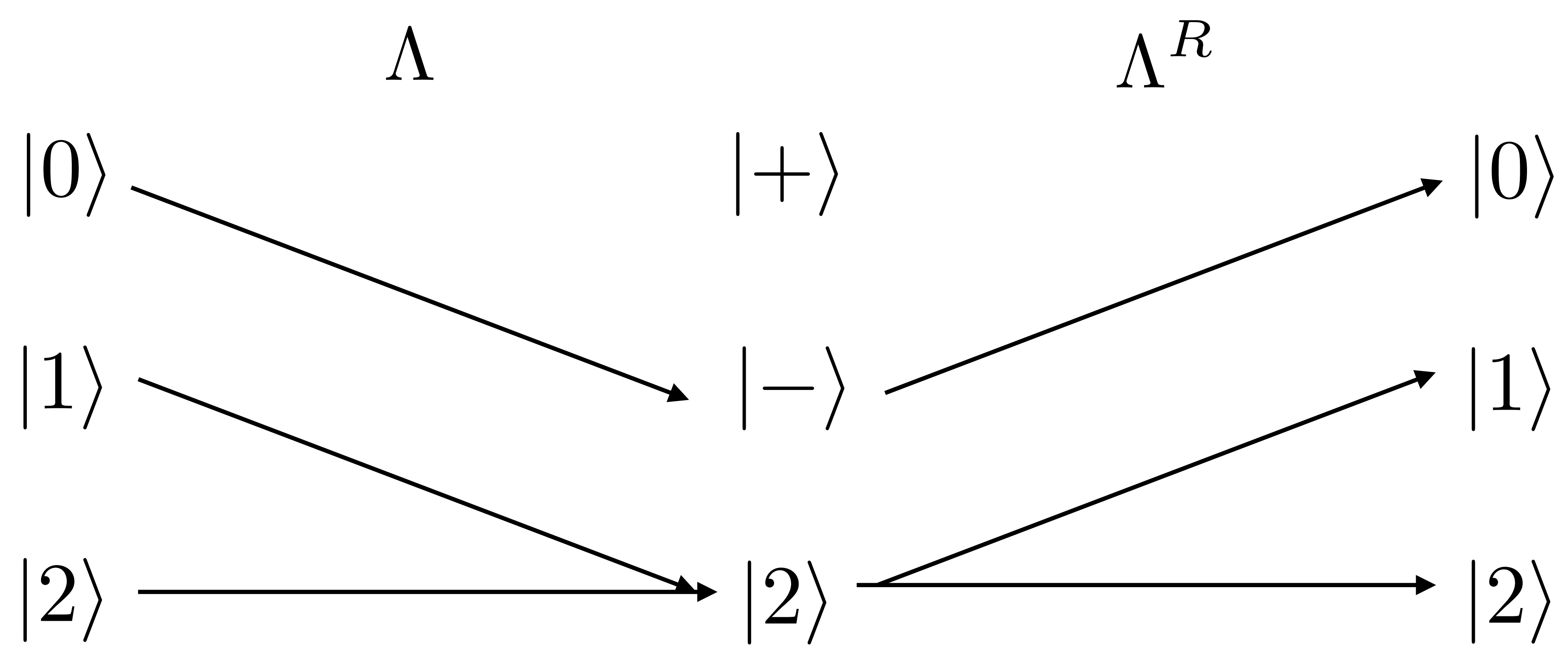


Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \qquad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

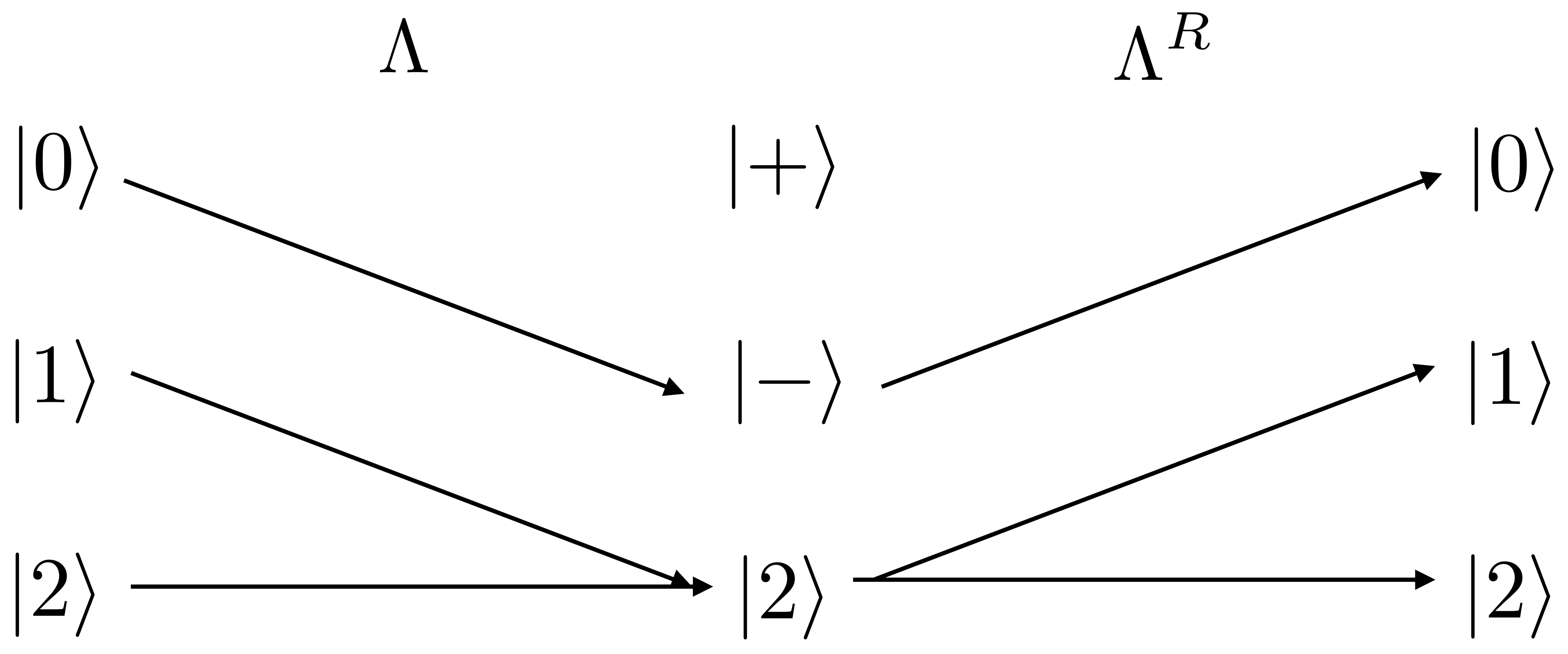
$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \qquad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

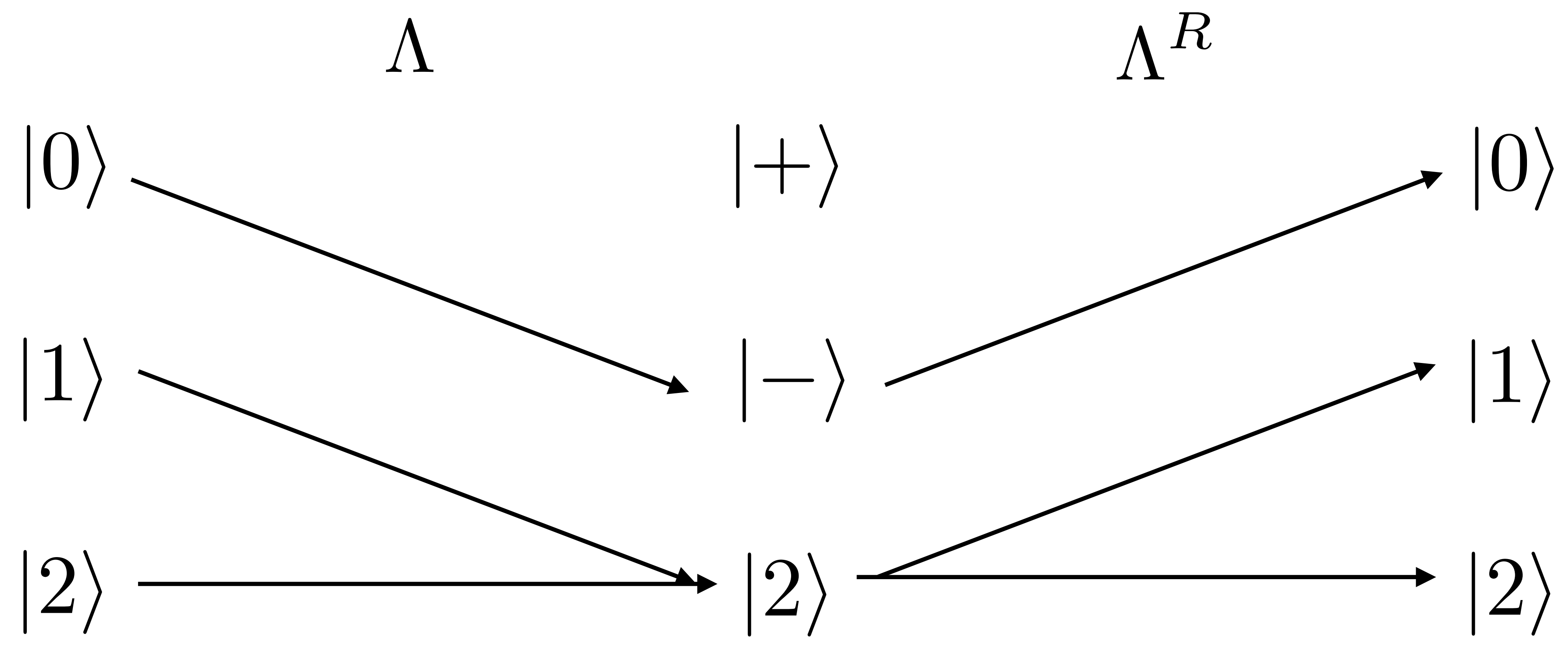
Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}}$$

$$\left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m \right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \rho_t \rho_t^{-1/2}) \rho_0^{1/2}$$

$$= \rho_0^{1/2} \sum_m K_m^\dagger K_m \rho_0^{1/2}$$

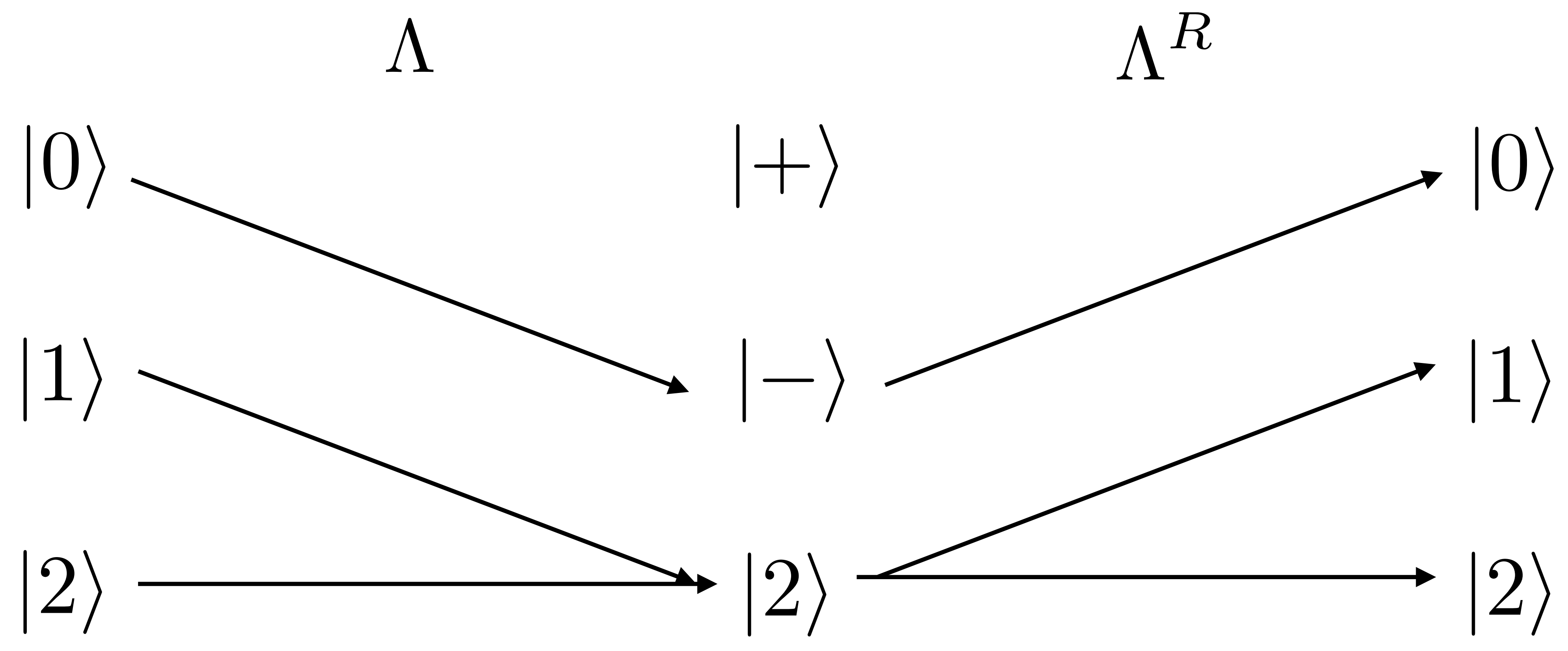
$$= \rho_0^{1/2} \mathbb{I} \rho_0^{1/2} = \rho_0$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert

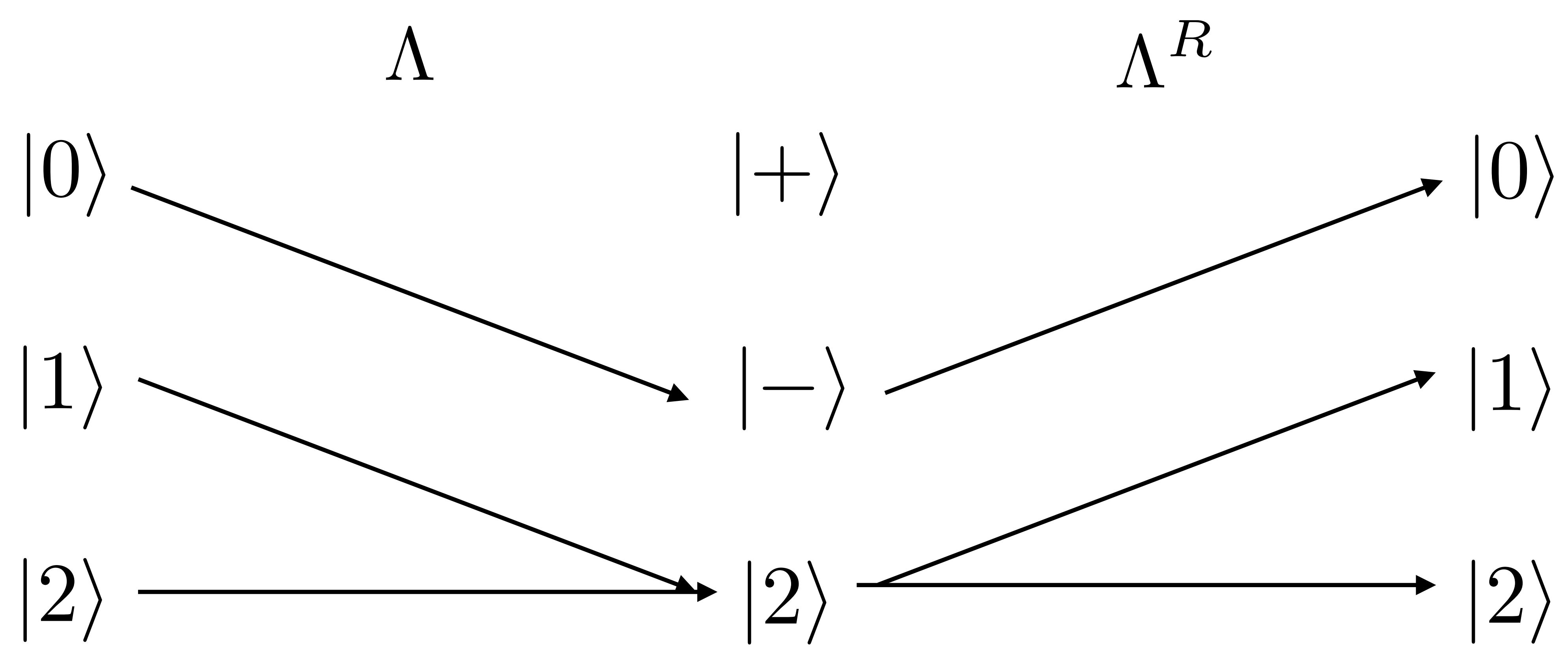
let $\Lambda(\cdot) = U \cdot U^\dagger$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m \right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \rho_t \rho_t^{-1/2}) \rho_0^{1/2}$$

$$= \rho_0^{1/2} \sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2} \mathbb{I} \rho_0^{1/2} = \rho_0$$

