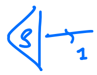
Higher Order Quantum maps

"Building blocks of quantum mechanics

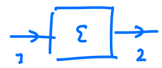
states $\rho \in L(\mathcal{H}_1)$

$\rho \geq 0$, $\operatorname{tr} \rho = 1$

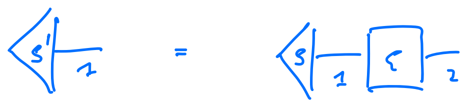


dynamics $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$

$\mathcal{E}$ is trace non-increasing and CP



Dynamics transform quantum states, i.e., $\rho' = \mathcal{E}[\rho]$



Are these all the transformations we can think of?

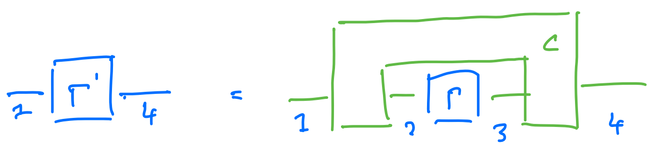
Why not "go up the ladder"?

Ex. ① Quantum superchannel $\mathcal{C} : \Gamma \to \Gamma'$, where

$\Gamma \in L(L(\mathcal{H}_2), L(\mathcal{H}_3))$, i.e. $\Gamma : L(\mathcal{H}_2) \to L(\mathcal{H}_3)$

$\Gamma' \in L(L(\mathcal{H}_1), L(\mathcal{H}_4))$, i.e. $\Gamma' : L(\mathcal{H}_1) \to L(\mathcal{H}_4)$

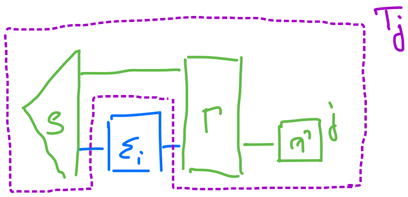
Graphically:



What properties would we demand from $\mathcal{C}$? Is it physical?

2.) Given two quantum channels $\{\mathcal{E}_1, \mathcal{E}_2\}$, what is the best strategy to distinguish them?

Resources for this task: states, dynamics and measurements.

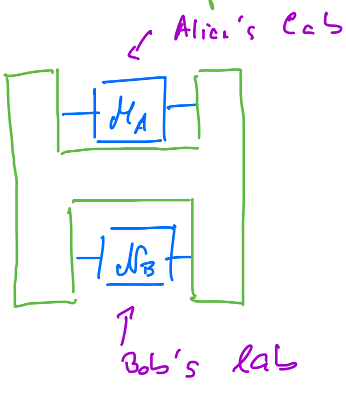


Task: $\max_{T=\{T_1,T_2\}} \sum_i p(j=i \mid i, T)$

→ "Measurement on a channel". What properties does $T=\{T_1, T_2\}$ have to satisfy?

3.) "Exotic" causal orders

Say, Alice and Bob sit in separate laboratories and they each receive a quantum system, manipulate it and send it forward.



What is the most general logically consistent causal relationship between them that quantum mechanics permits?

Higher order quantum maps and the link product

Recall: Channel $\mathcal{E}: L(\mathcal{H}_1) \to L(\mathcal{H}_2)$, state $S^{(1)} \in L(\mathcal{H}_1)$ — space it is defined on

Let $\eta^{(12)} \in L(\mathcal{H}_2 \otimes \mathcal{H}_1)$ be the Choi state of $\mathcal{E}$, i.e. $\eta^{(12)} \geq 0$ and $\mathrm{tr}_2\, \eta^{(21)} = \mathbb{1}^{(1)}$.

transpose over the space they share

$$\rho' = \mathcal{E}[\rho] = \mathrm{tr}_1\left(\eta^{(12)}\left(\rho^{(1)T} \otimes \mathbb{1}^{(2)}\right)\right) =: \eta^{(12)} * \rho^{(1)}$$

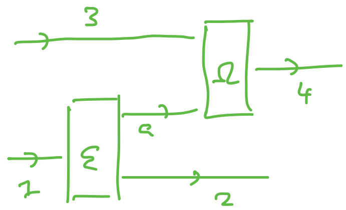
Choi state of $\mathcal{E}$

link product

<u>What about more general situations?</u>

Let $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_2 \otimes \mathcal{H}_a)$ and $\Omega : L(\mathcal{H}_3 \otimes \mathcal{H}_a) \to L(\mathcal{H}_4)$ be two CPTP maps. What is the Choi state of their concatenation?

