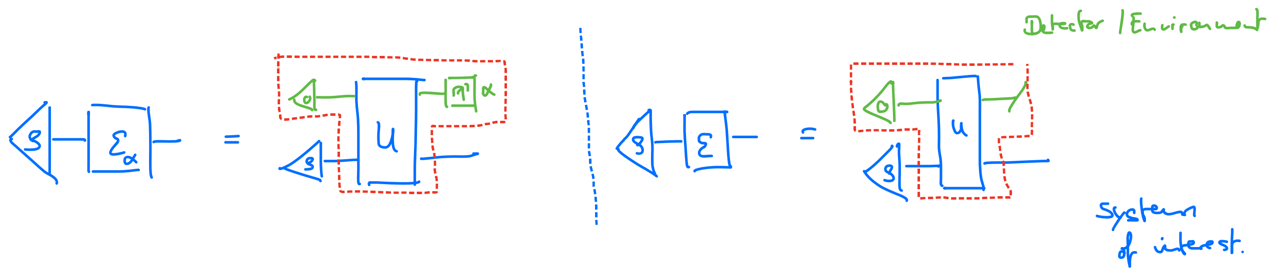
Recap:

Any map $\underbrace{\mathcal{E}_\alpha[\rho] = \sum_\alpha K_\alpha \rho K_\alpha^\dagger}_{\text{CP map}}$, with $\underbrace{\sum_\alpha K_\alpha^\dagger K_\alpha \leq \mathbb{1}}_{\text{trace non-increasing}}$ can be understood as coming from a unitary circuit with a measurement.

Any map $\underbrace{\mathcal{E}[\rho] = \sum_\alpha K_\alpha \rho K_\alpha^\dagger}_{\text{CP map}}$, with $\underbrace{\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}}_{\text{trace preserving}}$ can be understood as coming from a unitary circuit where the environment is discarded



3.1 CP and CPTP maps - proper introduction

What properties should the temporal evolution of a quantum state satisfy?

physical answer

axiomatic

It should follow from unitary dynamics

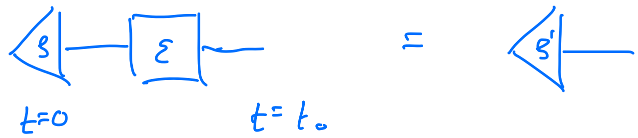
equivalent statements

It should "preserve the properties" of quantum states

Church of the larger Hilbert space

<u>Axiomatic answer:</u> Quantum states satisfy
$\rho \geq 0$, $tr(\rho) = 1$

<u>Evolution:</u>



<u>Requirements</u> $\mathcal{E}$ should be linear (always assumed)

$$\underbrace{\mathcal{E}[\rho] \geq 0 \quad \forall\, \rho \geq 0}_{\text{positive map}} \quad \& \quad \underbrace{tr[\mathcal{E}[\rho]] = tr[\rho] = 1}_{\text{trace preserving}}$$

<u>But:</u> Positivity should also be preserved when $\mathcal{E}$ only acts on a part of a larger state $\eta \in L(\mathcal{H}_1 \otimes \mathcal{H}_a)$, i.e.

$$(\mathcal{E} \otimes I_a)[\eta] \geq 0 \qquad \forall\, \eta \geq 0$$

where I_a is the identity map on $L(\mathcal{H}_a)$, and $\mathcal{H}_a$ is arbitrary. This property is called complete positivity (CP).

<u>Example of a positive but not completely positive map</u>

Transposition map T in a fixed basis, i.e.

$T[|i\rangle\langle j|] = |j\rangle\langle i|$. $\rightsquigarrow$ positive map.