2) Unitaries invert

let $\Lambda(\cdot) = U \cdot U^\dagger$

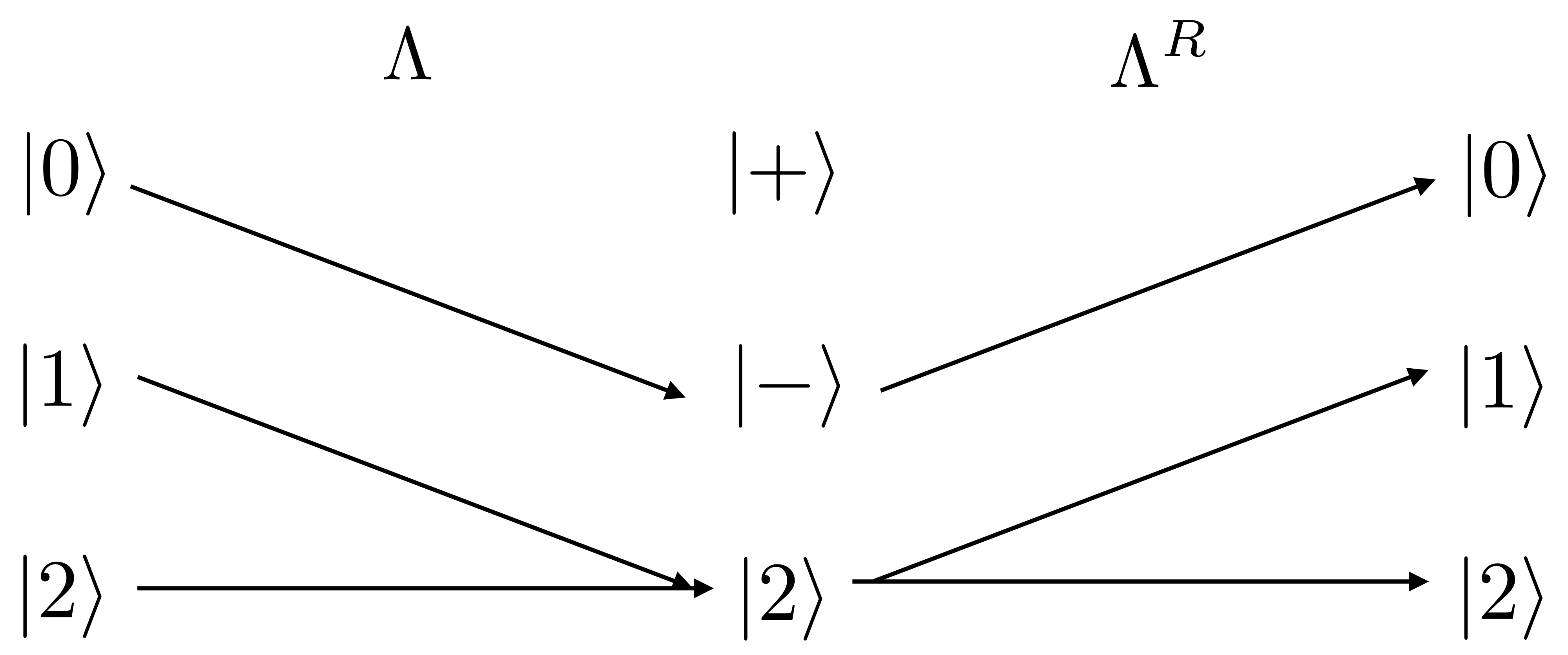
$$\Lambda^R(\cdot) = \rho_0 U^\dagger (U \rho_0^{-1/2} U^\dagger) \cdot (U \rho_0^{-1/2} U^\dagger) U \rho_0$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger(U\rho_0^{-1/2}U^\dagger) \cdot (U\rho_0^{-1/2}U^\dagger)U\rho_0$$

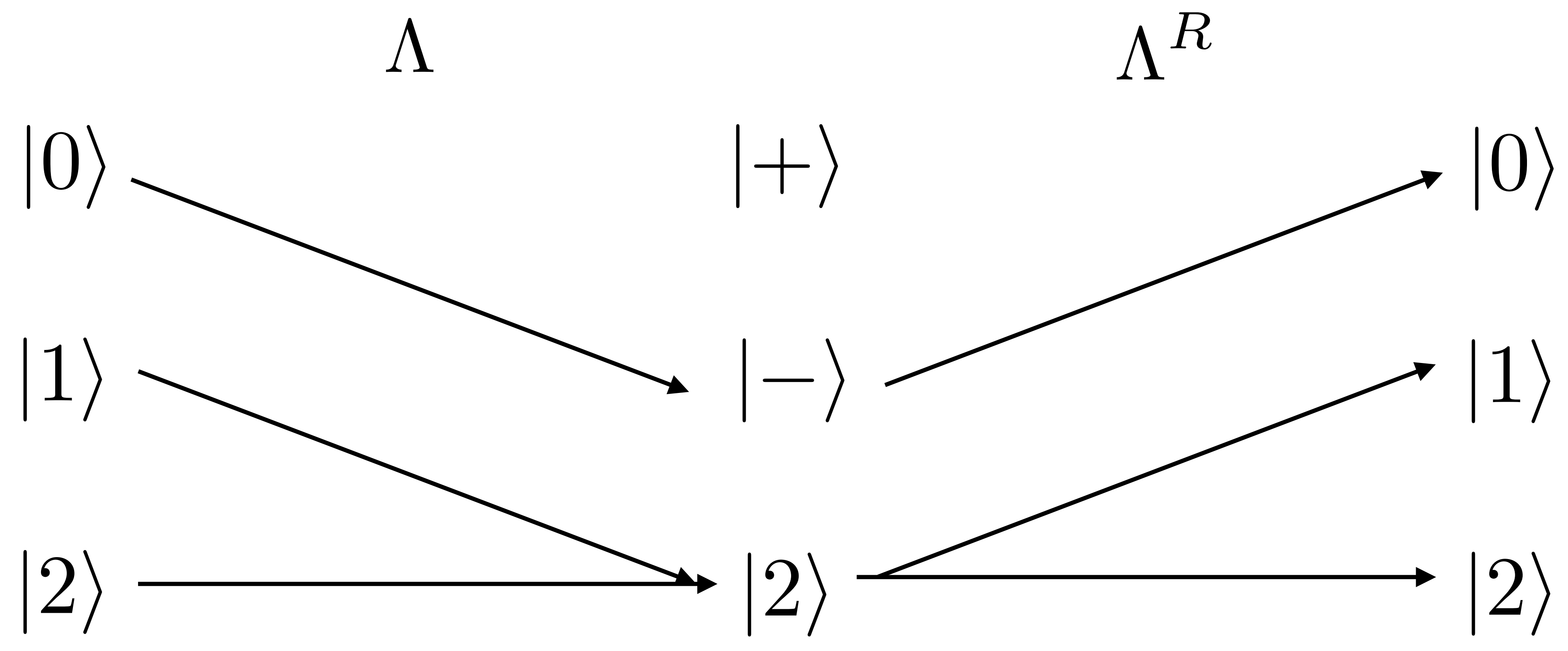
$$= \rho_0\rho_0^{-1/2}U^\dagger \cdot U\rho_0^{-1/2}\rho_0$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger(U\rho_0^{-1/2}U^\dagger) \cdot (U\rho_0^{-1/2}U^\dagger)U\rho_0$$

$$= \rho_0\rho_0^{-1/2}U^\dagger \cdot U\rho_0^{-1/2}\rho_0$$

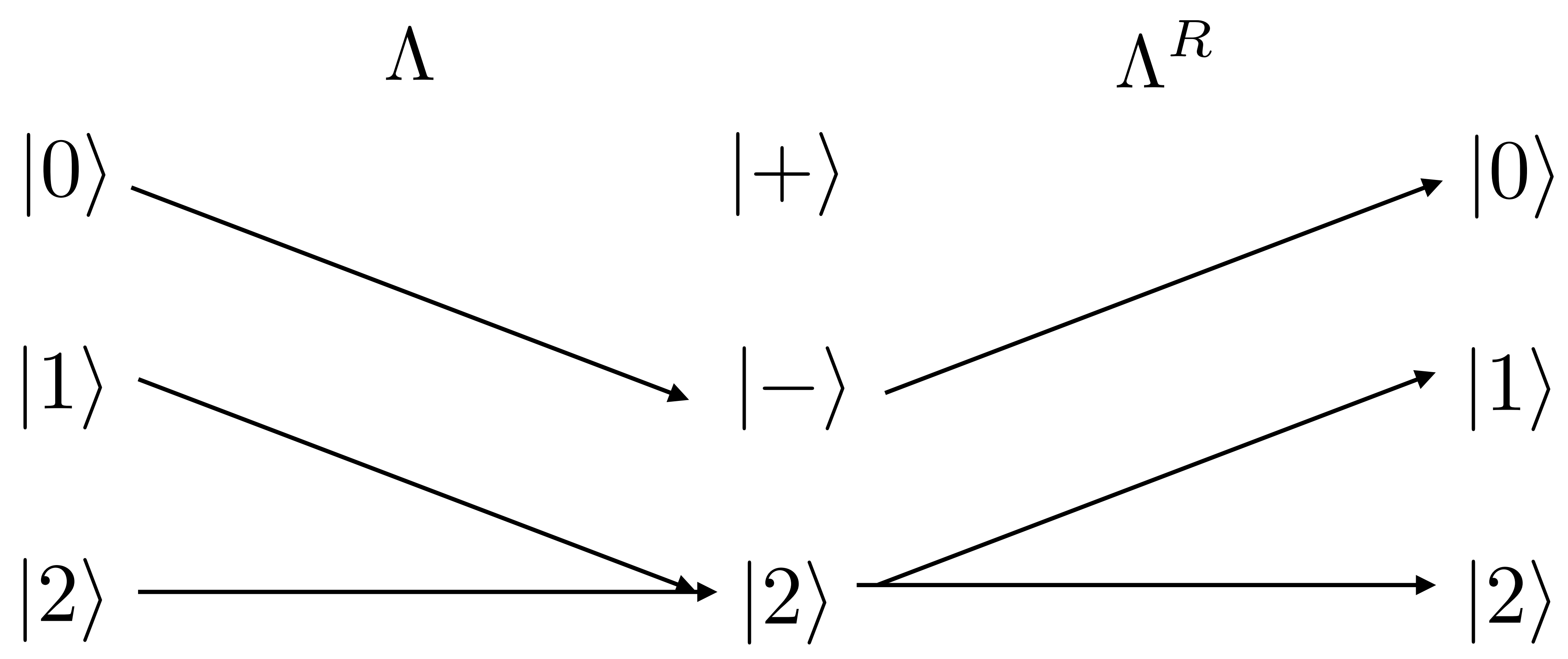
$$= U^\dagger \cdot U \quad = U^R \cdot U^{R\dagger}$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \qquad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert

let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger(U\rho_0^{-1/2}U^\dagger) \cdot (U\rho_0^{-1/2}U^\dagger)U\rho_0$$

$$= \rho_0\rho_0^{-1/2}U^\dagger \cdot U\rho_0^{-1/2}\rho_0$$

$$= U^\dagger \cdot U \quad = U^R \cdot U^{R\dagger}$$

3) Classical Bayes rule

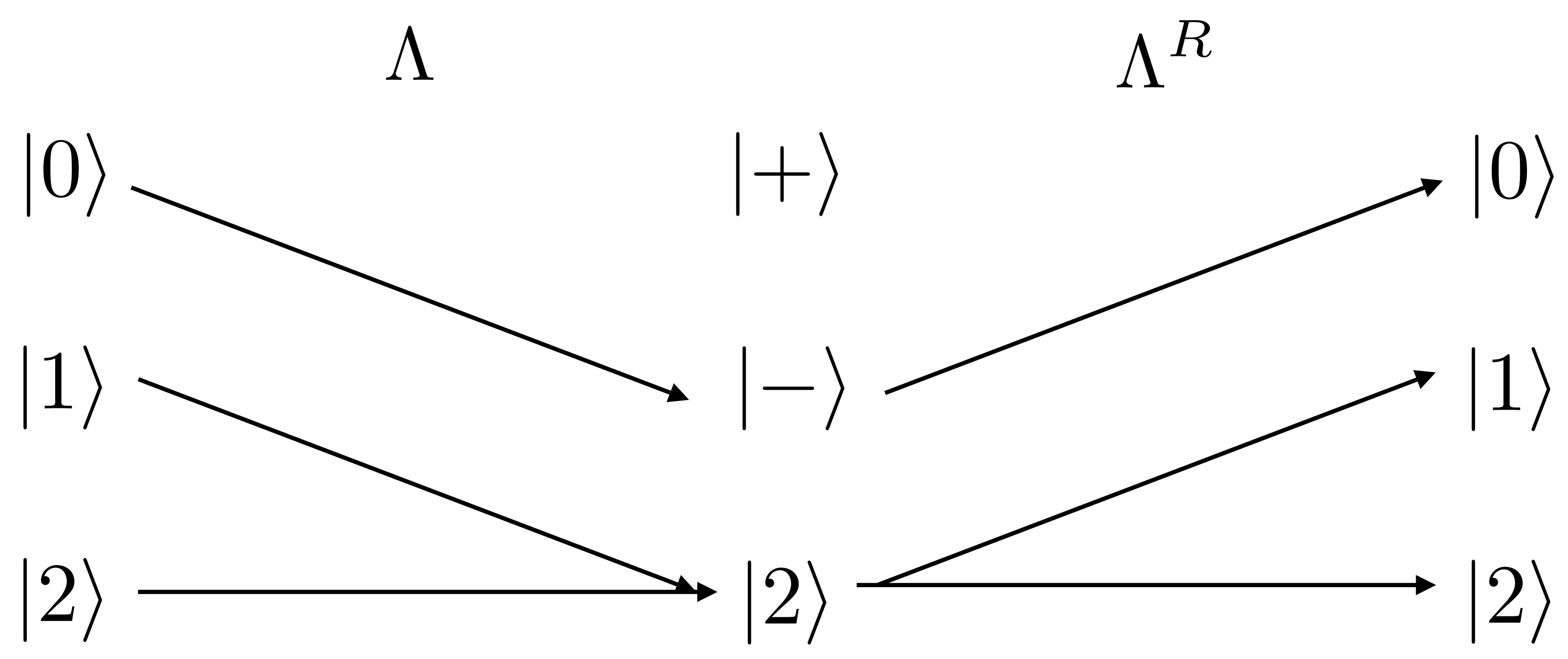
$$\Lambda(\cdot) = \sum_{i,j} T_{i\to j}|j\rangle\langle i| \cdot |i\rangle\langle j|$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert

let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger(U\rho_0^{-1/2}U^\dagger)\cdot(U\rho_0^{-1/2}U^\dagger)U\rho_0$$

$$= \rho_0\rho_0^{-1/2}U^\dagger \cdot U\rho_0^{-1/2}\rho_0$$

$$= U^\dagger \cdot U \quad = U^R \cdot U^{R\dagger}$$

3) Classical Bayes rule

$$\Lambda(\cdot) = \sum_{i,j} T_{i\to j}|j\rangle\langle i| \cdot |i\rangle\langle j|$$

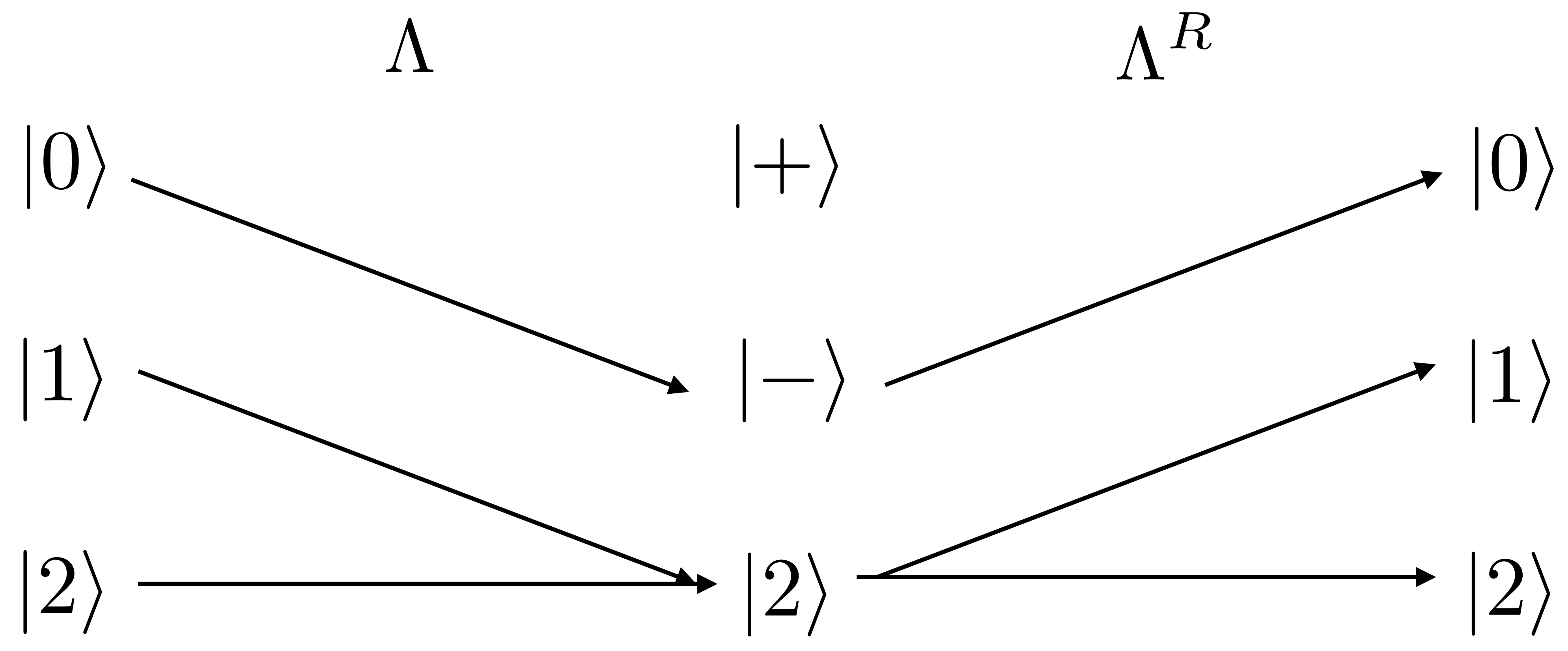
initial and final states in classical basis: $\rho_t^n|i\rangle = |i\rangle\,\Pr(S_t = i)^n$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert

let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger(U\rho_0^{-1/2}U^\dagger)\cdot(U\rho_0^{-1/2}U^\dagger)U\rho_0$$

$$= \rho_0\rho_0^{-1/2}U^\dagger \cdot U\rho_0^{-1/2}\rho_0$$

$$= U^\dagger \cdot U \quad = U^R \cdot U^{R\dagger}$$

3) Classical Bayes rule

$$\Lambda(\cdot) = \sum_{i,j} T_{i\to j}|j\rangle\langle i| \cdot |i\rangle\langle j|$$

initial and final states in classical basis: $\rho_t^n|i\rangle = |i\rangle \Pr(S_t = i)^n$