<u>Graphically</u>:

$\Omega \circ \mathcal{E} =$ 

Let $\eta^{(a21)} \in L(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_a)$ and $\omega^{(4a3)} \in L(\mathcal{H}_4 \otimes \mathcal{H}_3 \otimes \mathcal{H}_a)$ be the Choi states of $\mathcal{E}$ and Ω, respectively.

<u>Claim</u>: $\mathrm{Choi}(\Omega \circ \mathcal{E}) =: \zeta^{(1234)} = \mathrm{tr}_a\left[\left(\eta^{(a21)} \otimes \mathbb{1}^{(34)}\right)\left(\omega^{(4a3)T_a} \otimes \mathbb{1}^{(12)}\right)\right]$

$$=: \eta^{(a21)} * \omega^{(4a3)}$$

<u>NB</u>: "primed" spaces will denote copies of their unprimed counterparts, e.g., $\mathcal{H}_\alpha \cong \mathcal{H}_{\alpha'}$)

Proof: For the proof, we need the "relabelling" matrix $\mathbb{1}_{\alpha\to\alpha'} = \sum_i |i^{(\alpha')}\rangle\langle i^{(\alpha)}|$ that acts as

$$\mathbb{1}_{\alpha\to\alpha'}|\psi^{(\alpha)}\rangle = |\psi^{(\alpha')}\rangle .$$

We have $\mathrm{Choi}(\Omega\circ\mathcal{E}) =$

$$= (\Omega\circ\mathcal{E})\otimes I^{(1'3')}\left[\,|\Phi^{(11')}\rangle\langle\Phi^{(11')}| \otimes |\Phi^{(33')}\rangle\langle\Phi^{(33')}|\,\right]$$

$$= (\Omega\otimes I^{(13')})\left[\,\eta^{(a21)} \otimes |\Phi^{(33')}\rangle\langle\Phi^{(33')}|\,\right]$$

$$= (\Omega\otimes I^{(13')})\left[\,\mathbb{1}_{a'\to a}\,\eta^{(a'21)}\,\mathbb{1}_{a\to a'} \otimes |\Phi^{(33')}\rangle\langle\Phi^{(33')}|\,\right]$$

$$= \sum_{ij} (\Omega\otimes I^{(13')})\left[\,|i^{(a)}\rangle\langle i^{(a')}|\,\eta^{(a'21)}\,|j^{(a')}\rangle\langle j^{(a)}| \otimes |\Phi^{(33')}\rangle\langle\Phi^{(33')}|\,\right]$$

$$= \mathrm{tr}_{a'}\left[(\Omega\otimes I^{(13')})\left[\,\underbrace{|i^{(a)} i^{(a')}\rangle\langle j^{(a)} j^{(a')}|}_{|\Phi^{(aa')}\rangle\langle\Phi^{(aa')}|}\,\eta^{(a'21)T_{a'}} \otimes |\Phi^{(33')}\rangle\langle\Phi^{(33')}|\,\right]\right]$$

$$= \mathrm{tr}_{a'}\left[\,(\omega^{(4a'3)}\otimes\mathbb{1}^{(12)})(\eta^{(a'21)T_{a'}}\otimes\mathbb{1}^{(34)})\,\right]$$

$$= \mathrm{tr}_{a}\left[\,(\eta^{(a21)}\otimes\mathbb{1}^{(34)})(\omega^{(4a3)T_a}\otimes\mathbb{1}^{(12)})\,\right]$$

$$= \eta^{(a21)} * \omega^{(4a3)} \qquad \blacksquare$$

Def.: Link product

Let $\eta^{(ab)} \in L(\mathcal{H}_a\otimes\mathcal{H}_b)$ and $\omega^{(bc)} \in L(\mathcal{H}_b\otimes\mathcal{H}_c)$

Then, their link product $\eta^{(ab)} * \omega^{(bc)}$ is defined as

$$\boxed{\eta^{(ab)} * \omega^{(bc)} = \mathrm{tr}_b \left[\left(\eta^{(ab)} \otimes \mathbb{1}_c \right) \left(\omega^{(bc)\, T_b} \otimes \mathbb{1}_a \right) \right]}$$

i.e. it is an "index contraction" on the spaces η and ω share and tensor product on the remaining spaces.

