① Entropy and Mutual Information

Let us start with a random variable X with a probability distribution $P_X(x), x \in \mathcal{X}$

How can we measure the uncertainty that this random variable is carrying?

A possible way to do that was introduced by Shannon:

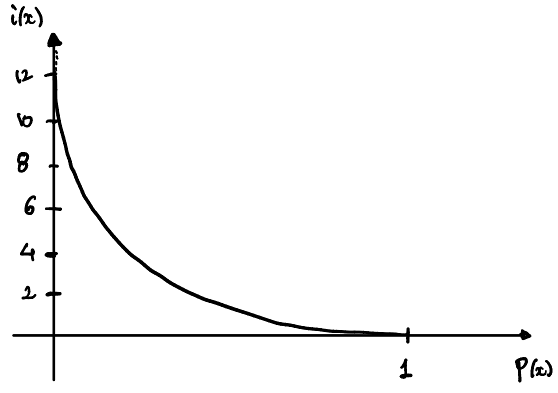
- An information bit is a measure of how much we learn from the outcome of a random event: information as surprise or uncertainty

Thus the information content of having observed $x \in \mathcal{X}$ is

$$i(x) = -\log(P_X(x))$$

Information in unit of bits

⊛ Log ~ logarithm in base 2



An event has a: lower surprise if it is more likely to occur;
- higher surprise if it is less likely to occur;

The amount of surprisal of a random variable X is the average of $i(x)$:

$$H(X) = \mathbb{E}_X\{i(x)\} = -\sum_x P_X(x)\log P_X(x)$$

We call this the entropy of X

Properties: 1) $H(X) \geq 0$; $H(X) = 0 \iff X$ is deterministic.

2) $H(X) \leq \log |X|$, equality $\iff X$ is uniform

• — • — • — •

We now move into considering two random variables X and Y possessed respectively by Alice and Bob. The two random var. might be correlated:

Let us say that Bob observe the outcome y and that communicate this to Alice. The information content that Alice has is now given as
uncertainty

$$i(x|y) = -\log \{ P_{X|Y}(x|y) \} \qquad P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

The average of this quantity is the Conditional Entropy

$$H(X|Y) = -\sum_{x,y} P_{X,Y}(x,y) \log \{ P_{X|Y}(x|y) \}$$

It corresponds to the average information gain on X by knowing Y.

Properties: 1) It seems reasonable that knowing Y should reduce uncertainty on X

$$H(X) \geq H(X|Y)$$

! ⊛ Not the same in quantum

In the case Bob doesn't know the outcome y → the uncertainty is simply measured by the Joint Entropy

$$H(X,Y) = \mathbb{E}_{X,Y} \{ i(X,Y) \}$$

Exc:
(*) Showing that

1) Chain rule

- $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$
- $H(X_1,\dots,X_n) = H(X_1) + H(X_2|X_1) + H(X_3|X_2X_1) + \dots + H(X_n|X_{n-1}\dots X_1)$

2) Subadditivity (Hint: use $H(X|Y) \leq H(X)$)

$$H(X_1,\dots,X_n) \leq \sum_{i=1}^{n} H(X_i) \text{ ; equality} \Leftrightarrow \{X_i\}_{i=1}^{n} \text{ are independent}$$

•—•—•—•

A measure of correlation corresponds to the amount of uncertainty reduction we have on X by knowing Y: the mutual information

$$I(X:Y) = H(X) - H(X|Y)$$

It is a measure of dependence ~ correlations of X and Y
It measure how much knowing Y reduce the uncertainty on X

Properties: 1) Symmetry $I(X:Y) = I(Y:X)$

2) Non-negativity $I(X:Y) \geq 0$; equality $\Leftrightarrow$ X and Y are independent

3) If X is a deterministic function of Y and viceversa $\rightarrow$

$$H(X|Y) = 0 \Rightarrow I(X:Y) = H(X)$$

4) $I(X:Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(XY)$
$= H(Y) - H(Y|X) =$