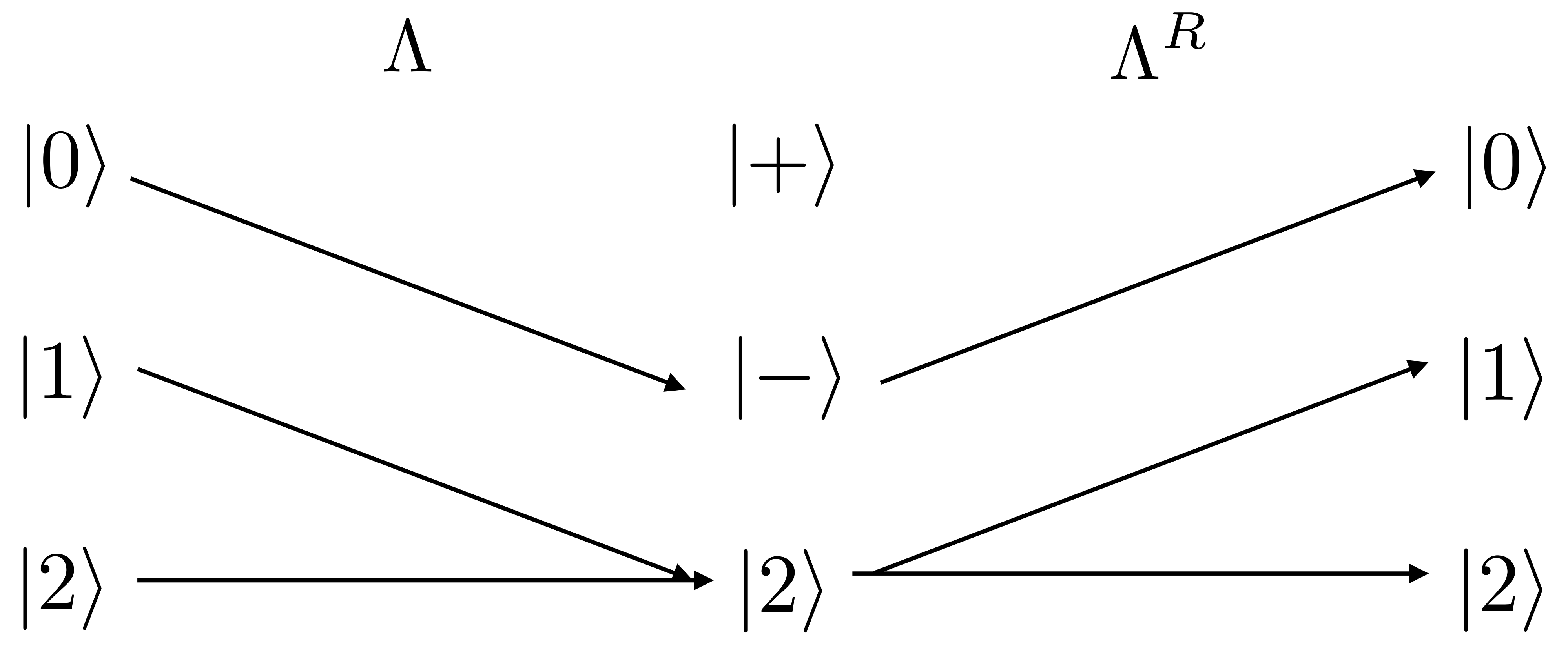
$$\Lambda^R(\cdot) = \sum_{i,j} T_{i\to j}\rho_0^{1/2}|i\rangle\langle j|\rho_t^{-1/2} \cdot \rho_t^{-1/2}|j\rangle\langle i|\rho_0^{1/2}$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2})\rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m\right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2}\Lambda^\dagger(\rho_t^{-1/2}\rho_t\rho_t^{-1/2})\rho_0^{1/2}$$

$$= \rho_0^{1/2}\sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2}\mathbb{I}\rho_0^{1/2} = \rho_0$$

2) Unitaries invert

let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger (U\rho_0^{-1/2}U^\dagger)\cdot(U\rho_0^{-1/2}U^\dagger)U\rho_0$$

$$= \rho_0\rho_0^{-1/2}U^\dagger \cdot U\rho_0^{-1/2}\rho_0$$

$$= U^\dagger \cdot U \quad = U^R \cdot U^{R\dagger}$$

3) Classical Bayes rule

$$\Lambda(\cdot) = \sum_{i,j} T_{i\to j}|j\rangle\langle i| \cdot |i\rangle\langle j|$$

initial and final states in classical basis: $\rho_t^n|i\rangle = |i\rangle \Pr(S_t = i)^n$

$$\Lambda^R(\cdot) = \sum_{i,j} T_{i\to j}\rho_0^{1/2}|i\rangle\langle j|\rho_t^{-1/2} \cdot \rho_t^{-1/2}|j\rangle\langle i|\rho_0^{1/2}$$

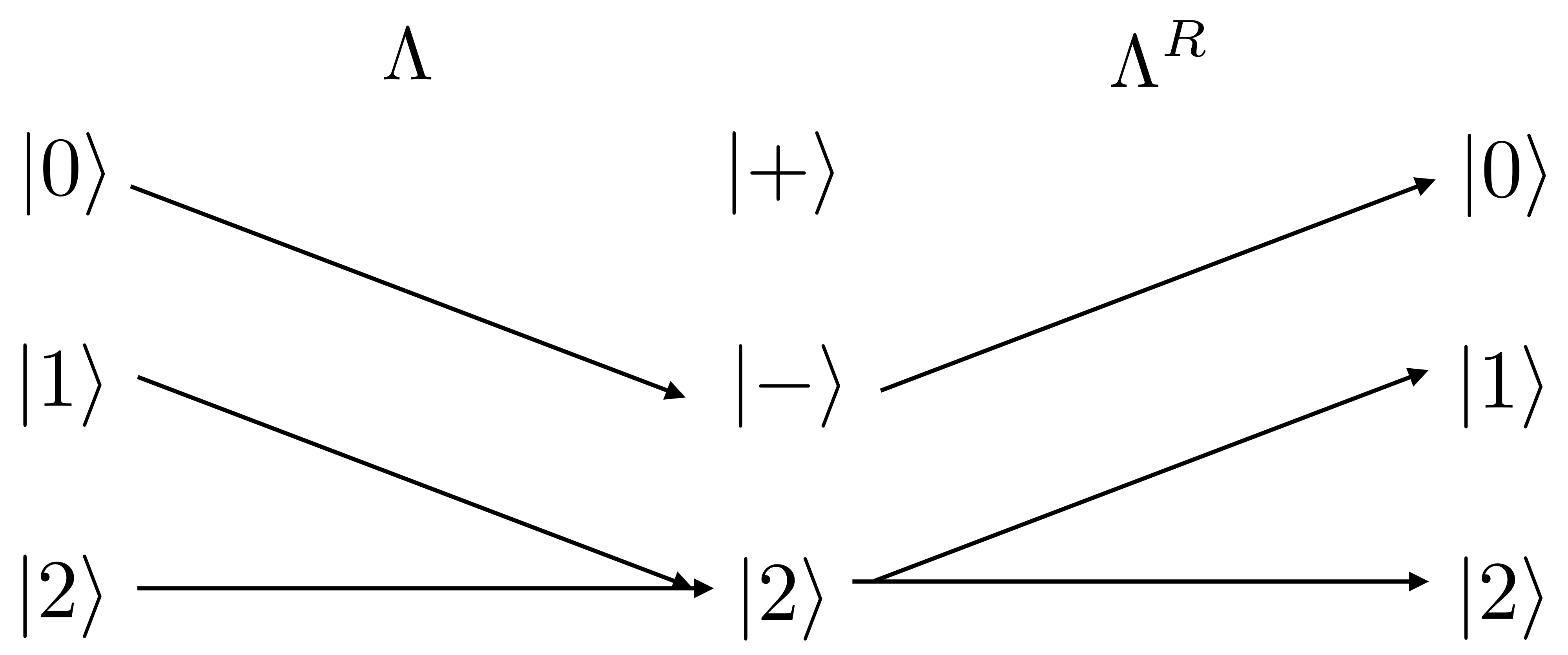
$$= \sum_{i,j} T_{i\to j}\Pr(S_0 = i)\Pr(S_t = j)^{-1}|i\rangle\langle j| \cdot |j\rangle\langle i|$$

Petz Recovery Map

From conditional quantum state we can recover the reverse operation

$$T_{S_0|S_t}(\Lambda^R) = T_{S_t|S_0}(\Lambda) \star (\rho_0 \otimes \rho_t^{-1}) \quad \leftrightarrow \quad \boxed{\Lambda^R(\cdot) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \cdot \rho_t^{-1/2}) \rho_0^{1/2}} \quad \left(\Lambda^\dagger(\cdot) \equiv \sum_m K_m^\dagger \cdot K_m \right)$$

Petz recovery requires definition of initial state: $\Lambda^R = \Lambda^P_{\rho_0}$



Properties:

1) Maps final to initial

$$\Lambda^R(\rho_t) \equiv \rho_0^{1/2} \Lambda^\dagger(\rho_t^{-1/2} \rho_t \rho_t^{-1/2}) \rho_0^{1/2}$$

$$= \rho_0^{1/2} \sum_m K_m^\dagger K_m \rho_0^{1/2}$$

$$= \rho_0^{1/2} \mathbb{I} \rho_0^{1/2} = \rho_0$$

2) Unitaries invert

let $\Lambda(\cdot) = U \cdot U^\dagger$

$$\Lambda^R(\cdot) = \rho_0 U^\dagger (U \rho_0^{-1/2} U^\dagger) \cdot (U \rho_0^{-1/2} U^\dagger) U \rho_0$$

$$= \rho_0 \rho_0^{-1/2} U^\dagger \cdot U \rho_0^{-1/2} \rho_0$$

$$= U^\dagger \cdot U \quad = U^R \cdot U^{R\dagger}$$

3) Classical Bayes rule

$$\Lambda(\cdot) = \sum_{i,j} T_{i \to j} |j\rangle\langle i| \cdot |i\rangle\langle j|$$

initial and final states in classical basis: $\rho_t^n |i\rangle = |i\rangle \Pr(S_t = i)^n$

$$\Lambda^R(\cdot) = \sum_{i,j} T_{i\to j} \rho_0^{1/2} |i\rangle\langle j| \rho_t^{-1/2} \cdot \rho_t^{-1/2} |j\rangle\langle i| \rho_0^{1/2}$$

$$= \sum_{i,j} T_{i \to j} \Pr(S_0 = i) \Pr(S_t = j)^{-1} |i\rangle\langle j| \cdot |j\rangle\langle i|$$

$$= \sum_{i,j} T^R_{j \to i} |i\rangle\langle j| \cdot |j\rangle\langle i|$$

Petz Recovery Theorem

Distinguishability of two states can be measured by relative entropy

$$D(\rho||\sigma) \equiv \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \geq 0$$

[Petz, Dénes. "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra." *Communications in mathematical physics* 105 (1986): 123-131.]

Petz Recovery Theorem

Distinguishability of two states can be measured by relative entropy

$$D(\rho||\sigma) \equiv \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \geq 0$$

Monotonicity $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma))$

[Petz, Dénes. "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra." *Communications in mathematical physics* 105 (1986): 123-131.]

Petz Recovery Theorem

Distinguishability of two states can be measured by relative entropy

$$D(\rho||\sigma) \equiv \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \geq 0$$

Monotonicity $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma))$

Petz reversibility theorem states that the following are equivalent for the tuple (Λ, ρ, σ)

1) equality is satisfied: $D(\rho||\sigma) = D(\Lambda(\rho)||\Lambda(\sigma))$

[Petz, Dénes. "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra." *Communications in mathematical physics* 105 (1986): 123-131.]

Petz Recovery Theorem

Distinguishability of two states can be measured by relative entropy

$$D(\rho||\sigma) \equiv \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \geq 0$$

Monotonicity $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma))$

Petz reversibility theorem states that the following are equivalent for the tuple (Λ, ρ, σ)

1) equality is satisfied: $D(\rho||\sigma) = D(\Lambda(\rho)||\Lambda(\sigma))$

2) The Petz recovery defined for σ returns both states to their original configuration

$\Lambda^P_\sigma(\Lambda(\rho)) = \rho$ and $\Lambda^P_\sigma(\Lambda(\sigma)) = \sigma$

[Petz, Dénes. "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra." *Communications in mathematical physics* 105 (1986): 123-131.]

Petz Recovery Theorem

Distinguishability of two states can be measured by relative entropy

$$D(\rho||\sigma) \equiv \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \geq 0$$

Monotonicity $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma))$

Petz reversibility theorem states that the following are equivalent for the tuple (Λ, ρ, σ)

1) equality is satisfied: $D(\rho||\sigma) = D(\Lambda(\rho)||\Lambda(\sigma))$

2) The Petz recovery defined for σ returns both states to their original configuration

$\Lambda^P_\sigma(\Lambda(\rho)) = \rho$ and $\Lambda^P_\sigma(\Lambda(\sigma)) = \sigma$

Proof:

2) => 1) $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma) \geq D(\Lambda^P_\sigma(\Lambda(\rho))||\Lambda^P_\sigma(\Lambda(\sigma))) = D(\rho||\sigma)$

[Petz, Dénes. "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra." *Communications in mathematical physics* 105 (1986): 123-131.]

Petz Recovery Theorem

Distinguishability of two states can be measured by relative entropy

$D(\rho||\sigma) \equiv \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \geq 0$

Monotonicity $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma))$

Petz reversibility theorem states that the following are equivalent for the tuple (Λ, ρ, σ)

1) equality is satisfied: $D(\rho||\sigma) = D(\Lambda(\rho)||\Lambda(\sigma))$

2) The Petz recovery defined for σ returns both states to their original configuration

$\Lambda^P_\sigma(\Lambda(\rho)) = \rho$ and $\Lambda^P_\sigma(\Lambda(\sigma)) = \sigma$

Proof:

2) => 1) $D(\rho||\sigma) \geq D(\Lambda(\rho)||\Lambda(\sigma) \geq D(\Lambda^P_\sigma(\Lambda(\rho))||\Lambda^P_\sigma(\Lambda(\sigma))) = D(\rho||\sigma)$

1) => 2) Petz magic

[Petz, Dénes. "Sufficient subalgebras and the relative entropy of states of a von Neumann algebra." *Communications in mathematical physics* 105 (1986): 123-131.]

Petz Recovery Theorem Example

Non-unitary and non-classical channel

	Λ		Λ^R	
$\|0\rangle$		$\|+\rangle$		$\|0\rangle$
$\|1\rangle$		$\|-\rangle$		$\|1\rangle$
$\|2\rangle$		$\|2\rangle$		$\|2\rangle$

Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

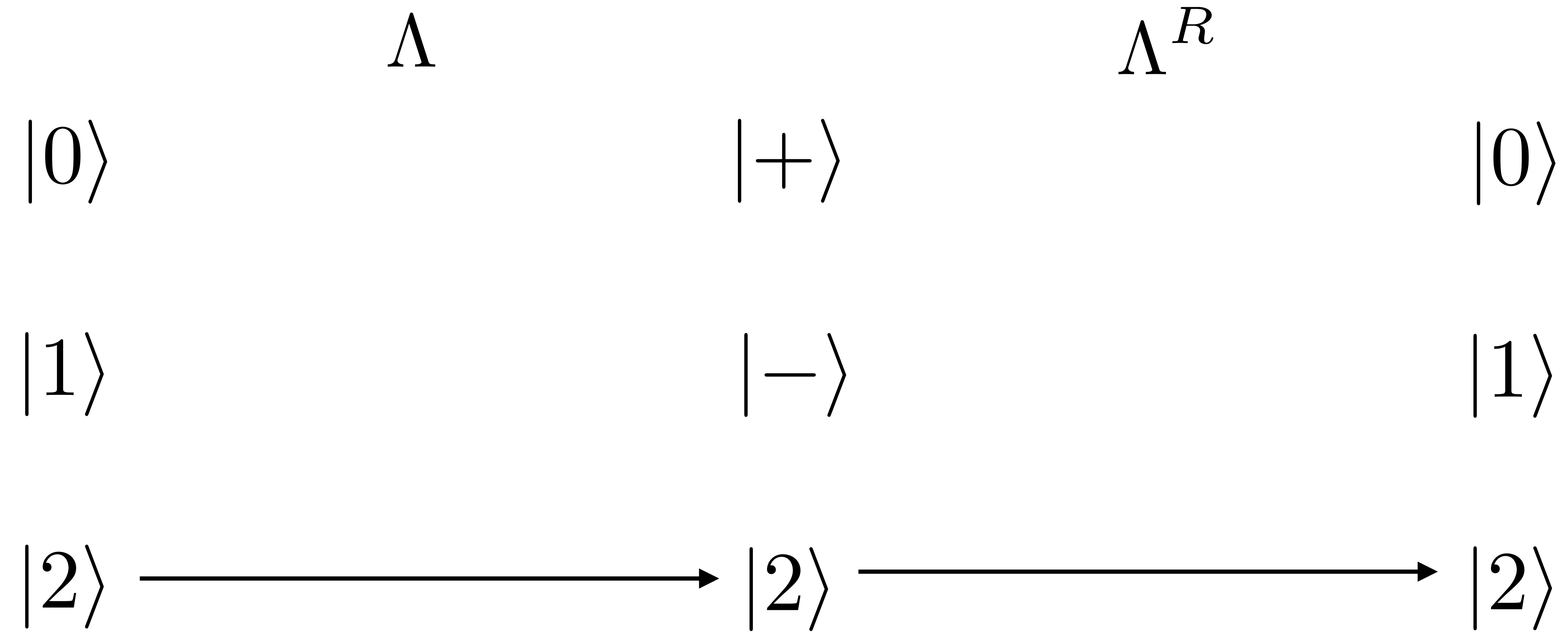
	Λ		Λ^R	
$\|0\rangle$		$\|+\rangle$		$\|0\rangle$
$\|1\rangle$		$\|-\rangle$		$\|1\rangle$
$\|2\rangle$		$\|2\rangle$		$\|2\rangle$

Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

$$K_0 = |2\rangle\langle 2|$$

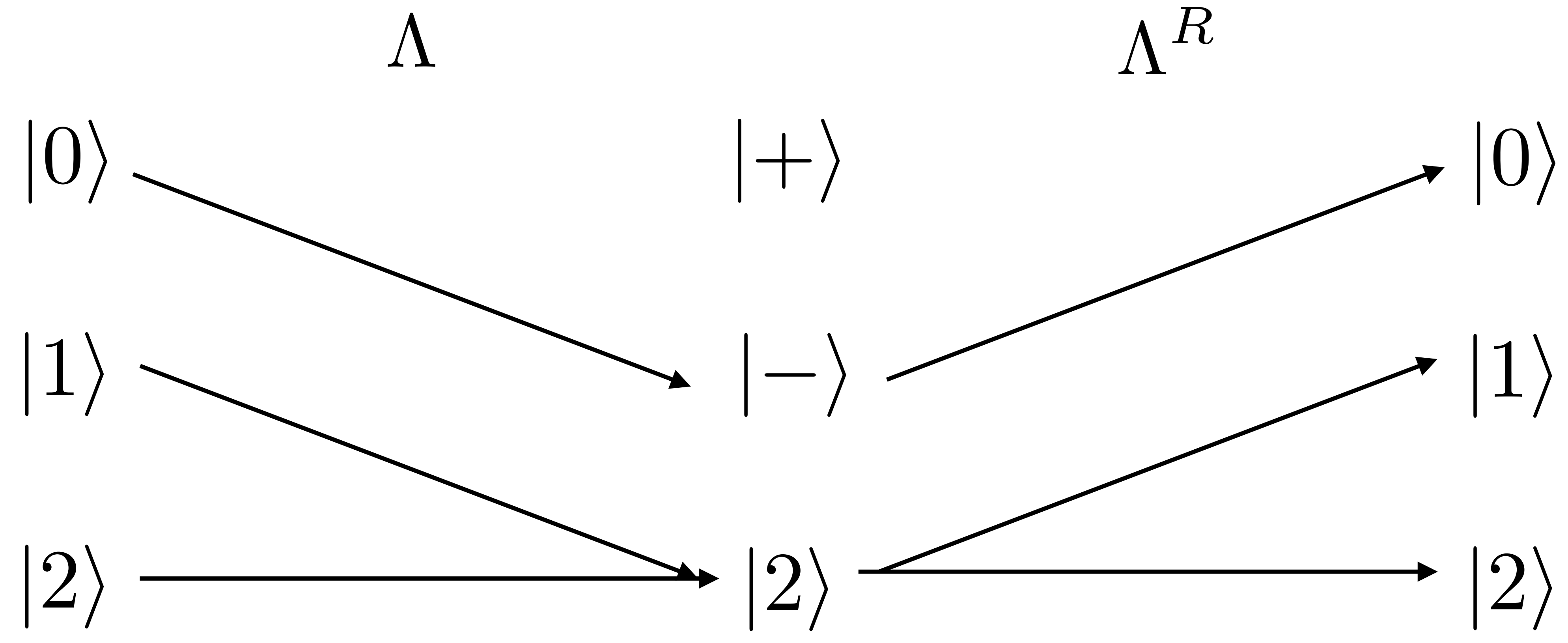


Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

$$K_0 = |2\rangle\langle 2| \quad K_1 = |-\rangle\langle 0| + |2\rangle\langle 1|$$