But: when acting on half of a maximally entangled state $|M\rangle\langle M| = \frac{1}{d}\sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j|$, we obtain:

$$(T\otimes I)[|M\rangle\langle M|] = \frac{1}{d}\sum_{ij} |j\rangle\langle i| \otimes |i\rangle\langle j| =: R$$

It is easy to check that R has eigenvalues $\{1/2, 1/2, 1/2, -1/2\}$ for two-qubit $|M\rangle$.

$$\boxed{\begin{array}{l}\textbf{Def.:}\ \text{A completely positive (CP) and trace preserving (TP) map } \mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_{1'}) \text{ is called} \\ \text{a quantum channel}\end{array}}$$

↳ Most general transformation that map states onto states.

Examples: unitary maps $\mathcal{U}: L(\mathcal{H}_1) \to L(\mathcal{H}_1)$ with $\mathcal{U}[\rho] = U\rho U^\dagger$

$$\mathrm{tr}[\mathcal{U}[\rho]] = \mathrm{tr}[U\rho U^\dagger] = \mathrm{tr}[U^\dagger U\rho] = \mathrm{tr}[\rho] \quad (\mathrm{TP})$$

$$(\mathcal{U}\otimes I)[\eta] = (U\otimes \mathbb{1})\,\eta\,(U^\dagger\otimes\mathbb{1}) =: \eta'$$

$$\hookrightarrow \langle x|\eta'|x\rangle = \langle x|(U\otimes\mathbb{1})\,\eta\,(U^\dagger\otimes\mathbb{1})|x\rangle$$

$$=: \langle x'|\eta|x'\rangle \geq 0 \quad \forall \eta \geq 0 \quad (\mathrm{CP})$$

trace map $tr : L(\mathcal{H}_n) \to \mathbb{C}$

↳ Homework: Show it is CPTP

Def.: A collection $\{\mathcal{E}_\alpha\}$ of CP, trace non-increasing maps that add up to a CPTP map (i.e. $\sum_\alpha \mathcal{E}_\alpha = \mathcal{E}$ is CPTP) is called an instrument.

↳ Are CP trace non-increasing and CPTP maps actually physical.

4. Representations of Quantum channels

4.1 Kraus representation

Thm: (Kraus decomposition)

A linear map $\mathcal{E} : L(\mathcal{H}_n) \to L(\mathcal{H}_{n'})$ is completely positive iff it can be represented as

$$\mathcal{E}[S] = \sum_\alpha K_\alpha S K_\alpha^\dagger \qquad (CP).$$

with $K_\alpha : \mathcal{H}_n \to \mathcal{H}_{n'}$.

In addition, it is trace non-increasing iff $\sum_\alpha K_\alpha^\dagger K_\alpha \leq \mathbb{1}$, and trace preserving iff $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}$.

Proof: "If" part ("Only if" further below).

Let $\mathcal{E}[\rho] = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$

Now, take arbitrary $\eta \in \mathcal{L}(\mathcal{H}_n \otimes \mathcal{H}_a)$ with $\eta \geq 0$.

Then $(\mathcal{E} \otimes \mathcal{I}_a)[\eta] = \sum_\alpha (K_\alpha \otimes \mathbb{1}_a)\, \rho\, (K_\alpha^\dagger \otimes \mathbb{1}_a) =: \eta'$

For arbitrary $|x\rangle \in \mathcal{H}_n \otimes \mathcal{H}_a$:

$$\langle x | \eta' | x\rangle = \sum_\alpha \langle x | (K_\alpha \otimes \mathbb{1}_a)\, \eta\, (K_\alpha^\dagger \otimes \mathbb{1}_a) | x\rangle$$

$$=: \sum_\alpha \underbrace{\langle x_\alpha' | \eta | x_\alpha' \rangle}_{\geq 0} \geq 0 \quad \checkmark$$

For trace preservation:

Observe that $\mathrm{tr}(\mathcal{E}[\rho]) = \sum_\alpha \mathrm{tr}[K_\alpha \rho K_\alpha^\dagger] =$

$$= \sum_\alpha \mathrm{tr}(K_\alpha^\dagger K_\alpha \rho) = \mathrm{tr}\left[\left(\sum_\alpha K_\alpha^\dagger K_\alpha\right) \rho\right]$$

if $\sum_\alpha K_\alpha^\dagger K_\alpha \leq \mathbb{1} \Rightarrow \mathrm{tr}\left(\sum_\alpha K_\alpha^\dagger K_\alpha \rho\right) \leq \mathrm{tr}(\rho)$

if $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1} \Rightarrow \mathrm{tr}\left(\sum_\alpha K_\alpha^\dagger K_\alpha \rho\right) = \mathrm{tr}(\rho)$ ∎

Example: Partially depolarizing channel

$$\mathcal{E}[\rho] = p \frac{\mathbb{1}}{d} \operatorname{tr}[\rho] + (1-p)\,\rho \qquad p \in [0,1]$$

$\rightarrow$ mixes the state with white noise.

$$\mathcal{E}[\rho] = \frac{p}{d} \sum_i |i\rangle\langle i| \sum_j \langle j|\rho|j\rangle + (1-p)\, \mathbb{1}\rho\mathbb{1}$$

$$= \sum_{ij} \sqrt{\tfrac{p}{d}}\, |i\rangle\langle j|\, \rho\, |j\rangle\langle i| \sqrt{\tfrac{p}{d}} + \sqrt{1-p}\, \mathbb{1}\, \rho\, \mathbb{1} \sqrt{1-p}$$

$K_0 := \sqrt{1-p}\, \mathbb{1}$, $K_{ij} := \sqrt{\tfrac{p}{d}}\, |i\rangle\langle j|$ $\Rightarrow$ CP. ✓

$$K_0^\dagger K_0 + \sum_{ij} K_{ij}^\dagger K_{ij} = (1-p)\mathbb{1} + \sum_{ij} \sqrt{\tfrac{p}{d}}\, |i\rangle\langle i|j\rangle\langle j| \sqrt{\tfrac{p}{d}}$$

$$= \mathbb{1}$$

From our previous considerations, we already know that <u>any</u> map of the form $\mathcal{E}[\rho] = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$ with $\sum_\alpha K_\alpha^\dagger K_\alpha \leq \mathbb{1}$ can be understood as coming from a unitary circuit (potentially with a projective measurement).

if TP: $\mathcal{E}[\rho] = \underset{\substack{\uparrow \\ \text{environment / detector.}}}{\operatorname{tr}_2} \left[U \left(\rho \otimes |0\rangle\langle 0|\right) U^\dagger \right]$

Thm.: (Stinespring dilation)

For any quantum channel $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_{1'})$ there exists a unitary $U \in L(\mathcal{H}_1 \otimes \mathcal{H}_a)$, such that

$$\mathcal{E}[\rho] = \mathrm{tr}_a \left[U (\rho \otimes |0\rangle\langle 0|) U^\dagger \right]$$

For any trace non-increasing CP map $\mathcal{E}_\alpha[\rho] = \sum_{\gamma=1}^{D} K_\alpha^{(\gamma)} \rho K_\alpha^{(\gamma)\dagger}$, there exists a unitary and a rank-D projector Π_γ such that

$$\mathcal{E}_\alpha[\rho] = \mathrm{tr}_a \left[U (\rho \otimes |0\rangle\langle 0|) U^\dagger (\mathbb{1}_1 \otimes \Pi_\gamma) \right]$$

Example for CPTP maps in Quantum Info:

Distinguishability under CPTP maps.