Quantity	Symbol	Meaning
Entropy	$H(X)$	Uncertainty / Surprisal on X
Conditional Entropy	$H(X\|Y)$	Uncertainty on X knowing Y
Joint entropy	$H(X,Y)$	Uncertainty on X and Y
Mutual information	$I(X:Y)$	Uncertainty gain on X by knowing Y

References: Wilde, Mark M. "From classical to quantum Shannon theory." *arXiv preprint arXiv:1106.1445* (2011)

https://arxiv.org/abs/1106.1445 → more specifically Chapter 10 of version n. 8

(2) Quantum Entropy and Quantum Mutual Information

Generalization of Shannon entropy by replacing prob. with density matrix :

Von Neumann entropy $\quad H(A)_\rho = -\operatorname{Tr}\{\rho_A \log \rho_A\} = H(\rho_A)$

Captures both classical and quantum uncertainty

Properties : 1) $H(\rho) \geq 0$; $H(\rho) \leq \log d$, d dim. of $\mathcal{H}$; eq $\Leftrightarrow \rho = \frac{\mathbb{1}_d}{d}$

2) $H(\rho) = 0 \Leftrightarrow \rho = |\psi\rangle\langle\psi|$ ~ analogous of deterministic random variable

3) Concavity

$$H(\rho) \geq \sum_x p_X(x) H(\rho_x) \quad \text{for} \quad \rho = \sum_x p_X(x) \rho_x$$

4) Isometric invariance $\quad H(\rho) = H(U\rho U^\dagger) \ \forall U$ unitary

Von Neumann entropy has a "classical" interpretation : it corresponds to the minimum Shannon Entropy when rank-one POVM is performed :

Given ρ, the quantum entropy of ρ, $H(\rho)$, is the minimum Shannon entropy among all the probab. distribution that can be obtained by ρ.

$$H(\rho) = \min_{\{\Lambda_y\}} H(Y) = \min_{\{\Lambda_y\}} \left\{ -\sum_y \operatorname{Tr}\{\Lambda_y \rho\} \log \operatorname{Tr}\{\Lambda_y \rho\} \right\}$$

the right question to ask, the one with the minimum uncertainty

↳ make more clear why $H(|\psi\rangle\langle\psi|) = 0$

- Joint quantum entropy is straight. generalization

$$H(\rho_{AB}) = -\mathrm{Tr}\{\rho_{AB} \log \rho_{AB}\}$$

- Conditional quantum entropy (measurement disturbance)

No formal notion of conditioning in quantum theory.

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$$

- Quantum Mutual Information

$$I(A:B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho =$$

$$= H(A)_\rho - H(A|B)_\rho =$$

$$= H(B)_\rho - H(B|A)_\rho$$

- Properties: 1) Non-negativity $I(A:B)_\rho \geq 0$

2) Upper-bound $I(A:B)_\rho \leq 2 \log [\min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}]$

3) monotonic under CPTP $\Rightarrow I(A:B)_{\mathcal{E}[\rho]} \leq I(A:B)_\rho$

All good, it seems that we have a notion of correlations at the quantum level that corresponds to a straightforward extension of classical notions.

So we are happy ... or maybe no?

What is the issue? Do these quantities really correspond to classical quantities?

References: Wilde, Mark M. "From classical to quantum Shannon theory." *arXiv preprint arXiv:1106.1445* (2011)

https://arxiv.org/abs/1106.1445 → Part of Chapter 11

- A communication problem/task between Alice and Bob

Chapter 12.1.1

References: Nielsen, Michael A., and Isaac L. Chuang. Quantum computation and quantum information. Cambridge university press, 2010.

Alice prepare an ensemble $\{p_X(x), \rho_x^B\}$ and send it to Bob.