Properties of the link product

(i) Hermiticity and positivity preserving

$\eta * \omega$ is Hermitian (positive) if η and ω are Hermitian (positive).

(ii) Associative

$$\eta * \omega * \xi = (\eta * \omega) * \xi = \eta * (\omega * \xi)$$ (if no spaces "occur" more than twice).

(iii) Commutativity

$$\eta * \omega = \omega * \eta$$ (up to reordering of spaces)

All relevant operations in QM can be phrased in terms of the link product.

Ex.: • $S' = \mathcal{E}[S] = \eta^{(12)} * S^{(1)} = S^{(1)} * \eta^{(12)}$

• Special case: $\mathcal{E}[\cdot] = \mathrm{tr}[\cdot] \Rightarrow \mathcal{E}[S] = \mathbb{1}^{(1)} * S^{(1)} = S^{(1)} * \mathbb{1}^{(1)}$

• $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$, $\Omega : L(\mathcal{H}_2) \to L(\mathcal{H}_3)$

$$\Rightarrow \mathrm{Choi}(\Omega \circ \mathcal{E}) = \omega^{(23)} * \eta^{(12)} = \eta^{(12)} * \omega^{(23)} \neq \mathrm{Choi}(\mathcal{E} \circ \Omega)$$

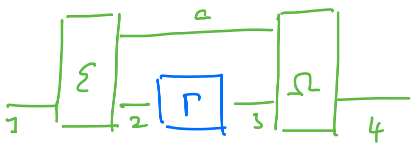
Example: Quantum superchannel

Let $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_2 \otimes \mathcal{H}_a)$, $\Gamma : L(\mathcal{H}_2) \to L(\mathcal{H}_3)$

$\Omega : L(\mathcal{H}_3 \otimes \mathcal{H}_a) \to L(\mathcal{H}_4)$

with corresponding Choi states $\eta^{(12a)}$, $\gamma^{(23)}$, and $\omega^{(3a4)}$

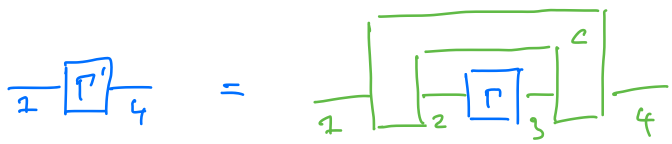
Graphically:



What is the Choi state of the resulting channel $\Gamma' = \Omega \circ \Gamma \circ \mathcal{E}$?

We have
$$\mathrm{Choi}(\Gamma') = \eta^{(12a)} * \gamma^{(23)} * \omega^{(3a4)}$$
$$= \left(\eta^{(12a)} * \omega^{(3a4)}\right) * \gamma^{(23)}$$
$$=: \underbrace{C^{(1234)}}_{\substack{\text{Choi state of a "superchannel"} \\ L(L(\mathcal{H}_2), L(\mathcal{H}_3)) \to L(L(\mathcal{H}_1), L(\mathcal{H}_4))}} * \gamma^{(23)}$$

$\rightarrow$ Choi state of Γ

2, 

What are the properties of $C^{(1234)}$?

(i) Positivity $C^{(1234)} \geq 0$ (link product of positive objects)

(ii) "Causal ordering": $\mathrm{tr}_4\, C^{(1234)} = \mathbb{1}^{(3)} \otimes C'^{(12)}$

$\mathrm{tr}_2\, C'^{(12)} = \mathbb{1}^{(1)}$

<u>Proof</u>: $\mathrm{tr}_4\, C^{(1234)} = \mathbb{1}^{(4)} * \eta^{(12a)} * w^{(3a4)}$

$= \eta^{(12a)} * \mathrm{tr}_4 \left[w^{(3a4)} \right]$

$= \eta^{(12a)} * \mathbb{1}^{(3)} * \mathbb{1}^{(a)}$

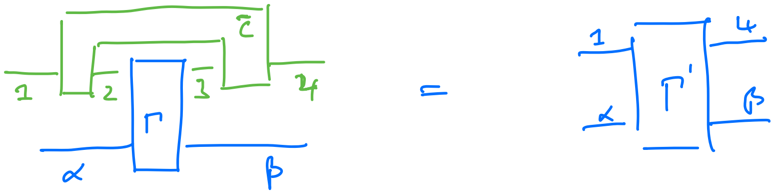
$= \mathrm{tr}_a \left(\eta^{(12a)} \right) \otimes \mathbb{1}^{(3)} =: C'^{(12)} \otimes \mathbb{1}^{(3)}$

$\mathrm{tr}_2 \left[C'^{(12)} \right] = \mathrm{tr}_{2a} \left(\eta^{(12a)} \right) = \mathbb{1}^{(1)}$

<u>What do these properties imply?</u>

(i) Positivity of $C^{(1234)}$: $C^{(1234)} * \zeta^{(23\alpha\beta)} \geq 0$ if $\zeta^{(12\alpha\beta)} \geq 0$

$\Rightarrow$ <u>every</u> CP map gets mapped onto CP map.



