Lecture 3 : Measurements & Dynamics in QM

Up to this point: Static / kinematic part of QM

Recall: A pure quantum state is given by a normalized vector $|\psi\rangle \in \mathbb{C}^n = \mathcal{H}$

A mixed quantum state is given by a matrix $\rho \in L(\mathcal{H})$, $\rho \geq 0$, $tr(\rho) = 1$

Which of the two is more general?

Obvious: $|\psi\rangle\langle\psi| \geq 0$ & $tr(|\psi\rangle\langle\psi|) = 1$

$\hookrightarrow$ density matrix

But also: For every $\rho = \sum_i \lambda_i |i\rangle\langle i| \in L(\mathcal{H})$

$\exists$ (infinitely many) pure states

$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |i\rangle|v_i\rangle \in \mathcal{H} \otimes \mathcal{H}_2$,

such that $tr_2(|\psi\rangle\langle\psi|) = \rho$ (Purification)

[Show that $\rho \geq 0 \Rightarrow \rho = \rho^\dagger$]

Technically: pure state QM "includes everything"

$$\boxed{\text{Church of the large Hilbert space}}$$

Why care about mixed states / non-unitary dynamics?

1.) Often: Only access to part of a system
(Recall: reduced states of entangled states are mixed)

2.) External noise / interactions lead to uncertainty about "what pure" state

↳ Necessitates ensemble description

$$\rho = \sum_i \lambda_i |i\rangle\langle i|$$

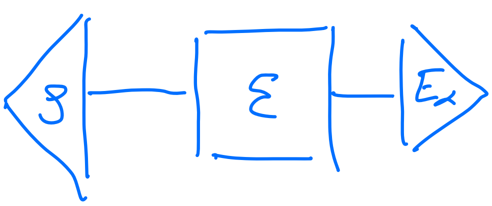
probability

pure state

3. Axiomatically: Mixed states as the most general object that contains all statistical information

↳ Gleason's theorem (see arXiv:quant-ph/9909073)

measurement



States evolution

Measurements:

1. Projective measurements

Recall: Let $A = A^\dagger$ be an observable with $A = \sum_\alpha a_\alpha |\alpha\rangle\langle\alpha|$, Set $\Pi_\alpha := |\alpha\rangle\langle\alpha|$. Then, measuring this observable on pure state $|\psi\rangle \in \mathbb{C}^d$ yields outcome a_α with probability

$$\underbrace{\boxed{\mathbb{P}(a_\alpha) = |\langle\alpha|\psi\rangle|^2 = \mathrm{tr}\left(|\psi\rangle\langle\psi|\,\Pi_\alpha\right)}}_{\text{"Born Rule"}}$$

By linearity, this extends to mixed states, i.e. the probability to observe a_α when measuring the observable A on a mixed state ρ is given by

$$\boxed{\mathbb{P}(a_\alpha) = \mathrm{tr}(\rho\,\Pi_\alpha)} \quad \text{with} \quad \underbrace{\boxed{\sum_\alpha \Pi_\alpha = \mathbb{1} \qquad \& \; \Pi_\alpha \Pi_{\alpha'} = \delta_{\alpha\alpha'} \Pi_\alpha}}_{\text{Projection-valued measure (PVM)}}$$

Each projector corresponds to an outcome

Normalization

$\Pi_\alpha \Pi_{\alpha'} = \delta_{\alpha\alpha'} \Pi_\alpha$

(all outcomes are perfectly distinguishable)

Are PVMs all there is?