Claim: $\| \mathcal{E}(\rho_1) - \mathcal{E}(\rho_2) \|_1 \leq \| \rho_1 - \rho_2 \|_1 \quad \forall$ CPTP maps $\mathcal{E}$

Data processing inequality

Def.: (dual map): The dual map $\mathcal{E}^\dagger: L(\mathcal{H}_{1'}) \to L(\mathcal{H}_1)$ of a linear map $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_{1'})$ is defined via $\mathrm{tr}(A^\dagger \mathcal{E}[B]) =: \mathrm{tr}((\mathcal{E}^\dagger[A])^\dagger B) \quad \forall A, B.$

For Hermitian matrices and Hermiticity preserving maps, we have $\mathrm{tr}(A\,\mathcal{E}[B]) =: \mathrm{tr}(\mathcal{E}^\dagger[A]\,B)$

Lemma: $\mathcal{E}$ is CP $\Longleftrightarrow$ $\mathcal{E}^\dagger$ is CP

$\mathcal{E}$ is TP $\Longleftrightarrow$ $\mathcal{E}^\dagger$ is unital, i.e. $\mathcal{E}[\mathbb{1}] = \mathbb{1}$

Hint: $\mathrm{tr}(A\,\mathcal{E}[B]) = \sum_\alpha \mathrm{tr}(A K_\alpha B K_\alpha^\dagger) =$

$$= \sum_\alpha \mathrm{tr}(\underbrace{K_\alpha^\dagger A K_\alpha}_{\mathcal{E}^\dagger[A]} B)$$

Lemma: A unital map maps a POVM element onto a POVM element.

$\Rightarrow$ Proof of the data processing inequality

$$\|\mathcal{E}(\rho_1) - \mathcal{E}(\rho_2)\|_1 = 2 \max_{\underset{\nearrow\,\text{POVM elements}}{\{E_\alpha\}}} \mathrm{tr}(E_\alpha(\mathcal{E}[\rho_1 - \rho_2]))$$

$$= 2 \max_{\{E_\alpha\}} \mathrm{tr}(\mathcal{E}^\dagger(E_\alpha)(\rho_1 - \rho_2)) =$$

$$= 2 \max_{\{E_\alpha'\}} \mathrm{tr}(E_\alpha'(\rho_1 - \rho_2)) \leq \|\rho_1 - \rho_2\|_1 \qquad \blacksquare$$

4.2 Liouville representation

Idea: Since $\mathcal{E}$ is linear, it should be representable as a matrix acting on a vector.

$\Rightarrow$ "vectorization" of $\rho \in L(\mathcal{H}_n)$ via its action on half of $|\Omega\rangle = \sum_i |ii\rangle$

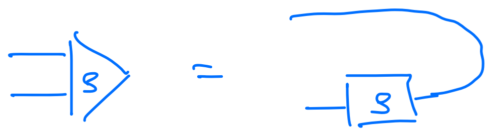
$$\Rightarrow \quad |\rho\rangle\rangle := (\rho \otimes \mathbb{1}) \sum_i |ii\rangle$$

$\uparrow$ vectorized version of ρ

$\hookrightarrow |\rho\rangle\rangle \in \mathcal{H}_n \otimes \mathcal{H}_n$

$\Rightarrow$ intuitively: $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j| \longleftrightarrow |\rho\rangle\rangle = \sum_{i,j} \rho_{ij} |i\rangle |j\rangle$

Pictographically:



What does the action of $\mathcal{E}$ look like in vectorized form?

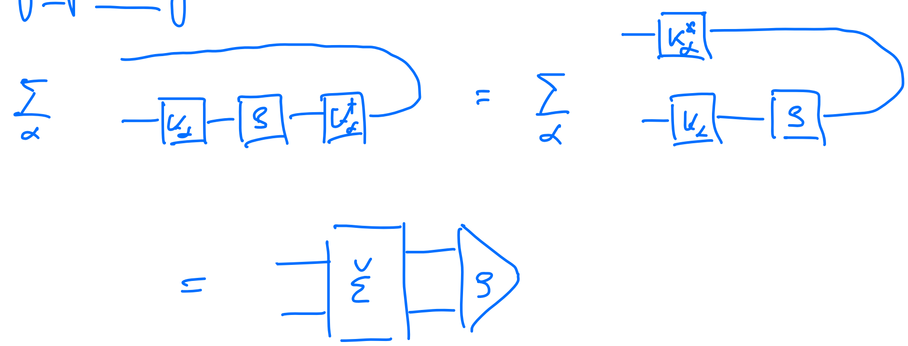
Let $\rho' = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$