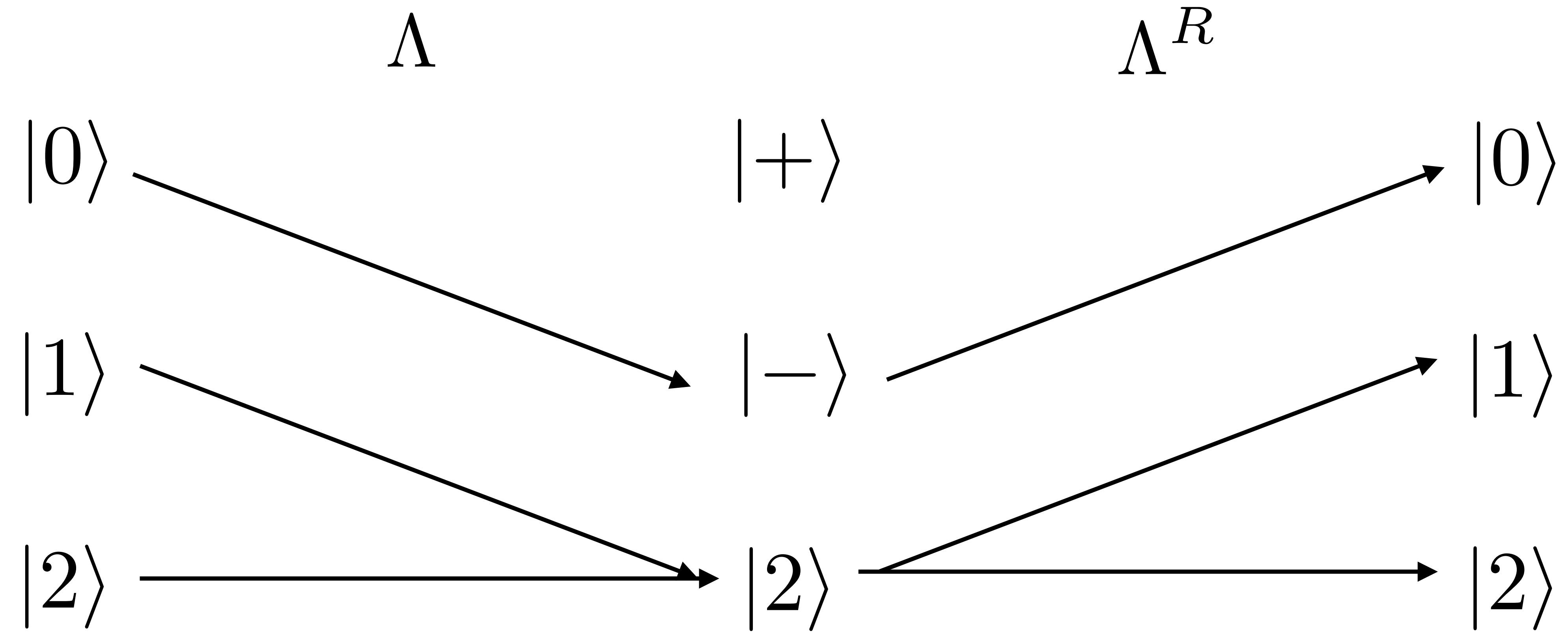


Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

$$K_0 = |2\rangle\langle 2| \qquad K_1 = |-\rangle\langle 0| + |2\rangle\langle 1|$$



Check that it is a valid CPTP

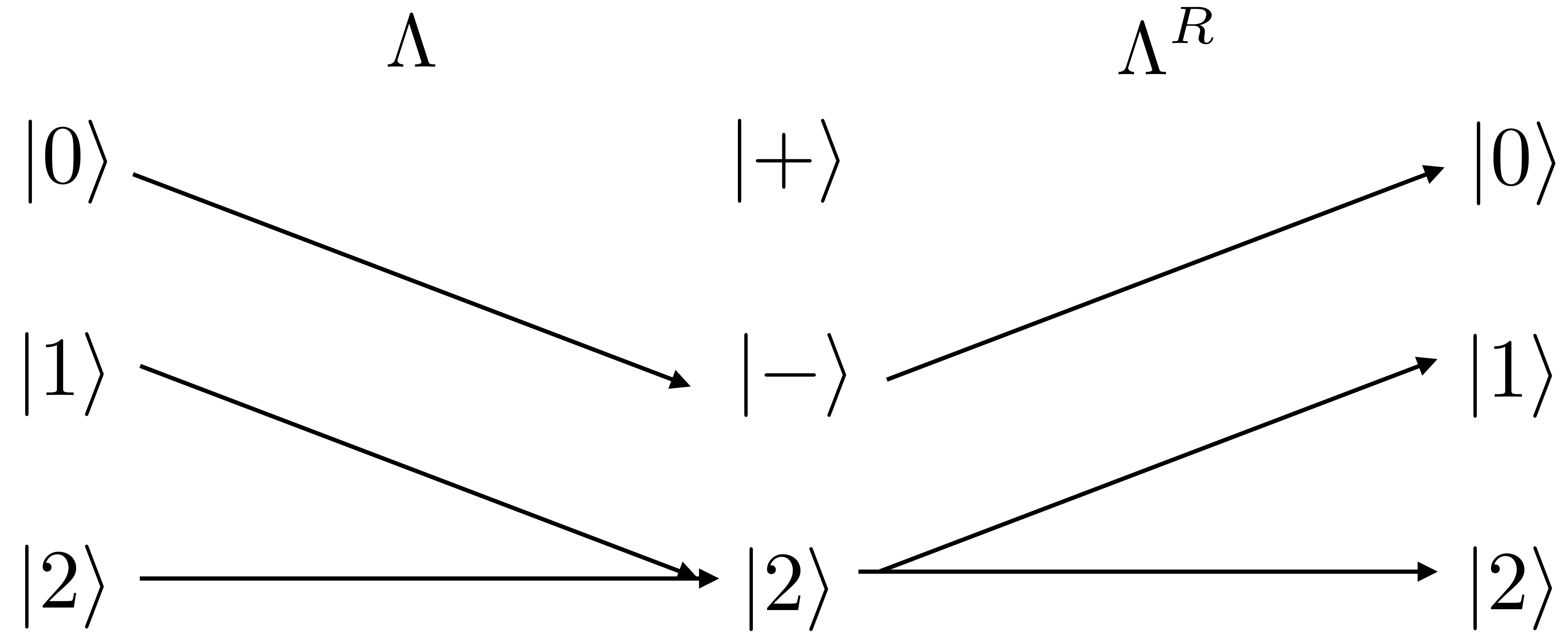
$$K_1^\dagger K_1 = (|0\rangle\langle -| + |1\rangle\langle 2|)(|-\rangle\langle 0| + |2\rangle\langle 1|)$$

Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

$$K_0 = |2\rangle\langle 2| \qquad K_1 = |-\rangle\langle 0| + |2\rangle\langle 1|$$



Check that it is a valid CPTP

$$K_1^\dagger K_1 = (|0\rangle\langle -| + |1\rangle\langle 2|)(|-\rangle\langle 0| + |2\rangle\langle 1|)$$

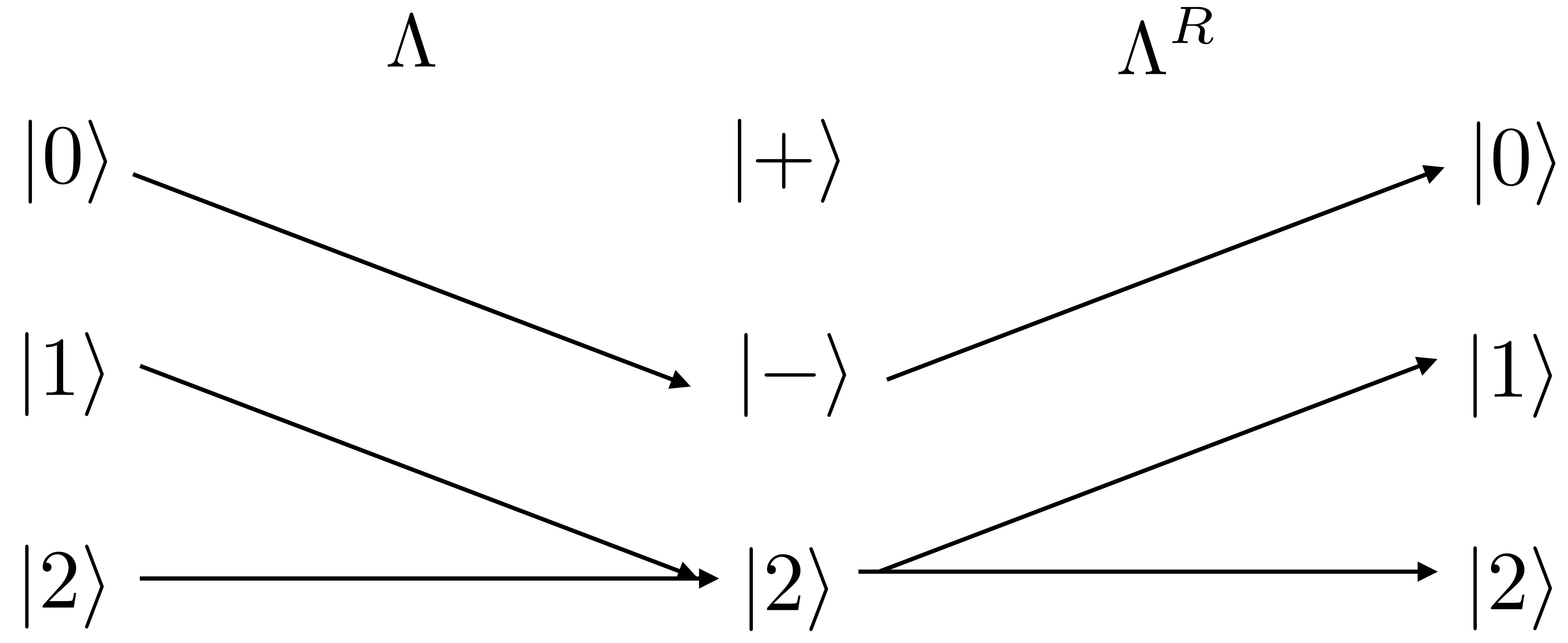
$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

$$K_0 = |2\rangle\langle 2| \qquad K_1 = |-\rangle\langle 0| + |2\rangle\langle 1|$$



Check that it is a valid CPTP

$$K_1^\dagger K_1 = (|0\rangle\langle -| + |1\rangle\langle 2|)(|-\rangle\langle 0| + |2\rangle\langle 1|)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

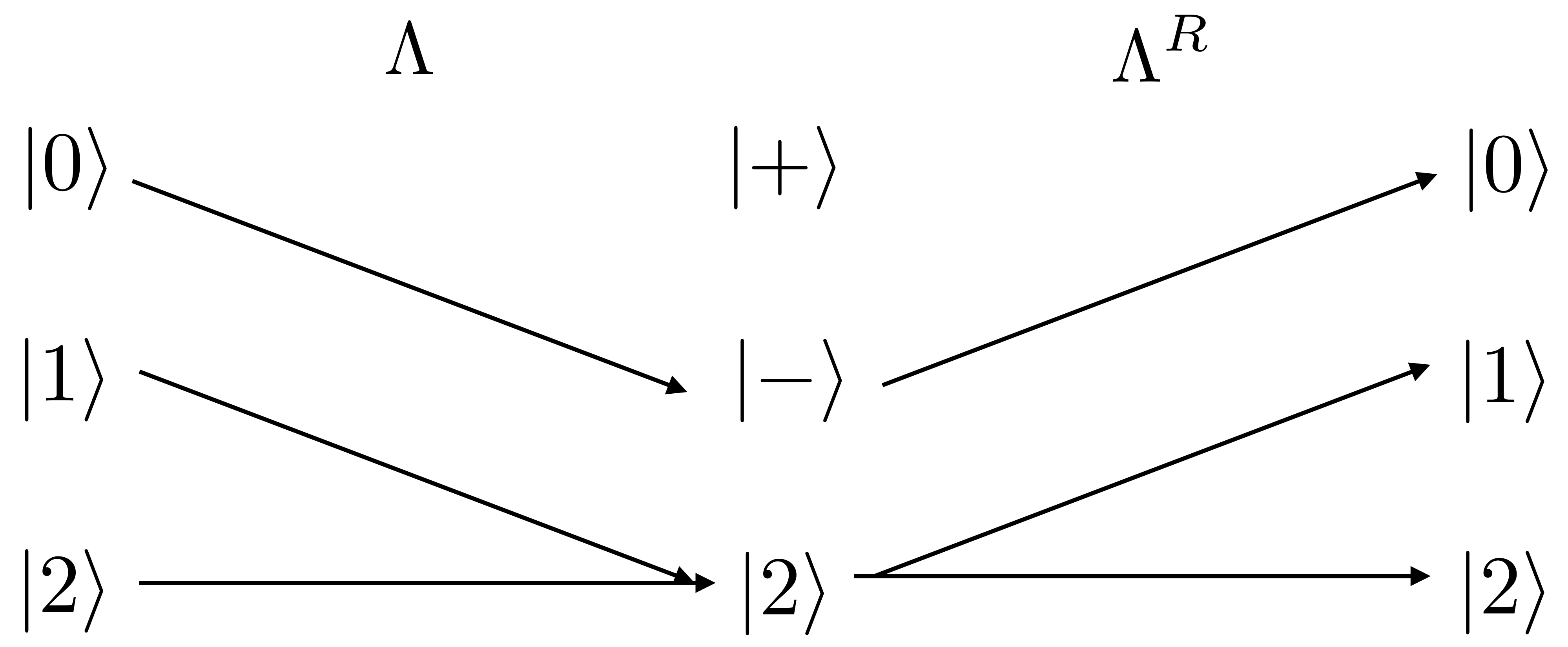
$$K_0^\dagger K_0 = |2\rangle\langle 2|$$

Petz Recovery Theorem Example

Non-unitary and non-classical channel

$$\Lambda(\cdot) = K_0 \cdot K_0^\dagger + K_1 \cdot K_1^\dagger$$

$$K_0 = |2\rangle\langle 2| \qquad K_1 = |-\rangle\langle 0| + |2\rangle\langle 1|$$



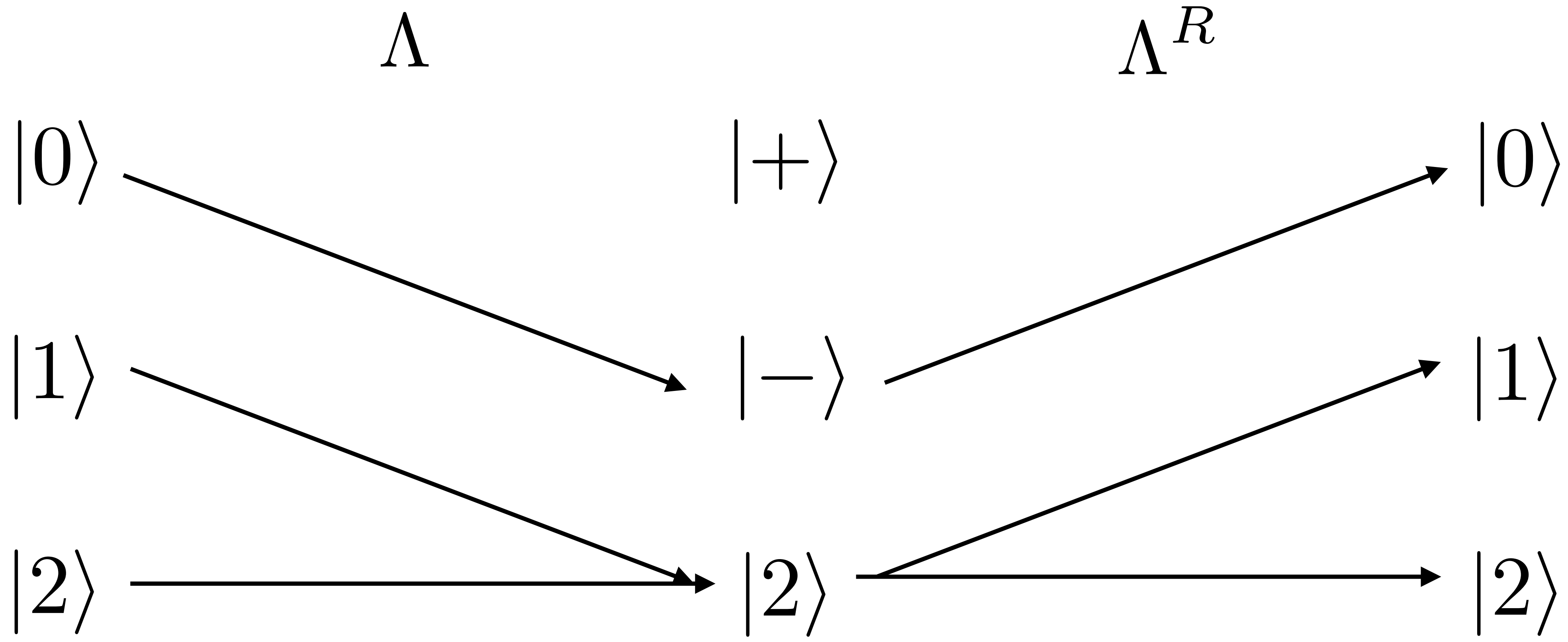
Check that it is a valid CPTP

$$K_1^\dagger K_1 = (|0\rangle\langle -| + |1\rangle\langle 2|)(|-\rangle\langle 0| + |2\rangle\langle 1|)$$
$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$K_0^\dagger K_0 = |2\rangle\langle 2| \qquad \sum_m K_m^\dagger K_m = \mathbb{I}$$

