Recap:

Choi - Jamiołkowski isomorphism

- Linear map $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_{1'})$ $\Longleftrightarrow$ matrix $\mathcal{E}_c \in L(\mathcal{H}_{1'} \otimes \mathcal{H}_1)$

- $\mathcal{E}_c = (\mathcal{E} \otimes \mathcal{I}_1)\,[\,|M\rangle\langle M|\,]$, where $|M\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$ is the maximally entangled state.

- $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_{1'})$ is CPTP $\Longleftrightarrow$ $\mathcal{E}_c \geq 0$ & $\mathrm{tr}_{1'}[\mathcal{E}_c] = \mathbb{1}_1$

  "$\Rightarrow$" By direct insertion

  "$\Leftarrow$" By proving that $\mathcal{E}_c \geq 0$ leads to a Kraus decomposition.

Note: $\mathcal{E}$ is CP $\Rightarrow$ $\mathcal{E}_c \geq 0$ $\Rightarrow$ Kraus decomposition

$\Rightarrow$ $\mathcal{E}$ is CP if and only if $\mathcal{E}$ admits a Kraus decomposition

above proof.

Examples: ("everything is Choi matrices")

a) partially dephasing map $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_{1'}) \cong L(\mathcal{H}_1)$

$$\mathcal{E}[\rho] = p \, \mathrm{tr}[\rho] \frac{\mathbb{1}_{1'}}{d} + (1-p)\rho$$

$$\hookrightarrow \mathcal{E}_c = p \underbrace{\frac{\mathbb{1}_{1'}}{d} \otimes \mathbb{1}_1}_{\text{Choi state of tr}} + (1-p) \underbrace{|\Omega\rangle\langle\Omega|}_{\text{Choi state of the identity channel}}$$

b) Let $\mathcal{R}: \mathbb{C} \to L(\mathcal{H}_{1'})$ be a map that satisfies

$$\mathcal{R}[z] \geq 0 \quad \forall z \geq 0 \quad (\underbrace{\text{complete}}_{\text{show this}} \text{ positivity})$$

and $\mathrm{tr}[\mathcal{R}[z]] = z$ (trace preservation)

The map $\mathcal{R}$ is called a <u>preparation</u> with corresponding Choi state $\mathcal{R}_c =: \rho \in L(\mathcal{H}_{1'} \otimes \mathbb{C}) = L(\mathcal{H}_{1'})$

it satisfies $\rho \geq 0$ and $\mathrm{tr}_{1'}[\rho] = \mathrm{tr}[\rho] = 1$

$\hookrightarrow$ Quantum states are Choi matrices of preparation maps.

↳ Let $\mathcal{F} : L(\mathcal{H}_n) \to L(\mathcal{H}_1) \cong \mathbb{C}$ be a linear map from matrices to numbers that satisfies $\mathcal{F}[\eta] \geq 0 \quad \forall \, \eta \geq 0$ (complete positivity)

show this

and $tr[\mathcal{F}[\eta]] \leq tr[\eta]$ (trace non-increasing)

$\mathcal{F}$ is called an effect. Its Choi state $F \in L(\mathcal{H}_n \otimes \mathbb{C}) \cong L(\mathcal{H}_n)$ satisfies

$$F \geq 0 \, , \quad tr_{\mathbb{C}}[F] = F \leq \mathbb{1}_1$$

if $\mathcal{F}$ is trace preserving, then

$$F \geq 0 \, , \quad tr[F] = \mathbb{1}_1$$

4.4 Choi representation and the non-uniqueness of the Kraus decomposition

Note: The Choi matrix $\mathcal{E}_c$ of a map $\mathcal{E}$ is unique but its Kraus decomposition is not.

↳ How many Kraus operators are required (at most), and how are different Kraus representations related?