This can be represented by $\rho_{AB} = \sum_x p_X(x)\, |x\rangle\langle x|_A \otimes \rho_x^B$

Bob measure using a POVM $\{M_y^B\}$: this corresponds to a random variable Y with probabilities $p_Y(y) = \mathrm{tr}\{\mathbb{1} \otimes M_y^B \rho_{AB}\}$

The goal of B is to find the value of x that Alice has prepared

On average this means that B should reproduce $p_X(x)$ with $p_Y(y)$: this corresponds to have a maximum mutual information since in this case it would mean that X is deterministic if Y is known and viceversa.

The classical mutual information $I(X:Y) \sim p_{XY}(x,y)$

Now, the measurement of Bob can be described by a quantum instrument

$$\mathcal{M}_y^B : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_B)$$

$$\mathcal{M}_y^B[\rho] = W\sqrt{M_y^B}\,\rho\,\sqrt{M_y^B}\,W^\dagger$$

$$p_Y(y)_\rho = \mathrm{tr}\{M_y^B \rho\} \qquad \rho_y = \frac{W\sqrt{M_y^B}\,\rho\,\sqrt{M_y^B}\,W^\dagger}{\mathrm{tr}\{M_y^B \rho\}}$$

$\mathcal{M}^B : \mathcal{L}(\mathcal{H}^B) \mapsto \mathcal{L}(\mathcal{H}^B \otimes \mathcal{H}^Q)$ $\quad$ H_Q a quantum register

$$\mathcal{M}^B[\rho] = \sum_y \mathcal{M}^B_y[\rho] \otimes |y\rangle\langle y|_Q = \sum_y P_Y(y)\ \rho_y \otimes |y\rangle\langle y|_Q$$

$$\mathcal{M}^B[\rho^B_x] = \sum_Y \underbrace{\operatorname{tr}\{M^B_y \rho^B_x\}}_{P_{Y|X}(y|x)} \underbrace{\frac{W\sqrt{M^B_y}\,\rho^B_x\,\sqrt{M^B_y}\,W^\dagger}{P_{Y|X}(y|x)}}_{=\rho^B_{y|x}} \otimes |y\rangle\langle y|_Q =$$

$$= \sum_y P_{Y|X}(y|x)\ \rho^B_{y|x} \otimes |y\rangle\langle y|_Q$$

So the overall state is given as

$$\rho^{ABQ} = [\mathbb{I}_A \otimes \mathcal{M}^B](\rho^{AB}) = \sum_x P_X(x)\ |x\rangle\langle x|_A \otimes \mathcal{M}^B[\rho^B_x] =$$

$$= \sum_{xy} P_X(x) P_{Y|X}(y|x)\ |x\rangle\langle x|_A \otimes \rho_{y|x} \otimes |y\rangle\langle y|_Q$$

Now $\quad I(A:Q)_{\rho_{AQ}}$ with

$$\rho_{AQ} = \operatorname{tr}_B\{\rho^{ABQ}\} = \sum_{xy} P_X(x)\ P_{Y|X}(y|x)\ |x\rangle\langle x|_A \otimes |y\rangle\langle y|_Q =$$

$$= \sum_{xy} P_{X,Y}(x,y)\ |x\rangle\langle x|_A \otimes |y\rangle\langle y|_Q$$

In this case $\quad I(A:Q)_{\rho_{AQ}} = I(X:Y)$

Now $\quad I(A:Q)_{\rho_{AQ}} \leq I(A:BQ)_{\rho_{ABQ}} \qquad$ since $\rho_{AQ} = \mathrm{Tr}\{\rho_{ABQ}\}$

$$I(A:BQ)_{\rho_{ABQ}} \leq I(A:B)_{\rho_{AB}} \qquad \text{since } \rho_{ABQ} = [\mathbb{I}_A \otimes \mathcal{N}^B][\rho_{AB}]$$