Petz Recovery Theorem Example

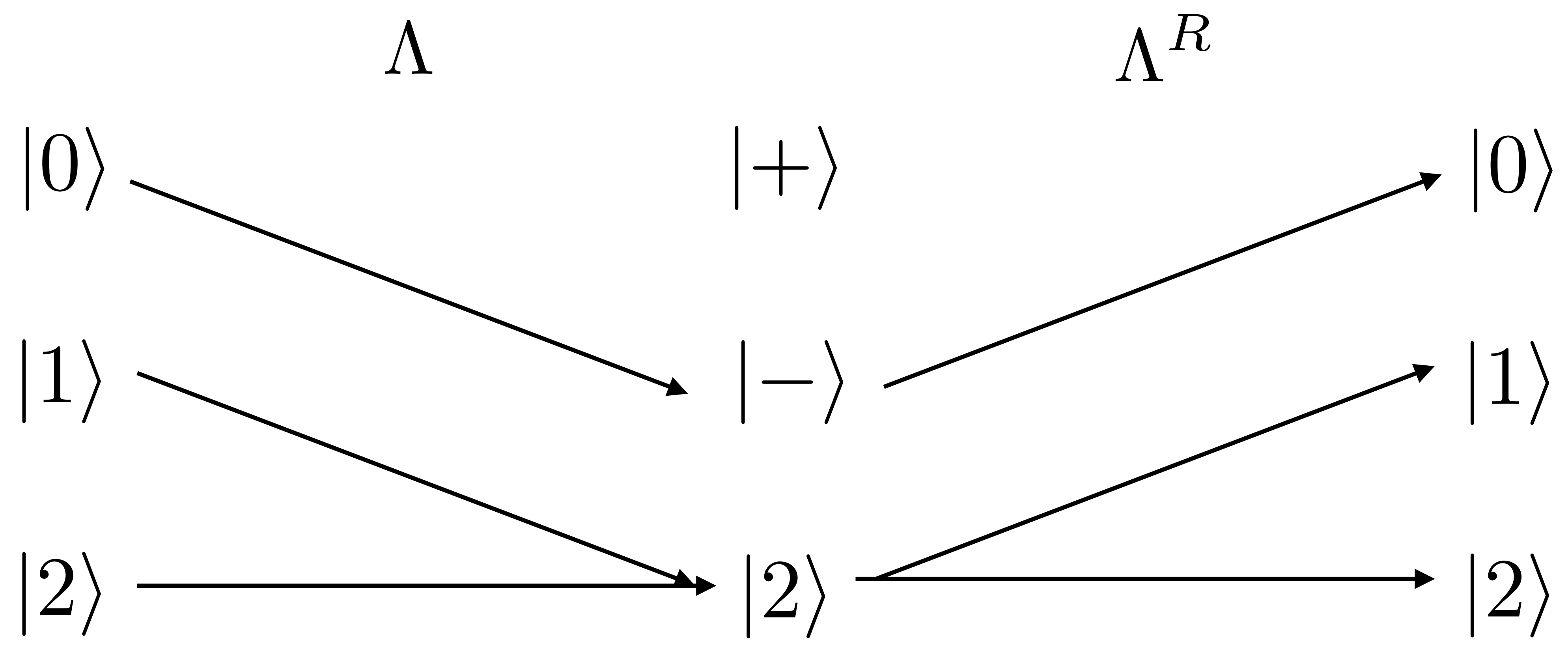
Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$



Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

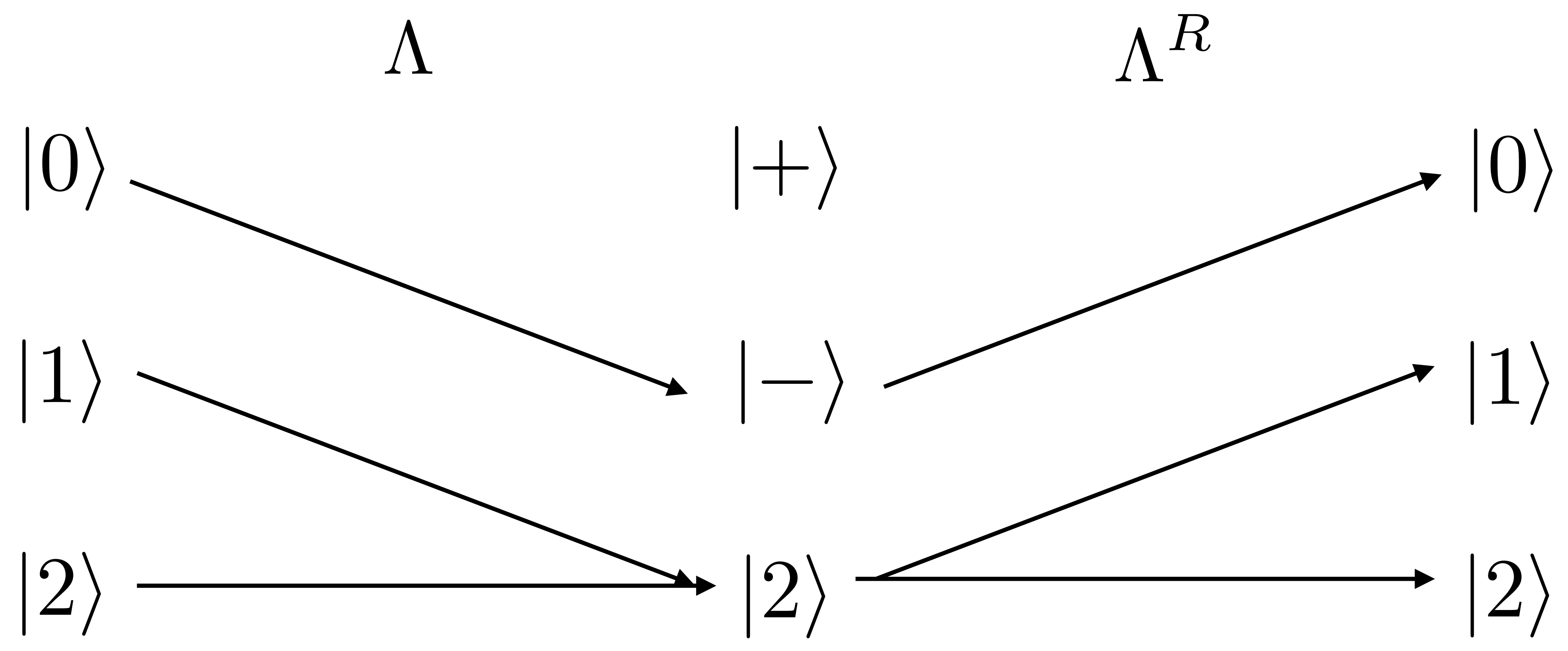
Final state $\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$



Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $$\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$$
$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$

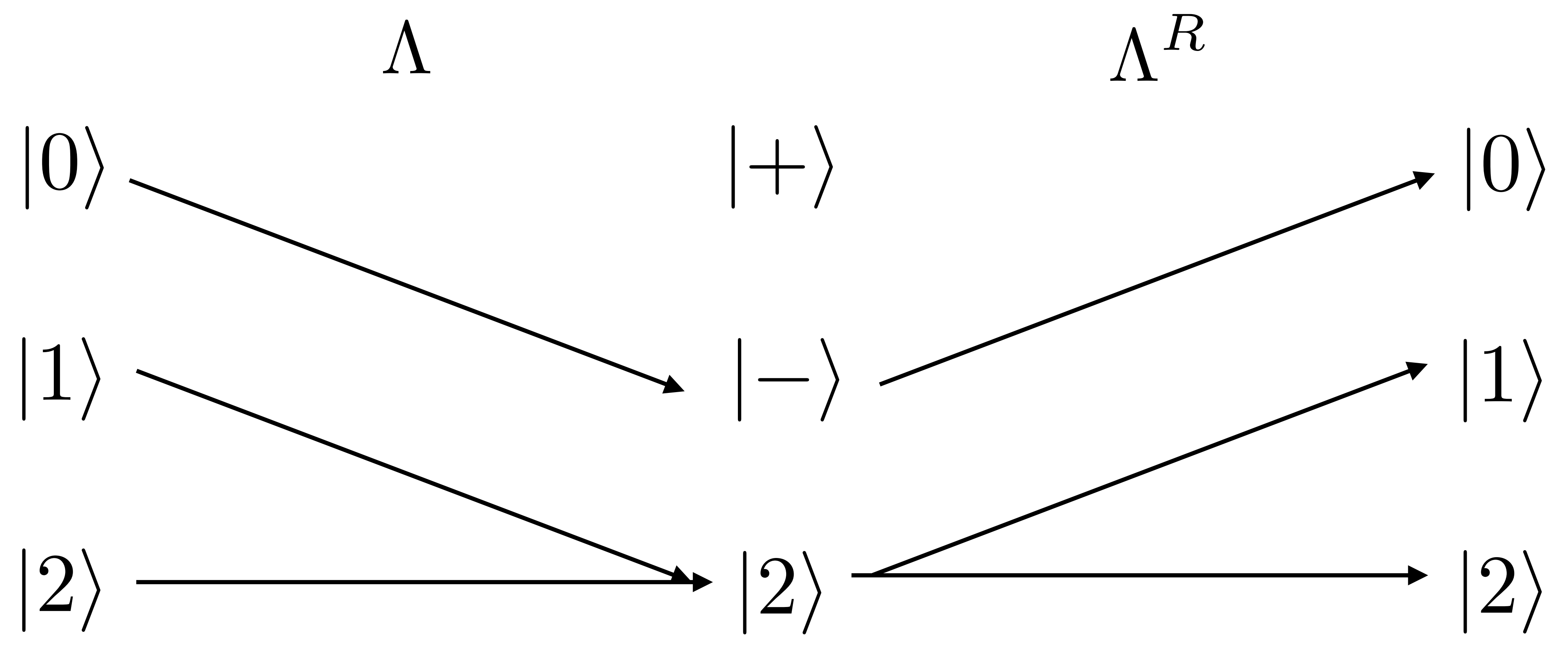


Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$

$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$



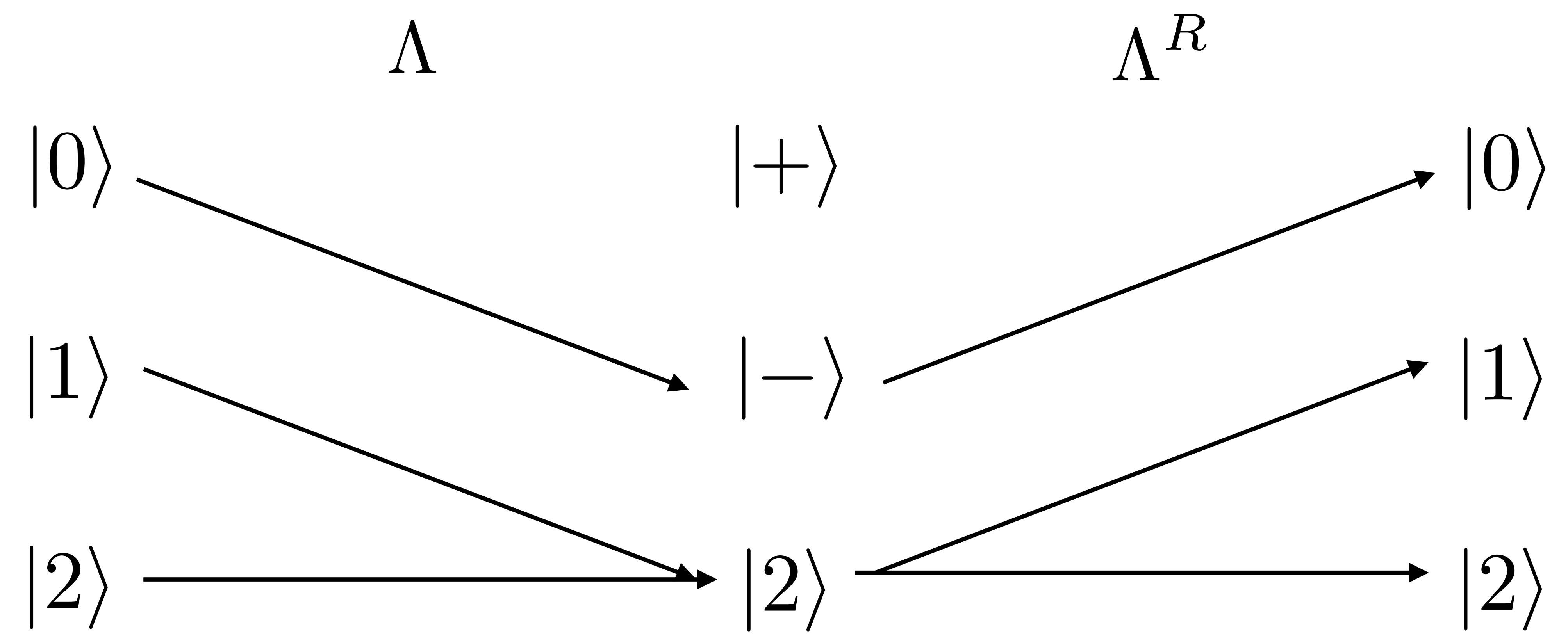
Recovery map:

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$

$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$



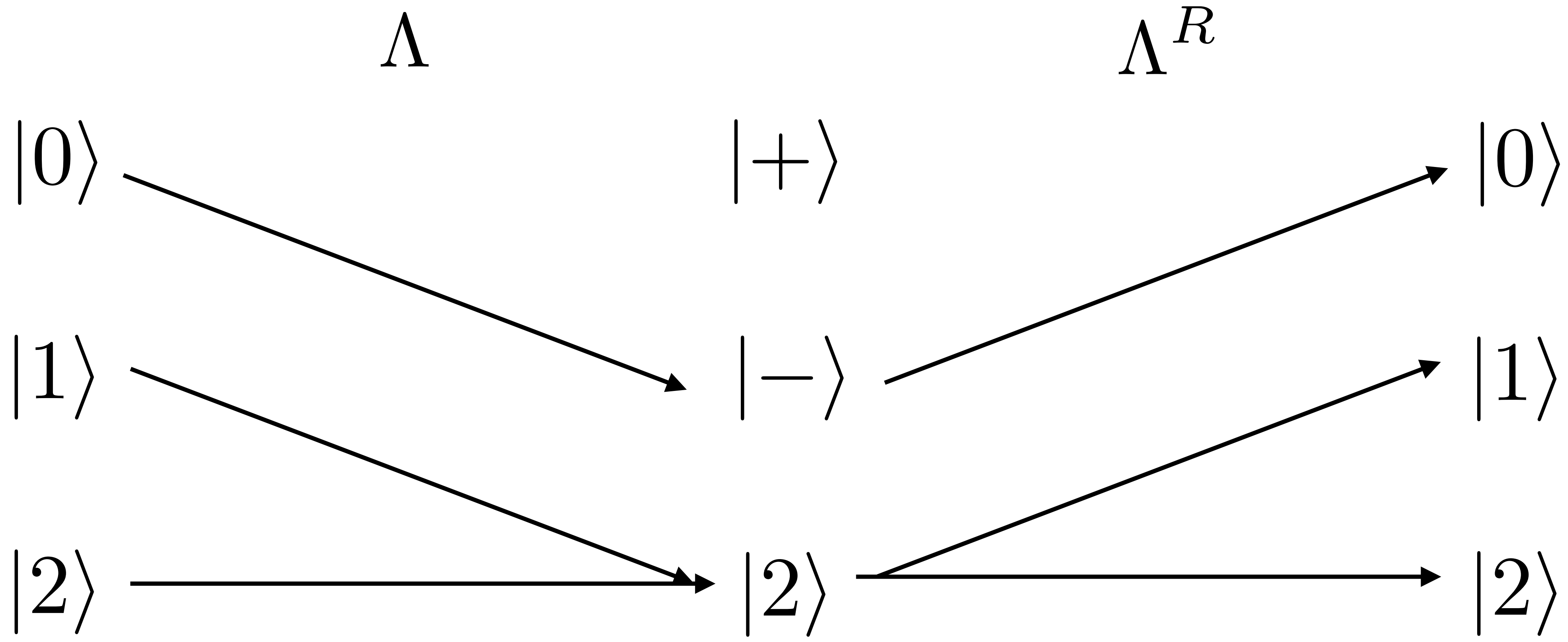
Recovery map: $\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2} = \frac{2}{p}|-\rangle\langle -|\langle -|M|-\rangle + \frac{2}{2-p}|2\rangle\langle 2|\langle 2|M|2\rangle\langle 2| + \text{und.} \times |+\rangle\langle +|\langle +|M|+\rangle$

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$

$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$



Recovery map: $\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2} = \frac{2}{p}|-\rangle\langle -|\langle -|M|-\rangle + \frac{2}{2-p}|2\rangle\langle 2|\langle 2|M|2\rangle\langle 2| + \text{und.} \times |+\rangle\langle +|\langle +|M|+\rangle$

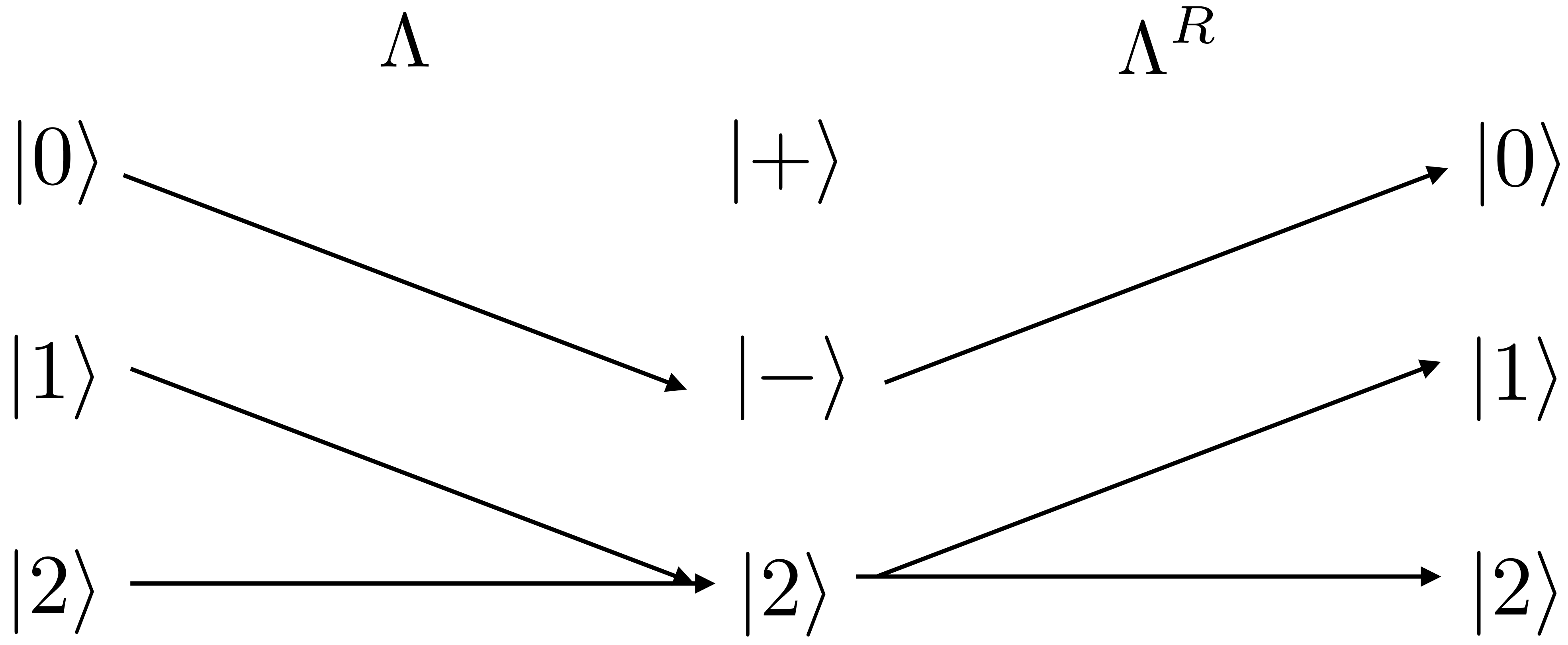
assume : $\langle +|M|+\rangle = 0$

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $$\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$$