Thm.: Every CP map $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_1')$ requires at most $d_1 \times d_{1'}$ Kraus operators $\{K_\alpha\}$ and they can be chosen such that

$$\underbrace{\mathrm{tr}(K_\alpha^\dagger K_\beta) = \delta_{\alpha\beta}\, \mathrm{tr}(K_\alpha^\dagger K_\beta)}_{\text{canonical Kraus decomposition}} \quad \forall \alpha, \beta$$

Proof: (for $d_1 = d_{1'} =: d$). Let $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_1')$ and $\mathcal{E}_C = \sum_{\alpha=1}^{d^2} \lambda_\alpha |\alpha\rangle\langle\alpha|$. Take the Kraus operators $K_\alpha = \sum_i \sqrt{\lambda_\alpha} \langle i|\alpha\rangle\langle i|$ from the previous theorem. There is at most d^2 of them and it is easy to check that they satisfy

$$\mathrm{tr}(K_\alpha^\dagger K_\beta) = \delta_{\alpha\beta}\, \mathrm{tr}(K_\alpha^\dagger K_\beta) \qquad \blacksquare$$

Def.: (Rank of a CP map):

The rank of a CP map is the minimal number of Kraus operators it requires for its representation.

Lemma: The rank of a CP map $\mathcal{E}$ coincides with the rank of its Choi matrix $\mathcal{E}_c$.

Proof: From above, we know $\exists$ Kraus decomposition $\{K_\alpha\}_{\alpha=1}^{D}$ with $D = \text{rank}(\mathcal{E}_c)$. Now, let $\{K'_\beta\}_{\beta=1}^{D'}$ be another Kraus decomposition of $\mathcal{E}$. Then:

$$\mathcal{E}_c = \sum_{\beta=1}^{D'} |K'_\beta\rangle\langle K'_\beta| \quad , \quad \text{with } |K'_\beta\rangle = (K'_\beta \otimes \mathbb{1})|M\rangle .$$

$\Rightarrow$ Since $\text{rank}(\mathcal{E}_c)$ is the minimal number of pure states required for such a decomposition of $\mathcal{E}_c$, we have $\text{rank}(\mathcal{E}_c) \leq D'$. $\Rightarrow$ $\text{rank}(\mathcal{E}_c)$ is the minimal number of required Kraus operators.

$\blacksquare$

↳ Important: Every choice of Kraus operators corresponds to a squareroot X of $\mathcal{E}_c = X^\dagger X$

↳ All Kraus operators are of the form

$$K_\alpha = \sum_i \langle i | K_\alpha \rangle \langle i | \, , \quad \text{where } \mathcal{E}_c = \sum_\alpha |K_\alpha\rangle\langle K_\alpha| .$$

↳ The canonical representation stemming from the unique positive squareroot $X = \sum_\alpha \sqrt{\lambda_\alpha}\, |\alpha\rangle\langle\alpha|$ of $\mathcal{E}_c$ is called minimal.

cf. dilations

Lemma: ("Kraus decompositions are related by isometries")

Denote the minimal Kraus decomposition of $\mathcal{E}$ by $\{K_\alpha\}$. All other Kraus decompositions $\{K'_\beta\}$ of $\mathcal{E}$ satisfy

$$K'_\beta = \sum_\alpha V_{\beta\alpha} K_\alpha ,$$

where V is an isometry, i.e. $V^\dagger V = \mathbb{1}$.

↳ Characterization of $\underline{\underline{\text{all}}}$ Kraus decompositions.

Proof: Let X be the unique positive squareroot of $\mathcal{E}_c$ and let X' be some other squareroot of $\mathcal{E}_c$. All squareroots of $\mathcal{E}_c$ are related to X by isometry, i.e. $\exists\, V$ with $V^\dagger V = \mathbb{1}$ and $X' = VX$ (show this).

Now, let $\{K_\alpha\}$ be the Kraus operators "stemming from" X and let $\{K'_\beta\}$ be those stemming from X'.

$$\Rightarrow \quad |K'_\beta\rangle_\gamma = X'_{\beta\gamma} = (VX)_{\beta\gamma} = \sum_\alpha V_{\beta\alpha} X_{\alpha\gamma} = \sum_\alpha V_{\beta\alpha} |K_\alpha\rangle_\gamma$$

$$\Rightarrow \quad |K'_\beta\rangle = \sum_\alpha V_{\beta\alpha} |K_\alpha\rangle$$

$$\hookrightarrow \quad K'_\beta = \sum_\alpha V_{\beta\alpha} K_\alpha$$

$\boxtimes$