Thus we have $\quad I(X:Y) = I(A:Q)_{\rho_{AQ}} \leq I(A:B)_{\rho_{AB}}$

We already notice that $\quad I(X:Y) \leq I(A:B)_{\rho_{AB}}$

We can rewrite $\quad I(A:B)_{\rho_{AB}} = H(A)_{\rho_A} + H(B)_{\rho_B} - H(AB)_{\rho_{AB}}$

$$H(A)_{\rho_A} = -\mathrm{tr}\{\rho_A \log \rho_A\} = -\sum_{x\,x'} \langle x| \rho_A \log p_X(x') |x'\rangle\langle x'|_A |x\rangle =$$

$$\rho_A = \sum_x p_X(x) |x\rangle\langle x|_A$$

$$= -\sum_x p_X(x) \log p_X(x) = H(X)$$

$$H(B)_{\rho_B} = -\mathrm{tr}\left\{\left[\sum_x p_X(x)\, \rho_x^B\right] \cdot \log\left[\sum_x p_X(x)\, \rho_x^B\right]\right\} = S\left(\sum_x p_X(x)\, \rho_x^B\right)$$

$$H(AB)_{\rho_{AB}} = -\mathrm{tr}\{\rho_{AB} \log \rho_{AB}\}$$

$$\log \rho_{AB} = \log\left\{\sum_x p_X(x)\, |x\rangle\langle x|_A \otimes \rho_x^B\right\} \overset{\text{block diagonal}}{=}$$

block diagonal

$|0\rangle\langle 0| \otimes (\quad)$

$|1\rangle\langle 1| \otimes (\quad)$ ...

$$= \sum_x \log\left\{p_X(x)\, |x\rangle\langle x|_A \otimes \rho_x^B\right\} \overset{\text{rank-1}}{=}$$

$$= \sum_x |x\rangle\langle x|_A \otimes \log\left\{p_X(x)\, \rho_x^B\right\} \overset{\text{product rule}}{=}$$

$$= \sum_x |x\rangle\langle x|_A \otimes \mathbb{I}^B \log p_X(x) + \sum_x |x\rangle\langle x|_A \otimes \log \rho_x^B$$

So we get $\quad \mathrm{tr}\{\rho_{AB}\log\rho_{AB}\} = \mathrm{tr}\{\rho_{AB}\sum_x |x\rangle\langle x|_A \otimes \mathbb{I}^B \log p_X(x)\} + \mathrm{tr}\{\rho_{AB}\sum_x |x\rangle\langle x|_A \otimes \log\rho_x^B\}$

$$= \sum_x \langle x|\rho_A|x\rangle_A \log p_X(x) + \sum_{xx'} \mathrm{tr}\{p_X(x')\, |x'\rangle\langle x'|_A \otimes \rho_{x'}^B \cdot |x\rangle\langle x|_A \otimes \log\rho_x^B\} =$$

$$= \sum_x p_X(x)\log p_X(x) + \sum_x p_X(x)\, \mathrm{tr}\{\rho_x^B \log\rho_x^B\}$$

$$= -H(X) - \sum_x p_X(x)\, H(B)_{\rho_x^B}$$

So we have $\quad I(A:B)_{\rho_{AB}} = H(A)_{\rho_A} + H(B)_{\rho_B} - H(AB)_{\rho_{AB}} =$

$$= \cancel{H(X)} + H(B)_{\rho_B} - \cancel{H(X)} - \sum_x p_X(x)\, H(B)_{\rho_x^B}$$

$$= H(B)_{\rho_B} - \sum_x p_X(x)\, H(B)_{\rho_x^B}$$

$$= \chi\left[\{p_X(x), \rho_x^B\}\right] = \chi[\mathcal{E}] \sim \text{Holevo Information}$$

- Holevo Bound $\qquad I(X:Y) \leq \chi[\mathcal{E}] \sim$ bound independent on the measurement on system B

We can maximize over measurement on B $\Rightarrow$

$$\max_{\{M_y^B\}} I(X:Y) = I_{acc}^B(X:B) \leq \chi[\mathcal{E}]$$

For some states $\quad I_{acc}^B(X:B) < \chi[\mathcal{E}]$ and

$$D[\mathcal{E}] = \chi[\mathcal{E}] - I_{acc}^B(X:B) = I(A:B)_{\rho_{AB}} - I_{acc}^B(X:B)$$

Ensemble of states which are not all orthogonal to each other.