$$\Rightarrow \quad |\rho'\rangle\rangle = \sum_\alpha (K_\alpha \rho K_\alpha^\dagger \otimes \mathbb{1}) |\Omega\rangle$$

$$= \sum_\alpha (K_\alpha \otimes K_\alpha^*) (\rho \otimes \mathbb{1}) |\Omega\rangle$$

$$= \sum_\alpha (K_\alpha \otimes K_\alpha^*)\, |S\rangle\rangle =: \underbrace{\check{\mathcal{E}}}_{\text{matrix}}\, \underbrace{|S\rangle\rangle}_{\text{vector}}$$

Pictographically:



$\Rightarrow$ $\check{\mathcal{E}}$: Liouvillian representation of $\mathcal{E}$

Pros: Action as matrix multiplication

Cons: Not "conceptually insightful"

4.3 Choi representation

Idea: Represent $\mathcal{E}$ as a matrix $\mathcal{E}_c$ that makes it easy to check the properties of $\mathcal{E}$.

Def.: (Choi-Jamiołkowski isomorphism (CJI))

Let $\mathcal{E}: L(\mathcal{H}_n) \to L(\mathcal{H}_{n'})$ be a linear map. Its Choi matrix $\mathcal{E}_c \in L(\mathcal{H}_{n'} \otimes \mathcal{H}_n)$ is given by

$$\mathcal{E}_c := \mathrm{Choi}(\mathcal{E}) := (\mathcal{E} \otimes I)\big[|M\rangle\langle M|\big]$$

where $|M\rangle = \sum_i |ii\rangle$ is the unnormalized maximally entangled state on $\mathcal{H}_n \otimes \mathcal{H}_n$

⇒ The matrix $\mathcal{E}_c$ is given by $\mathcal{E}_c = \sum_{ij} \mathcal{E}[|i\rangle\langle j|] \otimes |i\rangle\langle j|$

⇒ Mathematically:

$$\underbrace{L(L(\mathcal{H}_n), L(\mathcal{H}_{n'}))}_{\text{maps } \mathcal{E}: L(\mathcal{H}_n) \to L(\mathcal{H}_{n'})} \cong \underbrace{L(\mathcal{H}_n \otimes \mathcal{H}_{n'})}_{(d_n \times d_{n'}) \times (d_n \times d_{n'}) \text{ matrices}}$$

Important: from now on: good "accounting of spaces.

$L(\mathcal{H}_n)$: input space of $\mathcal{E}$
$L(\mathcal{H}_{n'})$: output space of $\mathcal{E}$

Lemma: (Action of $\mathcal{E}$ in terms of $\mathcal{E}_c$)

The action of a map $\mathcal{E}: L(\mathcal{H}_n) \to L(\mathcal{H}_{n'})$ on a state $S \in L(\mathcal{H}_n)$ can be expressed in terms of its Choi matrix $\mathcal{E}_c \in L(\mathcal{H}_n \otimes \mathcal{H}_{n'})$ as

$$\mathcal{E}[S] = \underbrace{tr_1}_{\substack{\text{trace over} \\ \text{the input space}}} [\mathcal{E}_c (\underbrace{\mathbb{1}_{n'}}_{\substack{\text{identity matrix} \\ \text{on the output space}}} \otimes \underbrace{S^T}_{\text{transpose of } S})]$$