2. POVMs

Recall: Projectors have positive eigenvalues (show this!)

i.e. $\Pi_\alpha \geq 0$

$\to$ Replace projectors in POVM with positive matrices $\{E_\alpha\}$, such that:

$$\boxed{E_\alpha \geq 0 \quad (\text{"positivity"}) \qquad \sum_\alpha E_\alpha = \mathbb{1} \quad (\text{normalization}) \qquad \& \quad \underbrace{P(\alpha) = \mathrm{tr}(E_\alpha \rho)}_{\text{"Born Rule"}}}$$

guarantees positivity of probabilities

Positive operator valued measure (POVM)

number of outcomes does not have to coincide with the dimension of ρ

Note: $E_\alpha E_{\alpha'} \neq \delta_{\alpha\alpha'} E_\alpha$ (unlike $\Pi_\alpha \Pi_{\alpha'} = \delta_{\alpha\alpha'} \Pi_\alpha$)

Obvious: $E_\alpha = \Pi_\alpha$ is a possible choice of POVM

$\hookrightarrow$ PVMs $\subseteq$ POVMs

Example: SIC - POVM : $\{|\phi_\alpha\rangle\langle\phi_\alpha|\}_{\alpha=0}^{3}$ on qubit.

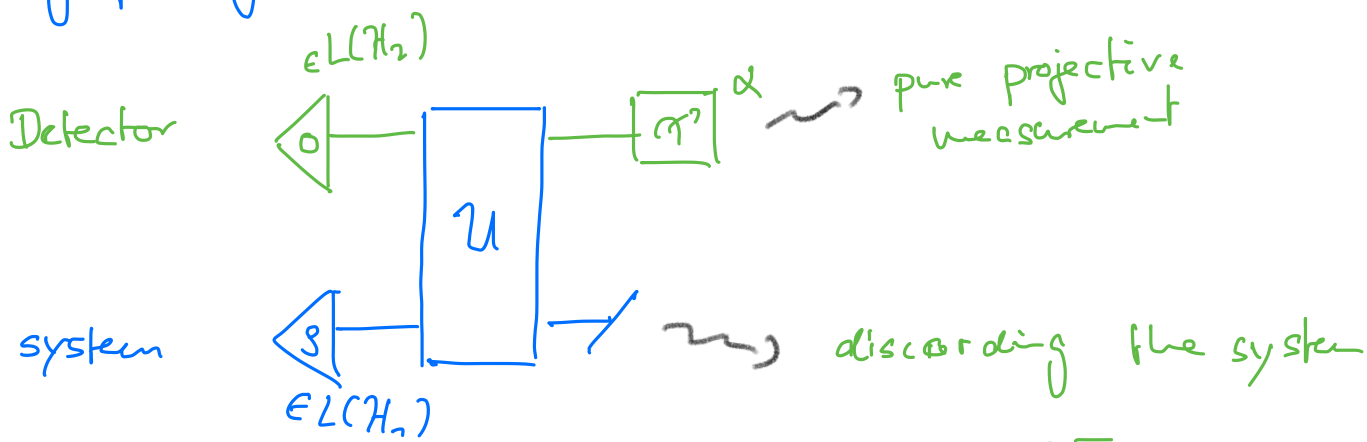
$\langle\phi_\alpha|\phi_\beta\rangle \neq \delta_{\alpha\beta}$ (check wikipedia)

Do POVMs make physical sense?

$\boxed{\text{von-Neumann prescription of a measurement:}}$

System is probed by coupling it to a detector and reading out the detector.

Pictographically:



$$\Rightarrow P(\alpha) = \mathrm{tr}\Big[\big(U(\rho\otimes|0\rangle\langle 0|)U^\dagger\big)\underbrace{\big(\mathbb{1}\otimes|\alpha\rangle\langle\alpha|\big)}_{\widehat{=}\,\Pi_\alpha}\Big] =$$

$$= \sum_{i\beta} \langle i\beta|U(\rho\otimes|0\rangle\langle 0|)U^\dagger(\mathbb{1}\otimes|\alpha\rangle\langle\alpha|)|i\beta\rangle$$

$\delta_{\kappa\beta}$

$$= \sum_i \langle i|\big(\langle\alpha|U|0\rangle\,\rho\,\langle 0|U^\dagger|\alpha\rangle\big)|i\rangle$$

$$= \mathrm{tr}\big(\langle\alpha|U|0\rangle\,\rho\,\langle 0|U^\dagger|\alpha\rangle\big)$$

$$= \mathrm{tr}\big(\rho\,\underbrace{\langle 0|U^\dagger|\alpha\rangle\langle\alpha|U|0\rangle}_{=:E_\alpha}\big) =: \mathrm{tr}(\rho E_\alpha)$$

i) $\sum_\alpha E_\alpha = \sum_\alpha \langle 0| U^\dagger |\alpha\rangle\langle\alpha| U |0\rangle = \mathbb{1}$ (normalization)