So the $I_{acc}^{B}(X:B) \leq H(B)_{\rho_B} - \sum_x P_X(x) H(B)_{\rho_x^B}$

$$\leq H(B)_{\rho_B}$$

Now if $\rho_B \sim \mathcal{H}^B \sim n$ qubits $\bigotimes^n \mathbb{C}^2 \Rightarrow$

$$H(B)_{\rho_B} \leq \log 2^n = n \quad \Rightarrow$$

$$I_{acc}^{B}(X:B) \leq n \quad \text{despite} \quad \dim \mathcal{H}^B = 2^n$$

The Holevo bound proves that given n qubits, although they can carry a larger amount of information thanks to quantum superposition, the amount of classical information that can be accessed can be up to n classical bits

This definition works for an ensemble $\mathcal{E} = \{ P_X(x), \rho_x^B \}$ and more specifically for states $\rho_{AB} = \sum_x P_X(x) \, |x\rangle\langle x|_A \otimes \rho_x^B$

More generally we can define the accessible information given a measurement on system B as

$$I(A:Y)_{\rho^{AB}} = H(A)_{\rho^A} - \overbrace{\sum_y P_Y(y) \, H(\rho_y^A)}^{H(A|Y)_{\rho^{AB}}}$$

with • $\rho^A = \mathrm{tr}_B \{ \rho^{AB} \}$

The probability of observing y $P_Y(y) = \mathrm{tr}_B \{ M_y^B \, \rho^B \}$

The state on A after observing outc. y $\rho_Y^A = \dfrac{\mathrm{tr}_B \{ M_y^B \, \rho^{AB} \}}{P_Y(y)}$

Then we have that the Discord for measurement on B is given as

$$D(A|B)_{\rho_{AB}} = I(A:B)_{\rho_{AB}} - \max_Y I(A:Y)_{\rho_{AB}}$$

↓

Quantifies the loss of quantum correlations once we measure system B → can be shown to represent the loss of quantum correlations when we try to copy them with LOCC.

Actually, we can bring back quantum Discord to Bohr's notion of non-disturbance

↳ for more details →

Wiseman, Howard M. "Quantum discord is Bohr's notion of non-mechanical disturbance introduced to counter the Einstein–Podolsky–Rosen argument." *Annals of Physics* 338 (2013): 361-374.

A system is disturbed by a measurement $\{M_i^B\}$ if the measurement influence the conditions for possible predictions regarding feature measurements

$\Downarrow$

A system is disturbed if the post-measurement state has lost some information on possible future prediction on all possible measurements

Example $\Rightarrow$ $\rho_{AB} = \frac{1}{2} |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2} |1\rangle\langle 1|_A \otimes |+\rangle\langle +|_B$

$\hookrightarrow$ separable state, no entanglement

If I measure system B $\rightarrow$ 1) $\{|0\rangle\langle 0|_B, |1\rangle\langle 1|_B\}$ $\longrightarrow$

$$\rightarrow \rho_{P.m.} = \frac{1}{2} |0\rangle\langle 0| \otimes |0\rangle\langle 0|_B + \frac{1}{2} |1\rangle\langle 1| \otimes \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \otimes \frac{1}{2} |1\rangle\langle 1|$$

$$= \frac{1}{2} |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{4} |1\rangle\langle 1| \otimes |0\rangle\langle 0| + \frac{1}{4} |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

We see that some information is lost $\Rightarrow$ measuring $\rho_{P.m}$ with an arbitrary measurement does not yield the same probability distribution

One can ask : maybe is just this specific measurement $\longrightarrow$

$\rightarrow$ this is why there is the max over accessible information $\Rightarrow$

$\Rightarrow$ If $D(A|B) \neq 0$ $\nexists$ no measurement such that the post measurement state can have the same predictions as the initial one.