Proof: By direct insertion

$$\mathrm{tr}_1 [\mathcal{E}_c (\mathbb{1}_{1'} \otimes S^T)] = \sum_{ij} \mathrm{tr}_1 [(\mathcal{E}[|i\rangle\langle j|] \otimes |i\rangle\langle j|)(\mathbb{1}_{1'} \otimes S^T)]$$

$$= \sum_{iju} \mathcal{E}[|i\rangle\langle j|] \langle u|i\rangle\langle j|S^T|u\rangle =$$

$$= \sum_{ij} \mathcal{E}[|i\rangle\langle j|] \langle j|S^T|i\rangle \underbrace{=}_{\text{linearity}} \mathcal{E}\Big[\sum_{ij} \langle j|S^T|i\rangle |i\rangle\langle j|\Big]$$

$$= \mathcal{E}\Big[\sum_{ij} S_{ij} |i\rangle\langle j|\Big] = \mathcal{E}[S]$$ ☒

Thm.: (Choi matrix of CPTP maps):

A linear map $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_{1'})$ is CP and TP iff its Choi state $\mathcal{E}_c \in L(\mathcal{H}_{1'} \otimes \mathcal{H}_1)$ satisfies:

$$\mathcal{E}_c \geq 0 \quad (CP) \quad \& \quad \mathrm{tr}_{1'}(\mathcal{E}_c) = \mathbb{1}_1$$

It is trace non-increasing iff $\mathrm{tr}_{1'}(\mathcal{E}_c) \leq \mathbb{1}_1$

Proof: if $\mathcal{E}$ is CP, then $(\mathcal{E} \otimes \mathcal{I})[|M\rangle\langle M|] \geq 0$.

if $\mathcal{E}$ is TP:

$$\mathrm{tr}[\mathcal{E}[S]] = \mathrm{tr}_{1'}[\mathrm{tr}_1(\mathcal{E}_c(\mathbb{1}_{1'} \otimes S^T))] =$$

$$= \mathrm{tr}[\mathrm{tr}_{1'}[\mathcal{E}_c] S] = \mathrm{tr}[S] \quad \forall S$$

$\hookrightarrow \mathrm{tr}_{1'}[\mathcal{E}_c] = \mathbb{1}_1$

"$\Leftarrow$" Let $\mathcal{E}_c \geq 0$ and $\mathrm{tr}_{1'}[\mathcal{E}_c] = \mathbb{1}_1$

TP: By direct insertion

CP: $\mathcal{E}_c \geq 0 \Rightarrow \mathcal{E}_c = \sum_\alpha \lambda_\alpha |\alpha\rangle\langle\alpha| = \sum_\alpha \sqrt{\lambda_\alpha} |\alpha\rangle\langle\alpha| \sqrt{\lambda_\alpha}$

$$\mathcal{E}[\rho] = \mathrm{tr}_1 [\mathcal{E}_c (\mathbb{1}_{1'} \otimes \rho^T)] =$$

$$= \sum_i \sum_\alpha \sqrt{\lambda_\alpha} \underbrace{\langle i|}_{\in \mathcal{H}_1} \underbrace{\alpha\rangle\langle\alpha|}_{\in \mathcal{H}_1 \otimes \mathcal{H}_{1'}} (\mathbb{1}_{1'} \otimes \rho^T) |i\rangle \sqrt{\lambda_\alpha}$$

$$= \sum_{ij} \sum_\alpha \sqrt{\lambda_\alpha} \langle i|\alpha\rangle\langle\alpha|j\rangle\langle j| \rho^T |i\rangle \sqrt{\lambda_\alpha}$$

$$= \sum_\alpha \underbrace{\sqrt{\lambda_\alpha} \sum_i \langle i|\alpha\rangle\langle i|}_{=: K_\alpha} \rho \underbrace{\sum_j |j\rangle\langle\alpha|j\rangle \sqrt{\lambda_\alpha}}_{K_\alpha^\dagger}$$

$$= \sum_\alpha K_\alpha \rho K_\alpha^\dagger \Rightarrow \text{completely positive}$$

$\boxtimes$