$\hookrightarrow \in \mathcal{H}_2$

ii) $E_\alpha = \langle 0| U^\dagger |\alpha\rangle\langle\alpha| U |0\rangle =: X_\alpha^\dagger X_\alpha \geq 0$ (positivity)

show that $A^\dagger A \geq 0 \quad \forall A$!

$$\boxed{\text{Every projective measurement leads to a POVM}}$$

Does the converse hold?

$\hookrightarrow$ Can every POVM be understood as coming from a projective measurement in a larger space

Naimark Dilation

Given a POVM $\{E_\alpha\}$, let us construct an isometry

$$V := \sum_\alpha \sqrt{E_\alpha} \otimes |\alpha\rangle \quad ; \quad \text{i.e.} \quad V_{i\alpha;j} = \langle i\alpha|V|j\rangle = \langle i|\sqrt{E_\alpha}|j\rangle$$

$$\sqrt{E_\alpha}^\dagger \sqrt{E_\alpha} = E_\alpha$$

This is indeed an isometry, since

$$V^\dagger V = \sum_{\gamma\alpha} \sqrt{E_\gamma}^\dagger \sqrt{E_\alpha} \underbrace{\langle\gamma|\alpha\rangle}_{\delta_{\gamma\alpha}} = \sum_\alpha E_\alpha = \mathbb{1}$$

Pictographically: $$V = \overbrace{\begin{pmatrix} \quad & \quad \\ & \\ & \end{pmatrix}}^{d_2} \left.\vphantom{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}\right\} d_1 \times d_2$$

d_2 = number of POVM elements

Due to $V^\dagger V = \mathbb{1}$, all columns of V are orthonormal and V can thus be understood as part of a unitary U on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$V_{i\alpha, j} = \langle i\alpha|V|j\rangle = \langle i\alpha|U|j0\rangle \quad \left(= \langle i|\sqrt{E_\alpha}|j\rangle\right)$$

For this unitary, by construction, we obtain

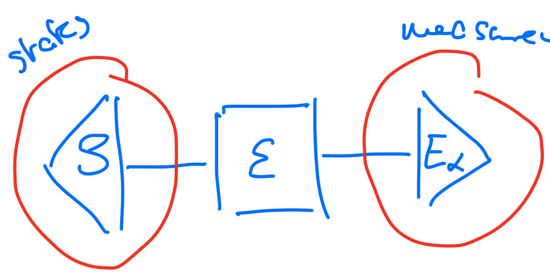
$$\boxed{\underbrace{\mathrm{tr}\left(U(\rho \otimes |0\rangle\langle 0|)U^\dagger \, (\mathbb{1} \otimes |\alpha\rangle\langle\alpha|)\right)}_{\text{von-Neumann-measurement}} = \underbrace{\mathrm{tr}(\rho E_\alpha)}_{\text{POVM}}}$$

> **Every POVM can be understood as a von-Neumann measurement on a larger space**

"Naimark dilation"

Church of the larger Hilbert space.

2) so far



3.1 CP and CPTP maps - a mathematical interlude

Above, we have actually proven more than "just" the Naimark dilation. For a given POVM element E_α, let K_α be such that $E_\alpha = K_\alpha^\dagger K_\alpha$ (i.e. K_α is one of the $\boxed{\text{non-unique}}$ squareroots of E_α)

→ exists, and are related by isometry

$$K_\alpha = V \underbrace{\sqrt{E_\alpha}}_{\text{unique positive root of } E_\alpha.}$$

Then, we have:

$$\boxed{tr(\rho E_\alpha) = tr(K_\alpha \rho K_\alpha^\dagger) =: tr(\mathcal{E}_\alpha[\rho])}$$

where $\mathcal{E}_\alpha : L(\mathcal{H}_1) \to L(\mathcal{H}_1)$ is a linear map

Not in general, but let's assume it for the moment.