- Discord zero states are: quantum-classical states

$$\rho^{AB} = \sum_y p_Y(y)\, \rho_y^A \otimes |y\rangle\langle y|_B$$

In this case $\max_Y I(A:Y) = I(A:B)_{\rho_{AB}}$, with $\{M_y^B\} = \{|y\rangle\langle y|\}$

- Discord zero both on A and B $\iff$ classical-classical states

$$\rho^{AB} = \sum_{x,y} p_{X,Y}(x,y)\, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B$$

- Detecting whether a given state has zero quantum discord can be solved in polynomial time

- Computing quantum discord is NP-complete (running time grows exponentially with dimension of Hilbert space).

A composite state that cannot be assembled by classical means is entangled. Similarly, a composite state that cannot be disassembled by classical means is still quantum—it has a nonvanishing discord, the quantum information lost in the process of deconstructing it into classical ingredients.

References: Streltsov, Alexander. "Quantum discord and its role in quantum information theory." arXiv preprint arXiv:1411.3208 (2014).

and references therein

For a reference on the volume of quantum states with zero discord

Ferraro, Alessandro, et al. "Almost all quantum states have nonclassical correlations." *Physical Review A* 81.5 (2010): 052318.

Why is this connected with disturbance?

We know show that we can not copy correlations for states with Quantum with LOCC operations: if QD is not zero, we will inevitably disturb the system. Further, QD quantify the loss of information in this task.
Analogous of no-cloning theorem for correlations.

Let us consider Alice, Bob and Charlie. We assume Alice and Bob correlated, while Charlie not:

$$\rho_{in}^{ABC} = \rho^{AB} \otimes \rho^{C}$$

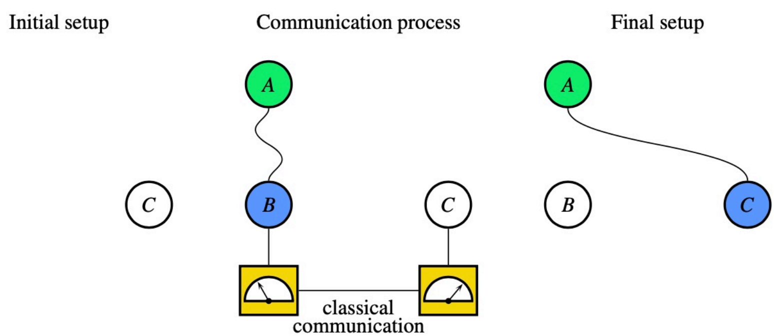
The goal is for Bob to transfer his state & correlations with Alice to Charlie using SEP

We want to find $\Lambda_{BC}^{SEP} : \mathcal{H}^B \otimes \mathcal{H}^C \mapsto \mathcal{H}^B \otimes \mathcal{H}^C$

with $\Lambda_{BC}^{SEP}[\bullet] = \sum_{i=1}^{m} B_i \otimes C_i \bullet B_i^\dagger \otimes C_i^\dagger$; $\sum_{i=1}^{m} B_i^\dagger B_i \otimes C_i^\dagger C_i = \mathbb{1}_B \otimes \mathbb{1}_C$

s.t. $\rho_{fi}^{ABC} = [\mathbb{1}_A \otimes \Lambda_{BC}^{LOCC}][\rho^{AB} \otimes \rho^{C}]$ with the property

$$\rho_f^{AC} = \mathrm{tr}_B\{\rho_f^{ABC}\} = \rho^{AB} \sim I^C[\rho^{AB}] = \sup_{\Lambda_{BC}^{sep}} I(A:C)_{\tilde{\rho}_f^{AC}}$$



We do not loose in generality if we choose $\rho_C = \frac{\mathbb{1}_C}{d_C}$ $\Rightarrow$

$$\rho_f^{ABC} = \frac{1}{d_c} \sum_{i=1}^{m} (\mathbb{1}_A \otimes B_i)\, \rho^{AB}\, (\mathbb{1}_A \otimes B_i^+) \otimes C_i C_i^+$$