$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$



Recovery map: $$\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2} = \frac{2}{p}|-\rangle\langle -|\langle -|M|-\rangle + \frac{2}{2-p}|2\rangle\langle 2|\langle 2|M|2\rangle\langle 2| + \text{und.} \times |+\rangle\langle +|\langle +|M|+\rangle$$

assume : $\langle +|M|+\rangle = 0$

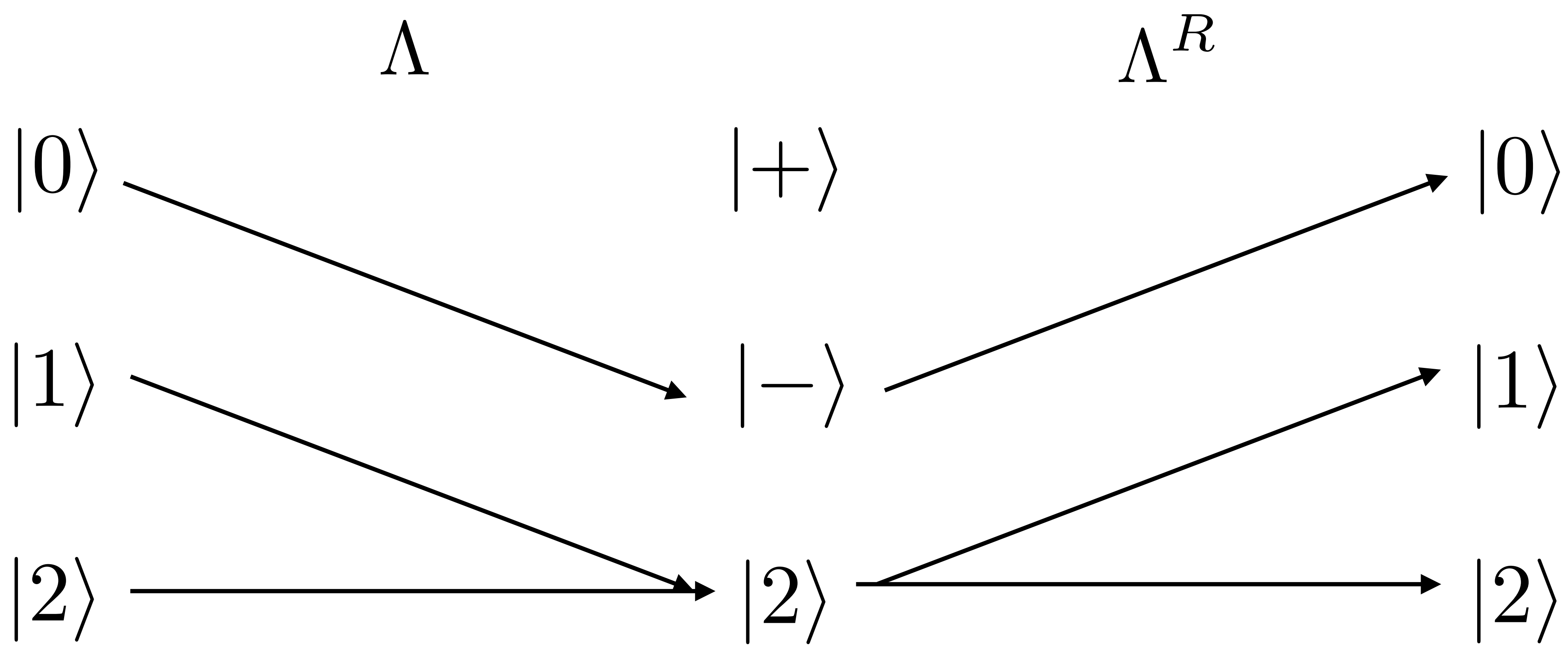
$$\Lambda^\dagger(\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2}) = \frac{2}{p}\Lambda^\dagger(|-\rangle\langle -|)\langle -|M|-\rangle + \frac{2}{2-p}\Lambda^\dagger(|2\rangle\langle 2|)\langle 2|M|2\rangle$$

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$

$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$



Recovery map: $\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2} = \frac{2}{p}|-\rangle\langle -|\langle -|M|-\rangle + \frac{2}{2-p}|2\rangle\langle 2|\langle 2|M|2\rangle\langle 2| + \text{und.} \times |+\rangle\langle +|\langle +|M|+\rangle$

assume : $\langle +|M|+\rangle = 0$

$$\Lambda^\dagger(\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2}) = \frac{2}{p}\Lambda^\dagger(|-\rangle\langle -|)\langle -|M|-\rangle + \frac{2}{2-p}\Lambda^\dagger(|2\rangle\langle 2|)\langle 2|M|2\rangle$$

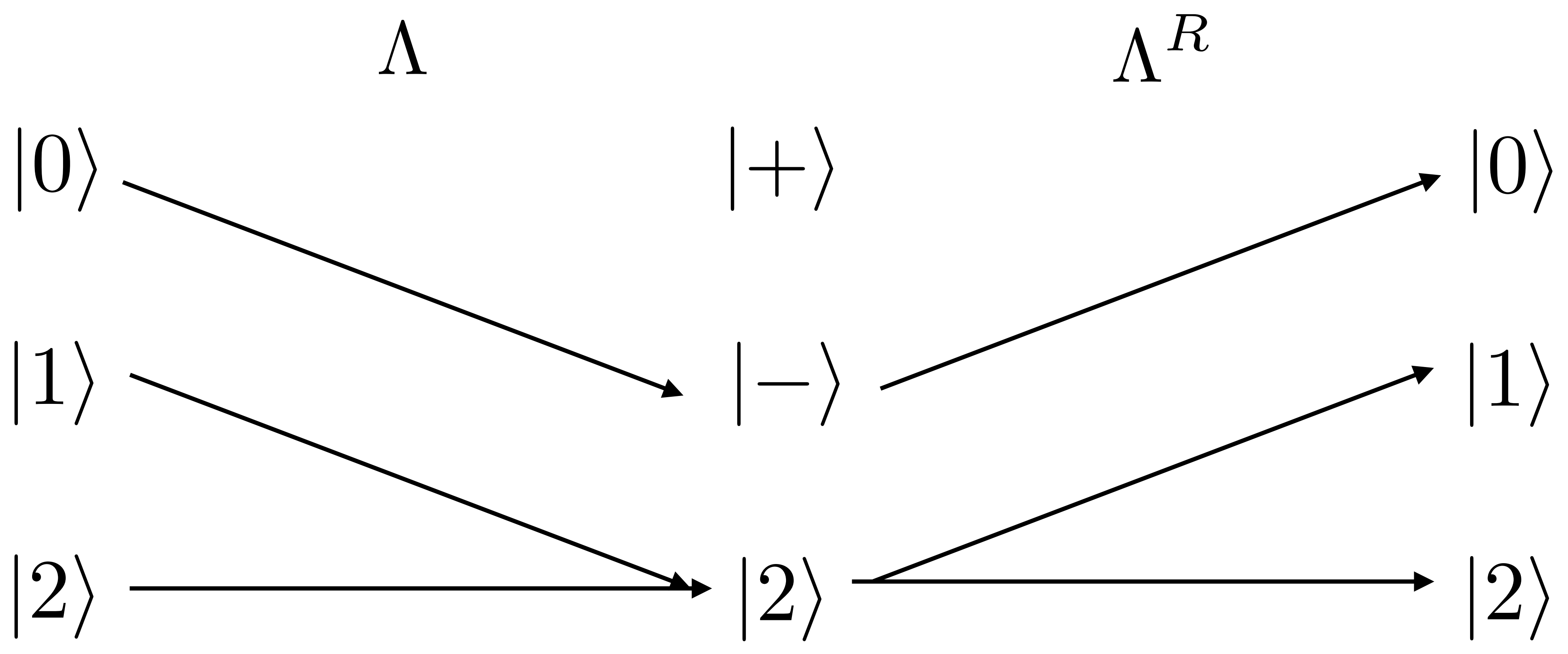
$$= \frac{2}{p}|0\rangle\langle 0|\langle -|M|-\rangle + \frac{2}{2-p}(|2\rangle\langle 2| + |1\rangle\langle 1|)\langle 2|M|2\rangle$$

Petz Recovery Theorem Example

Initial state $\sigma = p\dfrac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $$\Lambda(\sigma) = p\frac{|-\rangle\langle -| + |2\rangle\langle 2|}{2} + (1-p)|2\rangle\langle 2|$$

$$= \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$$



Recovery map: $$\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2} = \frac{2}{p}|-\rangle\langle -|\langle -|M|-\rangle + \frac{2}{2-p}|2\rangle\langle 2|\langle 2|M|2\rangle\langle 2| + \text{und.} \times |+\rangle\langle +|\langle +|M|+\rangle$$

$\text{assume}: \langle +|M|+\rangle = 0$

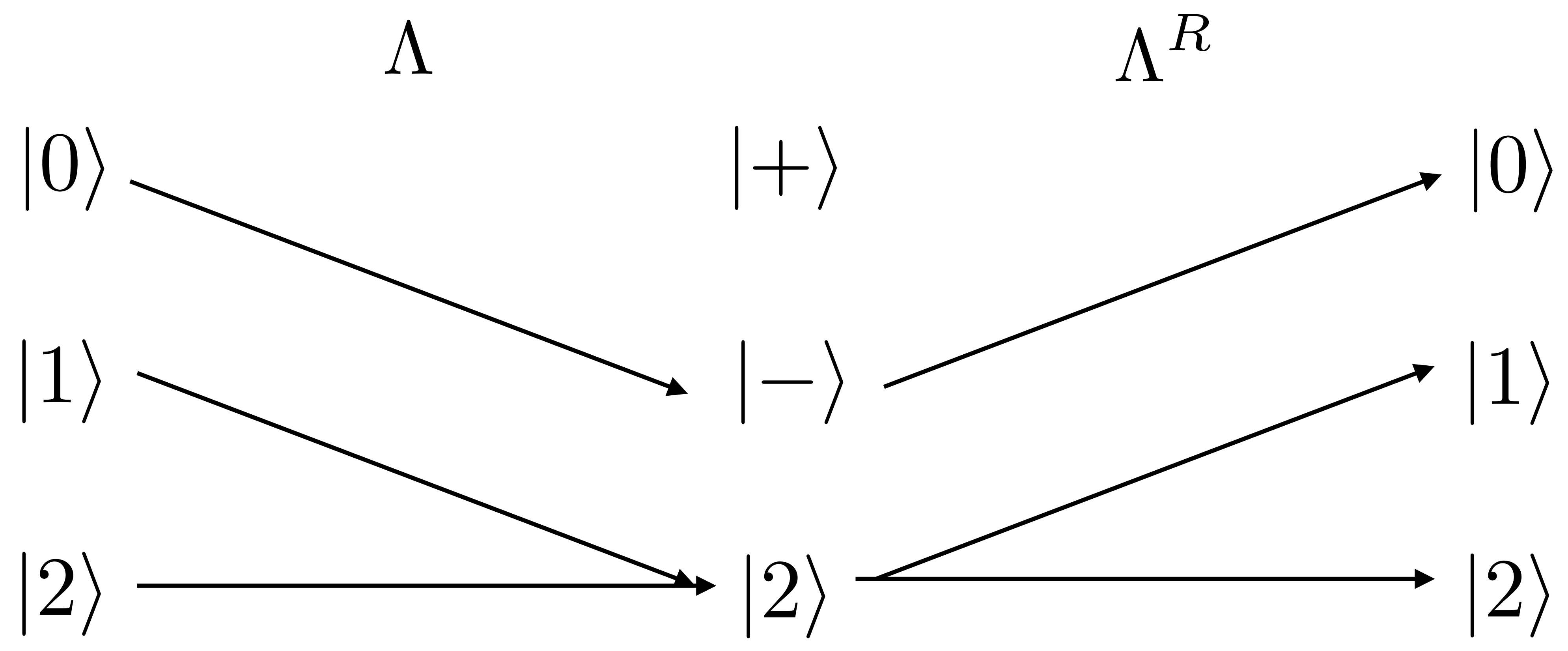
$$\Lambda^\dagger(\Lambda(\sigma)^{-1/2} M \Lambda(\sigma)^{-1/2}) = \frac{2}{p}\Lambda^\dagger(|-\rangle\langle -|)\langle -|M|-\rangle + \frac{2}{2-p}\Lambda^\dagger(|2\rangle\langle 2|)\langle 2|M|2\rangle$$

$$= \frac{2}{p}|0\rangle\langle 0|\langle -|M|-\rangle + \frac{2}{2-p}(|2\rangle\langle 2| + |1\rangle\langle 1|)\langle 2|M|2\rangle$$

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Petz Recovery Theorem Example

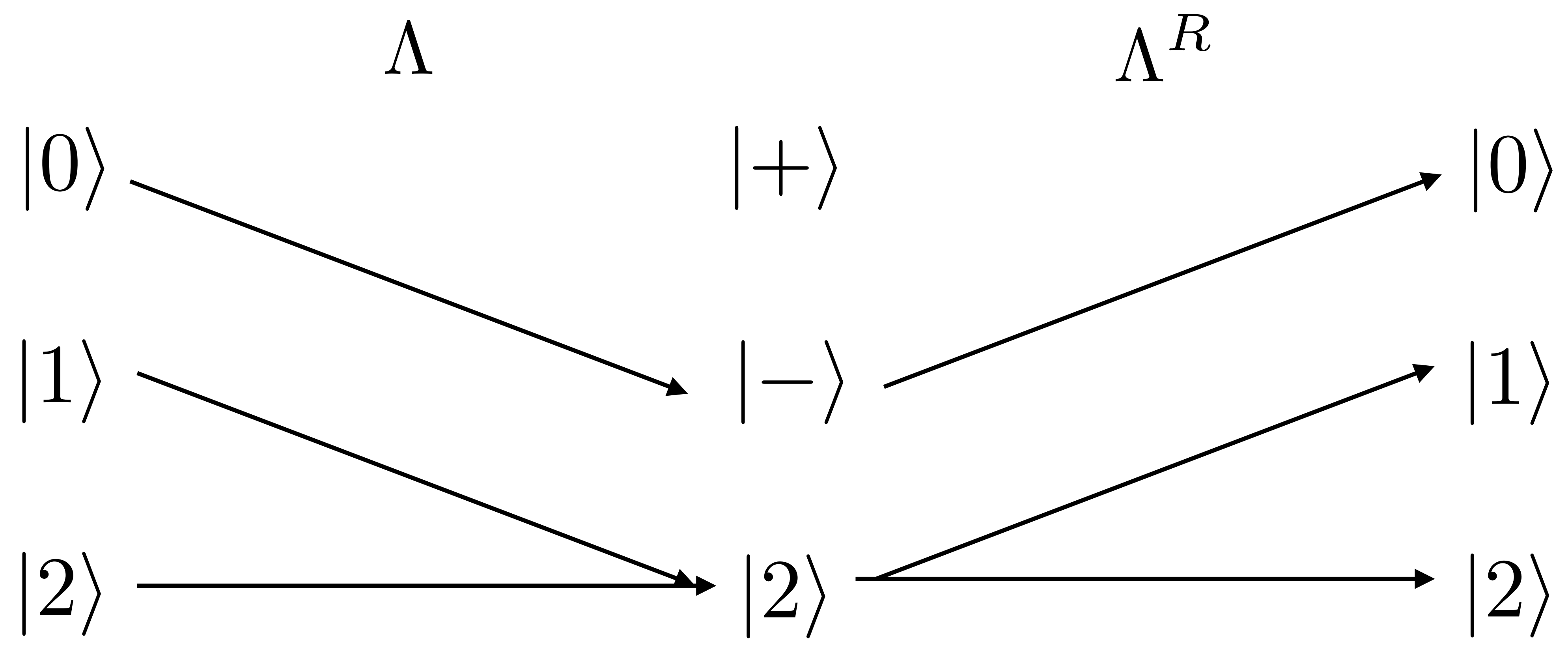
Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$



Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$



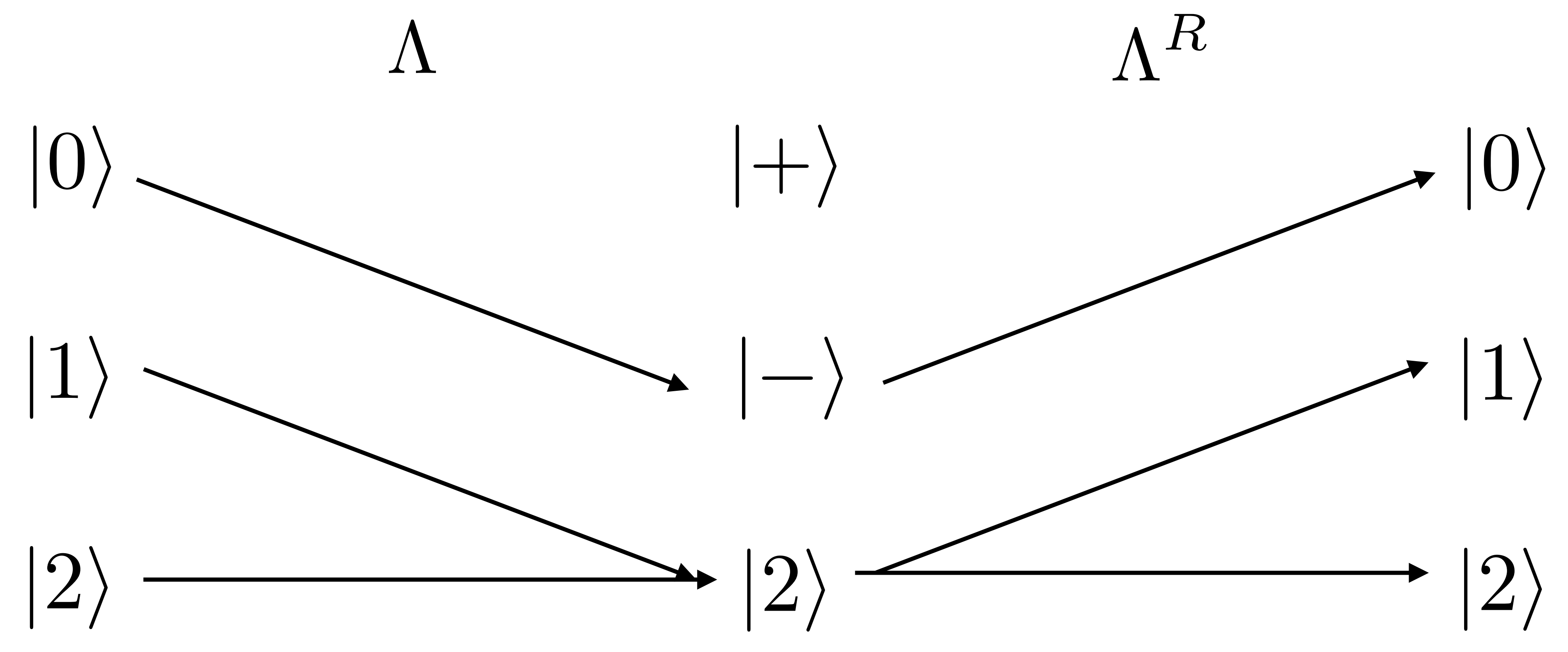
Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$



Petz Recovery Theorem Example

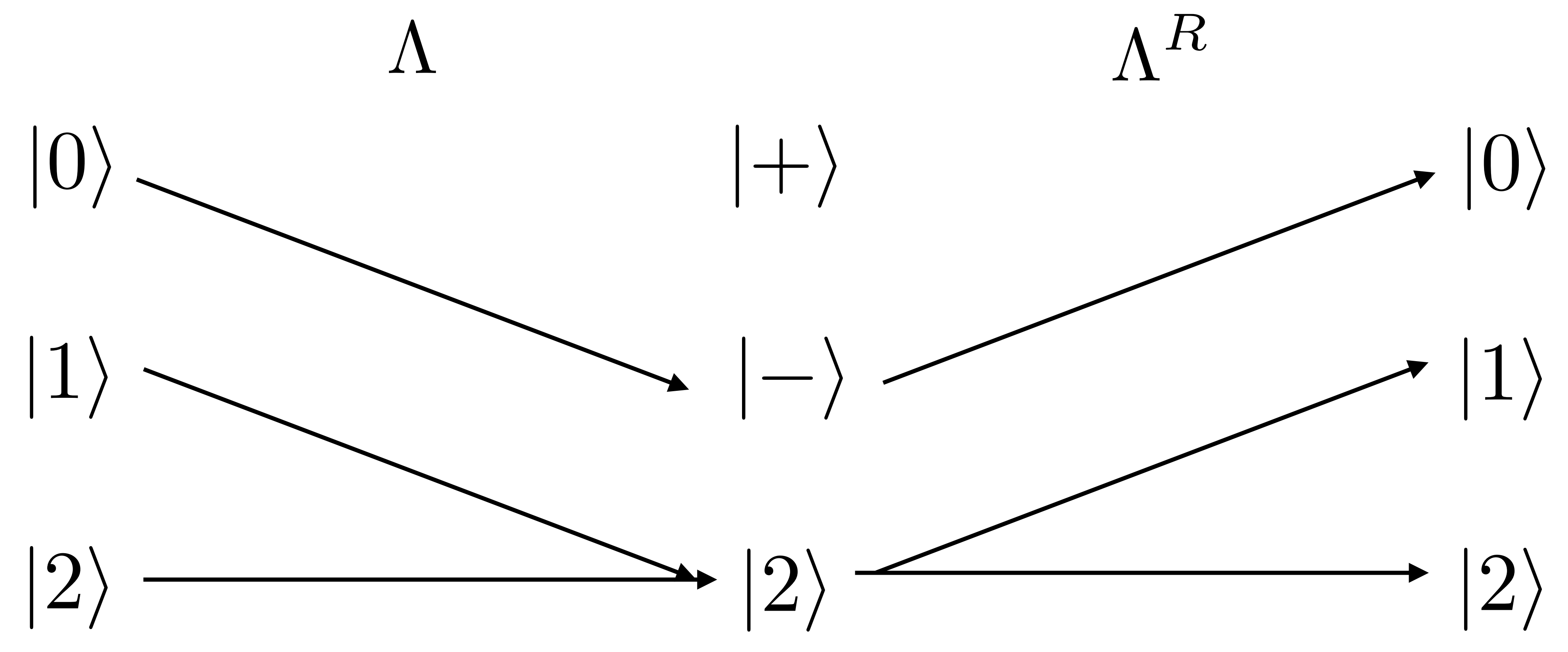
Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$



Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

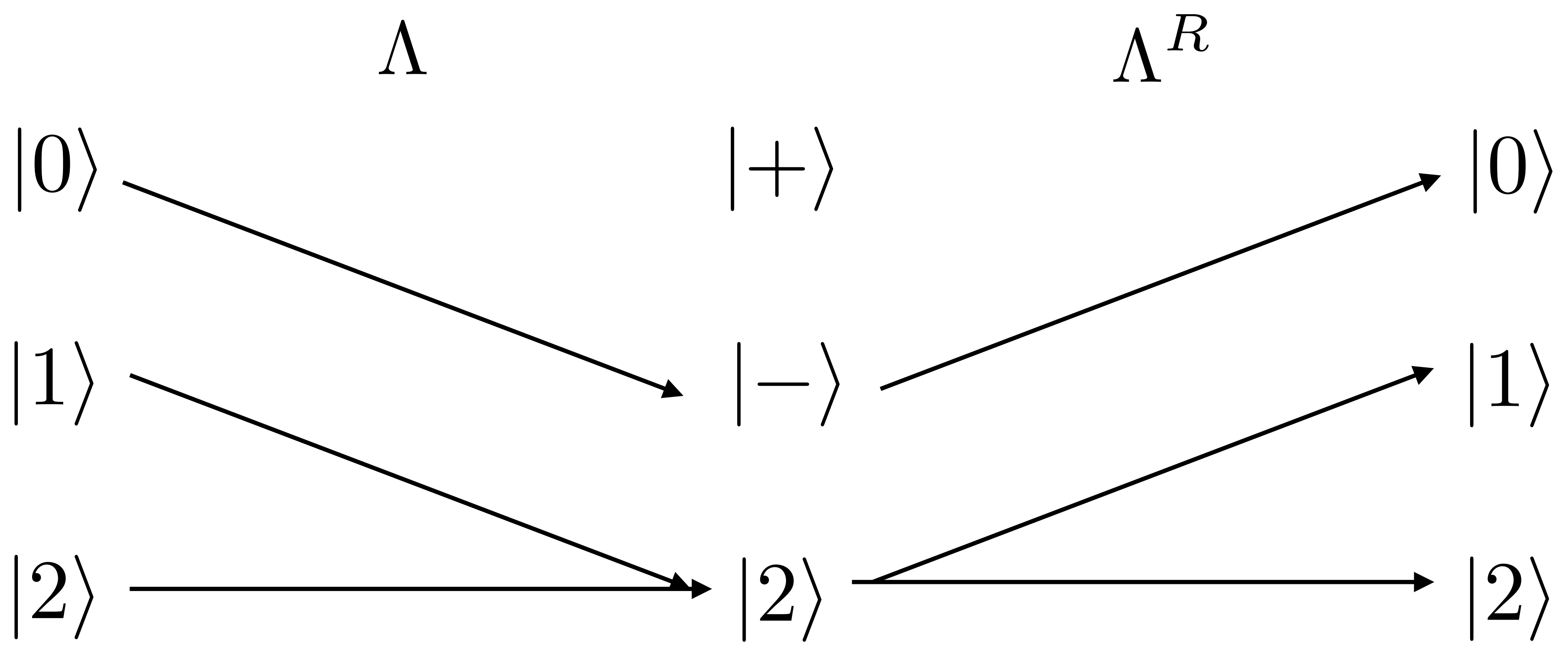
Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda^P_\sigma(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$



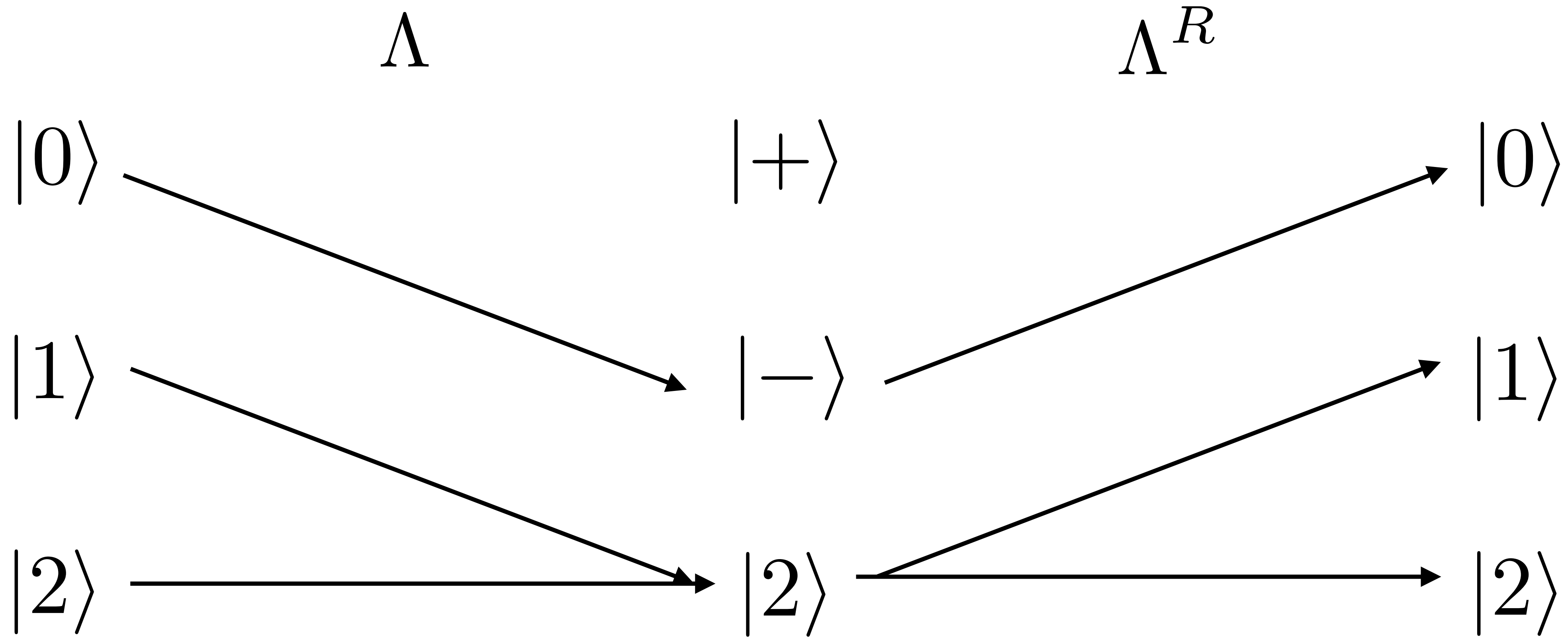
Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda^P_\sigma(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$



Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

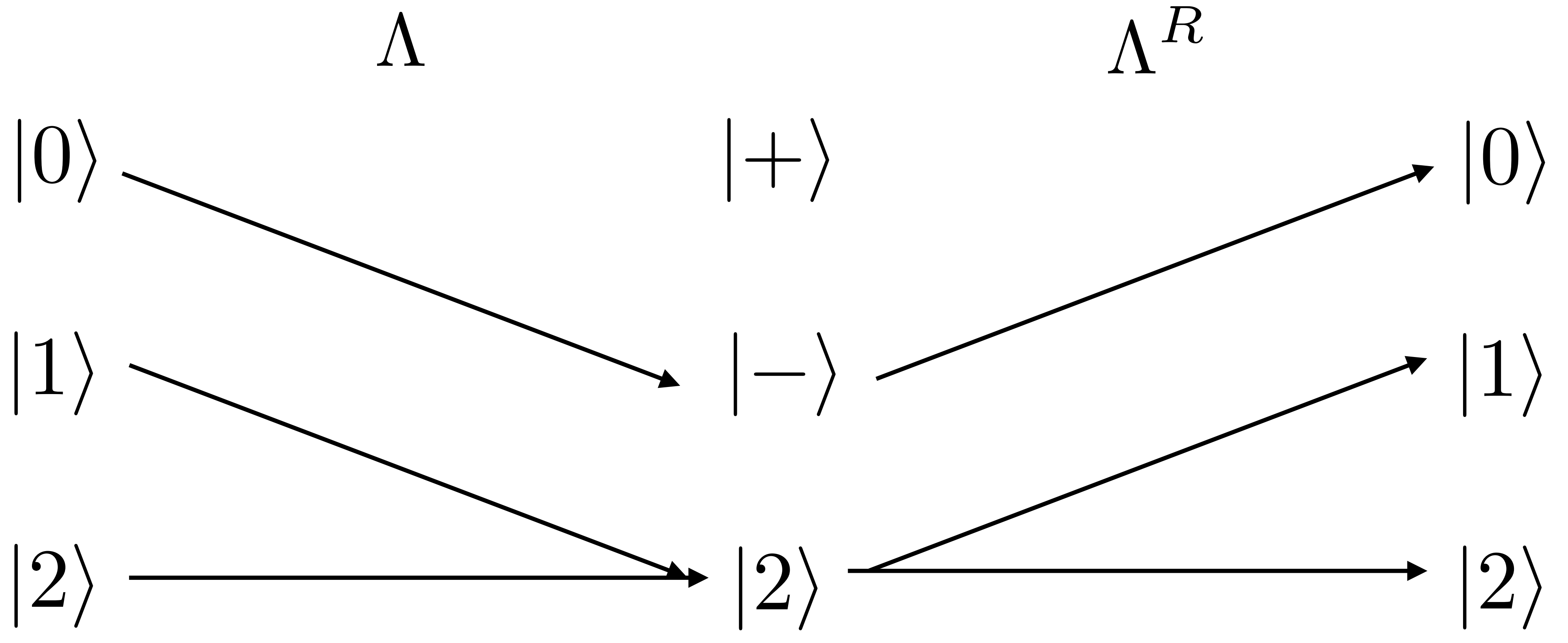
$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

$$\Lambda^P_\sigma(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$



Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

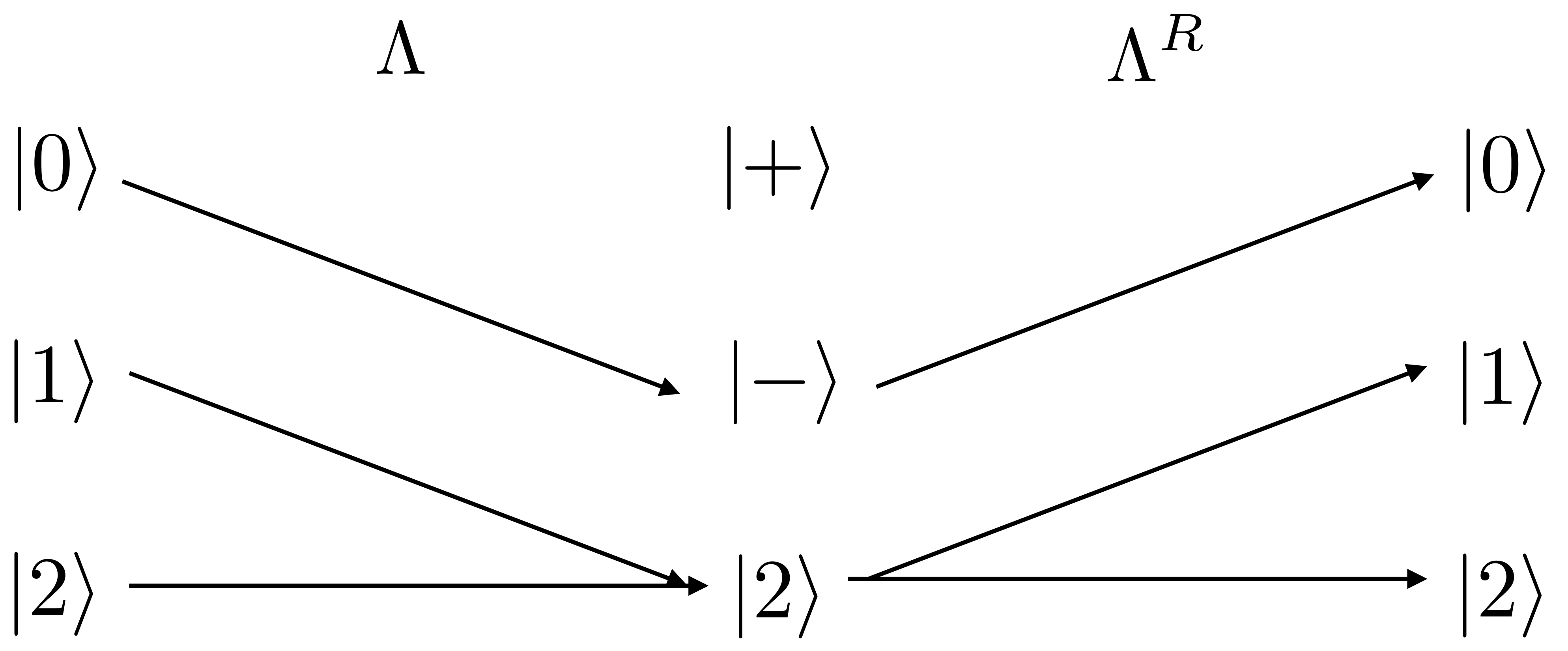
$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$



Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

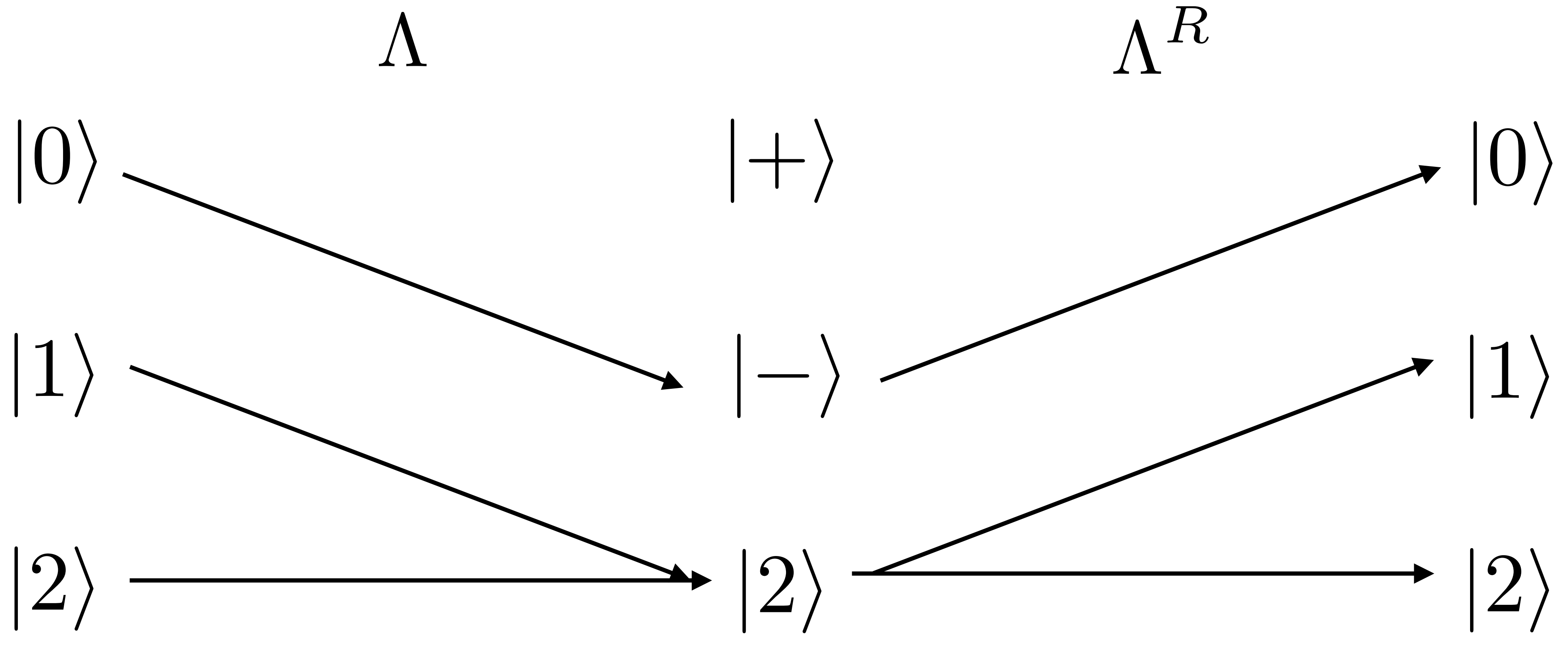
Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

$$p(1-b) = \frac{p(1-pb)}{2-p}$$

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$



Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

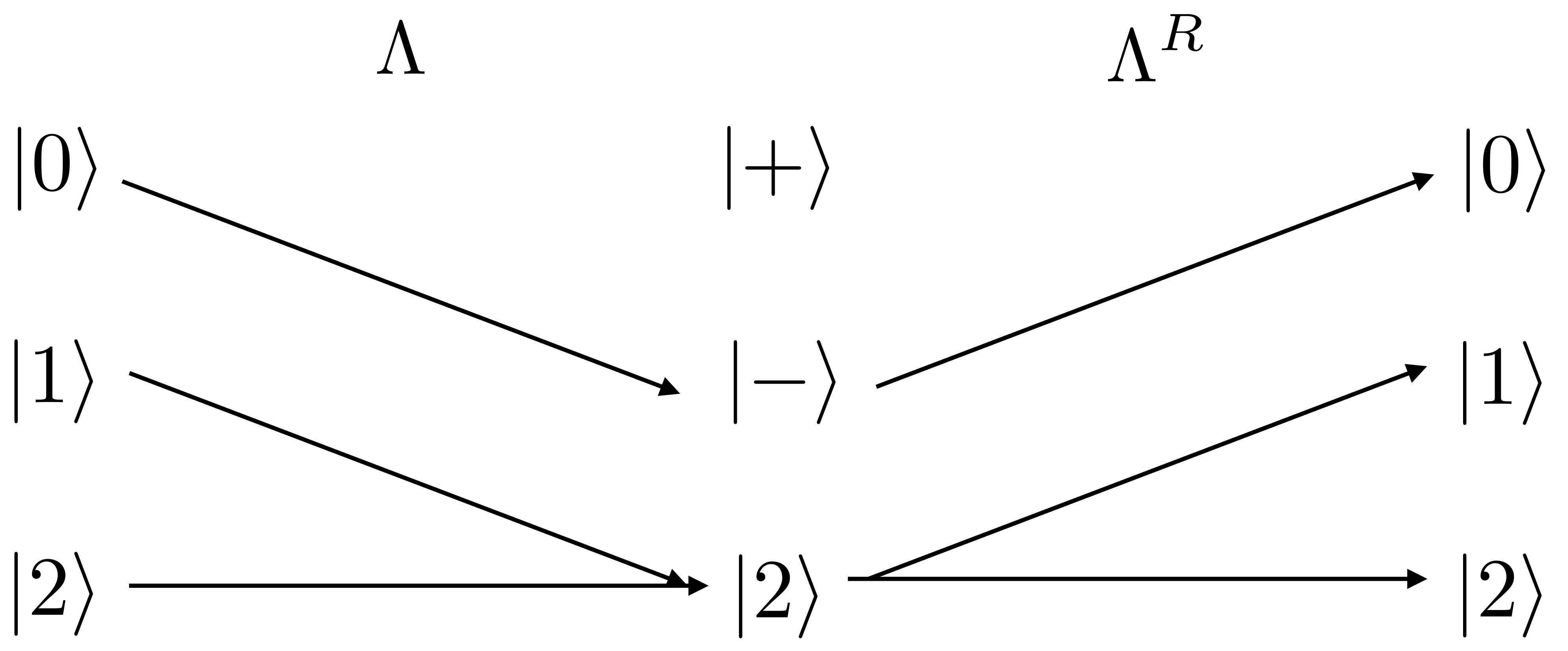
$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

$$p(1-b) = \frac{p(1-pb)}{2-p}$$

$$(p-p^2)(1-2b) = 0$$

Petz Recovery Theorem Example



Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

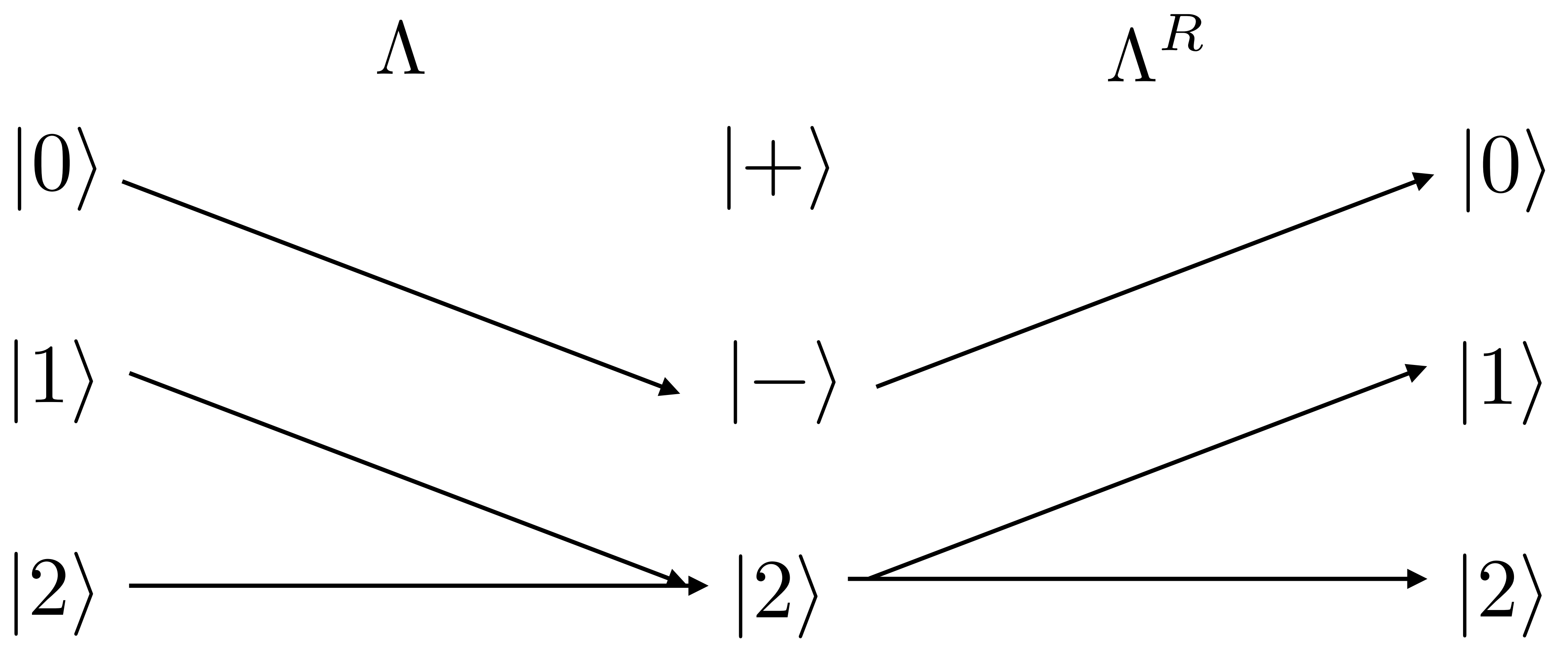
$$p(1-b) = \frac{p(1-pb)}{2-p}$$

$$(p-p^2)(1-2b) = 0$$

Solutions: $b = 1/2$

$p = 0$ or 1

Petz Recovery Theorem Example



Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

$$p(1-b) = \frac{p(1-pb)}{2-p}$$

$$(p-p^2)(1-2b) = 0$$

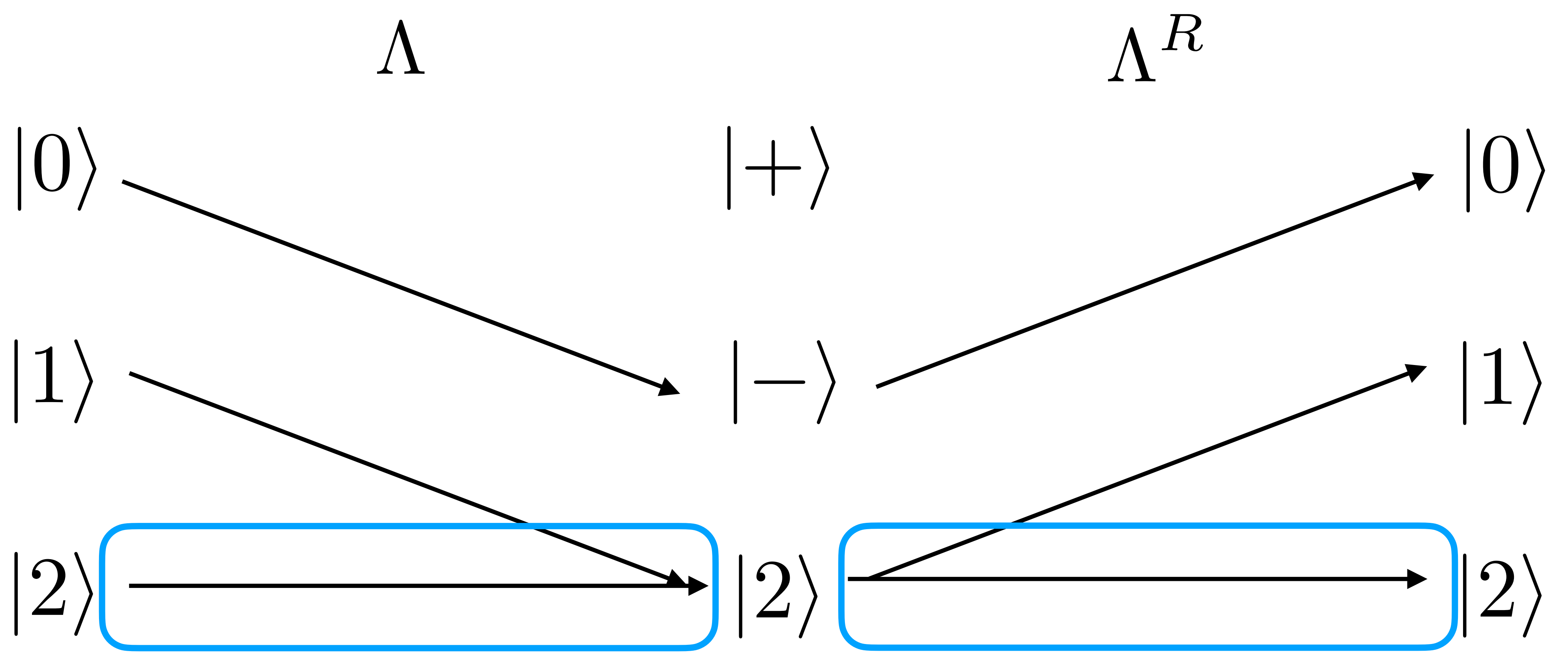
Solutions: $b = 1/2$ (trivial $\rho = \sigma$)

$p = 0$ or 1

Petz Recovery Theorem Example

Initial state $\sigma = p\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \frac{p}{2}|-\rangle\langle -| + \frac{2-p}{2}|2\rangle\langle 2|$



Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + (\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

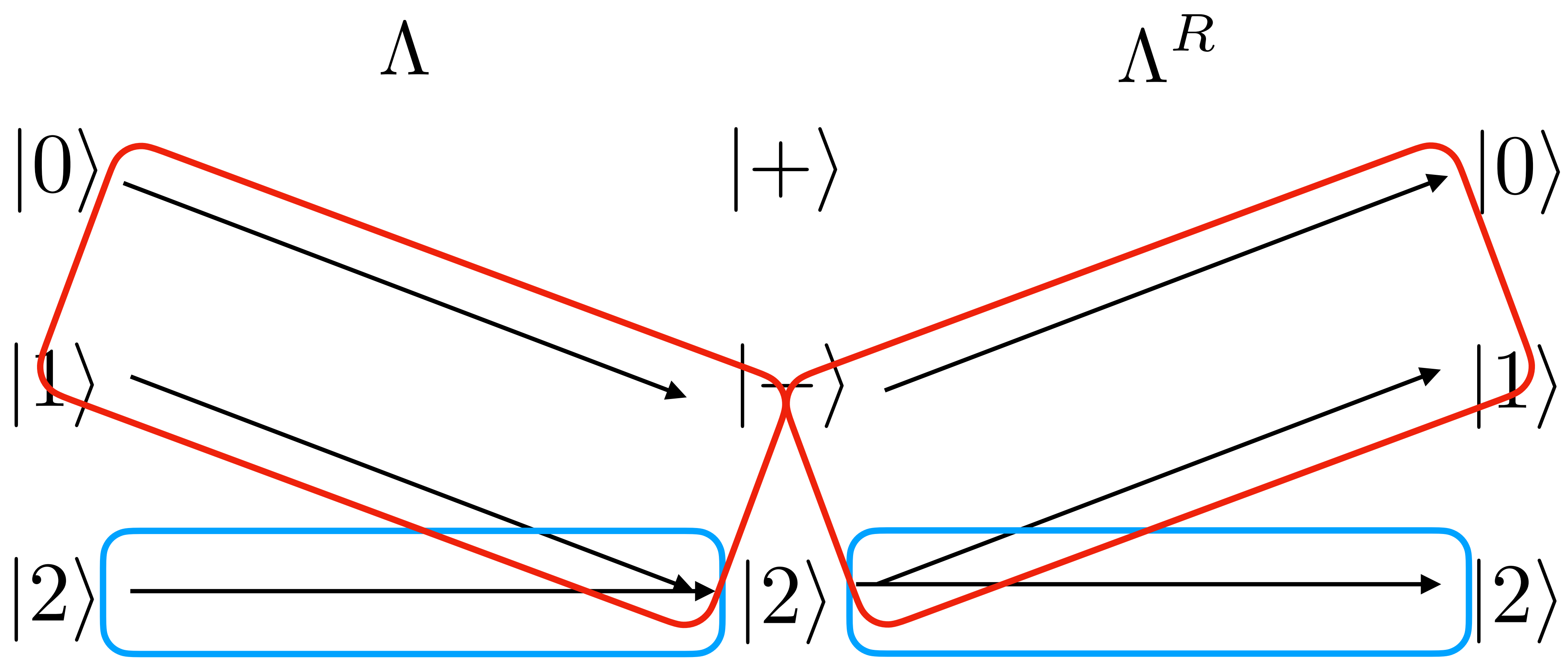
$$p(1-b) = \frac{p(1-pb)}{2-p}$$

$$(p-p^2)(1-2b) = 0$$

Solutions: $b = 1/2 \quad (\text{trivial } \rho = \sigma)$

$p = 0$ or 1

Petz Recovery Theorem Example



Initial state $\sigma = p\dfrac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + (1-p)|2\rangle\langle 2|$

Final state $\Lambda(\sigma) \quad = \dfrac{p}{2}|-\rangle\langle -| + \dfrac{2-p}{2}|2\rangle\langle 2|$

Recovery map:

$$\Lambda_\sigma^P(M) = |0\rangle\langle 0|\langle -|M|-\rangle + \left(\frac{2-2p}{2-p}|2\rangle\langle 2| + \frac{p}{2-p}|1\rangle\langle 1|\right)\langle 2|M|2\rangle$$

Alternate state: $\rho = pb|0\rangle\langle 0| + p(1-b)|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|$

$$\Lambda(\rho) = pb|-\rangle\langle -| + (1-pb)|2\rangle\langle 2|$$

$$\Lambda_\sigma^P(\Lambda(\rho)) = pb|0\rangle\langle 0| + \frac{p(1-pb)}{2-p}|1\rangle\langle 1| + \frac{(2-2p)(1-pb)}{2-p}|2\rangle\langle 2|$$

Consequence of Petz Recovery Thm: Distinguishability is only preserved if: $\Lambda_\sigma^P(\Lambda(\rho)) = \rho$

$$p(1-b) = \frac{p(1-pb)}{2-p}$$

$$(p-p^2)(1-2b) = 0$$

Solutions:

$b = 1/2 \quad$ (trivial $\rho = \sigma$)

$p = 0$ or 1

Review

-How do we time reverse operations in quantum mechanics?
-Unitaries are like classical permutations of state space
-This gives us some insight into how to go beyond quantum time reversal.
-What if we implement something that is NOT a permutation, like erasure (something IRREVERSIBLE). Can we reverse it?
-For this we introduce probabilities and Bayes rule.
-Results from distribution over time
-Question: what is a quantum Bayes rule?
-Answer: the Petz Recovery

Denes Petz