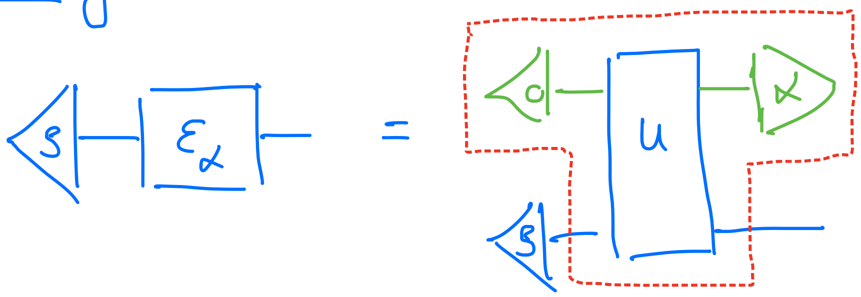
⇒ Above, we have proven that, for such a map $\mathcal{E}_\alpha[\rho] = K_\alpha \rho K_\alpha^\dagger$, there exists a unitary $U \in L(\mathcal{H}_1 \otimes \mathcal{H}_2)$

with $K_\alpha = \langle \alpha | U | 0 \rangle$. Consequently, we obtain:

$$\boxed{\mathcal{E}_\alpha[\rho] = tr_2 [U (\rho \otimes |0\rangle\langle 0|) U^\dagger (\mathbb{1} \otimes |\alpha\rangle\langle\alpha|)]}$$

(show this)

Pictographically:



Describes the transformation of the state ρ upon observing outcome α.

The probability of an outcome is given by

$$\boxed{\mathbb{P}(\alpha) = \operatorname{tr}\left(\mathcal{E}_\alpha[\rho]\right)} \qquad \left(= \operatorname{tr}\left(K_\alpha \rho K_\alpha^\dagger\right) = \operatorname{tr}(\rho E_\alpha)\right)$$

Since $0 \leq \mathbb{P}(\alpha) \leq 1$, we see that $\mathcal{E}_\alpha$ is trace non-increasing and from $\sum_\alpha E_\alpha = \mathbb{1}$, we obtain

$$\boxed{\sum_\alpha \operatorname{tr}\left(\mathcal{E}_\alpha[\rho]\right) = \operatorname{tr}\left(\sum_\alpha \mathcal{E}_\alpha[\rho]\right) =: \operatorname{tr}\left(\mathcal{E}[\rho]\right) = \operatorname{tr}(\rho)}$$

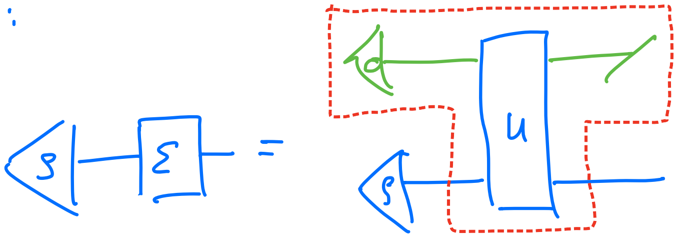
the map $\mathcal{E}$ is trace preserving.

We see that

$$\mathcal{E}[\rho] = \sum_\alpha \operatorname{tr}_2\left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger\left(\mathbb{1} \otimes |\alpha\rangle\langle\alpha|\right)\right]$$

$$= \operatorname{tr}_2\left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger\right]$$

Pictographically:



What we have shown is thus, that any map $\boxed{\mathcal{E}_\alpha[\rho] = K_\alpha \rho K_\alpha^\dagger}$ that satisfies $\boxed{K_\alpha^\dagger K_\alpha \leq \mathbb{1}}$ can be understood as coming from a unitary dynamics and a projective measurement on a larger Hilbert space.

Analogously, any map $\boxed{\mathcal{E}[\rho] = \sum_\alpha K_\alpha \rho K_\alpha^\dagger \quad \text{with} \ \sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}}$

can be understood as coming from a unitary dynamics on a large Hilbert space and a discarding of the auxiliary degrees of freedom.