I can define $q_i = \mathrm{Tr}\{C_i C_i^+\}$ and $\sigma_i = \frac{C_i C_i^+}{q_i}$ $\Rightarrow$

$$\Rightarrow \quad \rho_f^{ABC} = \sum_{i=1}^{m} \frac{q_i}{d_c} (\mathbb{1}_A \otimes B_i)\, \rho^{AB} (\mathbb{1}_A \otimes B_i^+) \otimes \sigma_i =$$

$$= \sum_{i=1}^{m} (\mathbb{1}_A \otimes E_i^B)\, \rho^{AB}\, (\mathbb{1}_A \otimes E_i^{B+}) \otimes \sigma_i$$

$$E_i^B = \sqrt{\frac{q_i}{d_c}}\, B_i$$

Now the state shared by Alice and Charlie is

$$\rho_f^{AC} = \mathrm{Tr}_B \{\rho_f^{ABC}\} = \sum_{i=1}^{m} \mathrm{tr}_B \{\mathbb{1}_A \otimes E_i^B\, \rho^{AB}\, \mathbb{1}_A \otimes E_i^{B+}\} \otimes \sigma_i =$$

$$= \sum_{i=1}^{m} \mathrm{tr}_B \{\mathbb{1}_A \otimes \underbrace{E_i^{B+} E_i^B}_{M_i^B}\, \rho^{AB}\} \otimes \sigma_i =$$

M_i^B is a POVM $\rightsquigarrow$ can be checked.

We will now show that the quantum mutual information is bounded above $\rightarrow$

We will do that in 2 steps:

① Upper bound on $I^C[\rho^{AB}]$

Let us consider the state $\tau^{A\tilde{C}} = \sum_{i=1}^{m} \mathrm{Tr}_B\{\mathbb{1}_A \otimes M_i^B \rho^{AB}\} \otimes |i\rangle\langle i|_{\tilde{C}}$

Let us consider a measure-and-prepare map $\Lambda_{\tilde{C}} : \mathcal{H}^{\hat{C}} \to \mathcal{H}^{C}$ s.t.

$$\Lambda_{\tilde{C}}[\bullet] = \sum_{ab} K_{ab}^{\hat{C}} \bullet K_{ab}^{\hat{C}\dagger} \qquad \text{with} \qquad K_{ab}^{\hat{C}} = \sqrt{\underset{\sigma_b^C}{\underset{\shortparallel}{\sigma_b^{\hat{C}}}}}\ |a\rangle\langle b|_{\tilde{C}}$$

We have that $\rho_f^{AC} = [\mathbb{1}_A \otimes \Lambda_{\tilde{C}}](J^{A\hat{C}})$

This means that

$$I(A:C)_{\rho_f^{AC}} = I(A:\tilde{C})_{(\mathbb{1}_A \otimes \Lambda_{\tilde{C}})(J^{A\hat{C}})} \overset{\text{monotonicity under CPTP}}{\leq}$$

$$\leq I(A:\tilde{C})_{J^{A\hat{C}}} \overset{\text{quantum-classical states}}{=}$$

$$= I(A:B)_{\{M_i^B\}}$$

Taking the sup of $\Lambda_B \leftrightarrow$ sup of $\{M_i^B\}$ we obtain

$$\sup_{\Lambda_B} I(A:C)_{\rho_f^{AC}} = I^c[\rho^{AB}] \leq$$

$$\leq \sup_{\{M_i^B\}} I(A:B)_{\{M_i^B\}} =$$

$$= I_{oa}(A:B) = I(A:B)_{\rho_{AB}} - D(A|B)_{\rho_{AB}}$$

Thus we conclude that $I^c[\rho^{AB}] \leq I(A:B)_{\rho_{AB}} - D(A|B)_{\rho_{AB}}$

2 Lower bound on $I^c[\rho^{AB}]$

Let us consider $C_i C_i^{\dagger} = |i\rangle\langle i|_C \Rightarrow \sigma_i = \frac{C_i C_i^{\dagger}}{q_i} = |i\rangle\langle i|_C \Rightarrow$

$$\Rightarrow \hat{\rho}_f^{AC} = \sum_{i=1}^{d_C} \mathrm{Tr}_B\{M_i^B \rho^{AB}\} \otimes |i\rangle\langle i|_C$$

We notice that

$$I(A:C)_{\tilde{\rho}_f^{AC}} = I(A:\tilde{C})_{\tilde{\rho}^{A\tilde{C}}} = I(A:B)_{\{M_i^B\}}$$

Given that this is a specific protocol $\Rightarrow$

$$I(A:C)_{\tilde{\rho}_f^{AC}} \leq \sup_{\Lambda_{BC}^{SEP}} I(A:C)_{\rho_f^{AC}} = I^c[\rho^{AB}]$$

Thus we have that

$$I^c[\rho^{AB}] \geq I(A:B)_{\{M_i^B\}} \underset{|}{=} I_{acc}(A:B) = I(A:B)_{\rho^{AB}} - D(A|B)_{\rho^{AB}}$$

$\{M_i^B\}$ is the one that correspond to the maximum

Thus we have proved that

$$I(A:B) - D(A|B) \leq I^c[\rho^{AB}] \leq I(A:B) - D(A|B) \Rightarrow$$

$$\Rightarrow I^c[\rho^{AB}] = I(A:B) - D(A|B)$$

Thus we conclude that correlations can be copied iff $D(A|B) = 0$

Streltsov, Alexander. "Quantum discord and its role in quantum information theory." arXiv preprint arXiv:1411.3208 (2014).

This result is connected with the problem of local Broadcastability

The problem of "Broadcasting Information" is given as follows:

The NO-Broadcasting theorem is the generalization of the no-cloning theorem for mixed states

Given an ensemble $\{p_i, \rho_i\}$ is it possible to find a CPTP Λ s.t.

$$\mathrm{tr}_S \Lambda[\rho_i^S \otimes \sigma_E] = \mathrm{tr}_E \Lambda[\rho_i^S \otimes \sigma_E] = \rho_i$$

if and only if $[\rho_i, \rho_j] = 0 \quad \forall i,j.$

The broadcasting of a single system can be generalized to bipartite state and local operations:

ρ_{AB} is locally broadcastable $\Leftrightarrow$

$$\exists\, \Lambda_{AA'} : \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^{A'}) \mapsto \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^{A'})$$

$$\Lambda_{BB'} : \mathcal{L}(\mathcal{H}^B \otimes \mathcal{H}^{B'}) \mapsto \mathcal{L}(\mathcal{H}^B \otimes \mathcal{H}^{B'})$$

such that

$$\mathrm{tr}_{AB} [\Lambda_{AA'} \otimes \Lambda_{BB'}](\rho_{AB} \otimes \sigma_{A'} \otimes \sigma_{B'}) = \mathrm{tr}_{A'B'}\{ \text{''} \} = \rho_{AB}$$

Th. ρ_{AB} is locally broadcastable $\Longleftrightarrow$ ρ_{AB} is classical-classical

Reference: Piani, Marco, Paweł Horodecki, and Ryszard Horodecki. "No-local-broadcasting theorem for multipartite quantum correlations." Physical review letters 100.9 (2008): 090502.

A list of the references on quantum discord:

- An introduction to classical and quantum information theory can be found in Chapter 10 and 11 of:

- Wilde, Mark M. "From classical to quantum Shannon theory." arXiv preprint arXiv:1106.1445 (2011).

- This is a pedagogical review on quantum discord and on some applications for quantum information processing:

- Streltsov, Alexander. "Quantum discord and its role in quantum information theory." arXiv preprint arXiv:1411.3208 (2014).

- This a review on all the results on quantum discord, from quantum computation, quantum communication, measurement disturbance, open quantum systems and quantum phase transition. Advanced read:

- Bera, Anindita, et al. "Quantum discord and its allies: a review of recent progress." Reports on Progress in Physics 81.2 (2017): 024001.