$\Rightarrow$ $C^{(1234)} \geq 0 \iff \tilde{C}$ is completely complete positivity preserving

corresponding superchannel

(ii) "Causal ordering"

Let $\zeta^{(23\alpha\beta)}$ be the Choi state of a TP map $\Gamma : L(\mathcal{H}_2 \otimes \mathcal{H}_\alpha) \to L(\mathcal{H}_3 \otimes \mathcal{H}_\beta)$, then:

$\zeta'^{(1\alpha 4\beta)} = C^{(1234)} * \zeta^{(23\alpha\beta)}$ is the Choi state of a TP map:

$$\mathrm{tr}_{4\beta}\left(\zeta'^{(1\alpha 4\beta)}\right) = \mathrm{tr}_\beta\left(\mathrm{tr}_4\left[C^{(1234)}\right] * \zeta^{(23\alpha\beta)}\right)$$

$$= \left(\mathbb{1}^{(3)} \otimes C'^{(12)}\right) * \mathrm{tr}_\beta\left[\zeta^{(23\alpha\beta)}\right]$$

$$= C'^{(12)} * \underbrace{\mathrm{tr}_{3\beta}\left[\zeta^{(23\alpha\beta)}\right]}_{= \mathbb{1}^{(2\alpha)}} = \mathbb{1}^{(2\alpha)}$$

$\Rightarrow$ $\zeta'^{(1\alpha 4\beta)}$ is TP whenever $\zeta^{(23\alpha\beta)}$ is TP

↳ $C^{(1234)}$ satisfies 'causality constraints' $\Longleftrightarrow$ $\tilde{C}$ is completely TP preserving

2) Are there more general maps that have these properties?

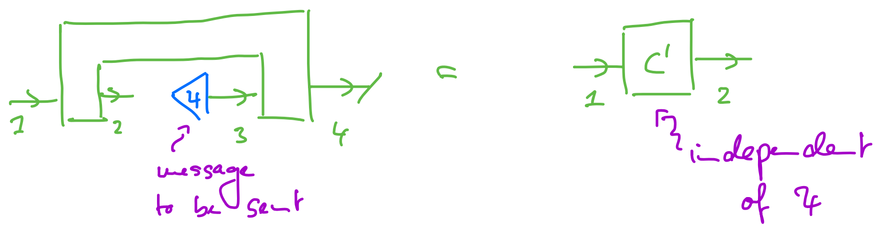
Thm.: A superchannel $\tilde{C}: L(L(\mathcal{H}_2), L(\mathcal{H}_3)) \to L(L(\mathcal{H}_1), L(\mathcal{H}_4))$ is completely complete positivity preserving and completely TP preserving iff its Choi state $C^{(1234)}$ satisfies

$$C^{(1234)} \geq 0, \quad tr_4\, C^{(1234)} = \mathbb{1}_3 \otimes C'^{(12)} \quad \& \quad tr_2\, C'^{(12)} = \mathbb{1}^{(1)}$$

NB: Why "causality constraints"

↳ They guarantee that no information can be sent backwards in time.

graphically:



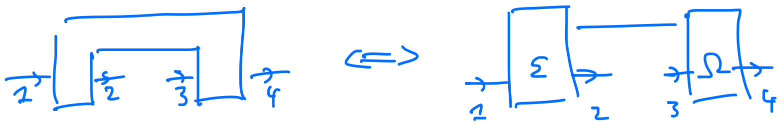
Proof: $\Psi^{(3)} * C^{(1234)} * \mathbb{1}^{(4)} = \Psi^{(3)} * (\mathbb{1}^{(3)} \otimes C'^{(12)})$

$= tr(\Psi^{(3)})\, C'^{(12)} = C'^{(12)}$ ⇝ independent of Ψ.

Thm.: For every proper superchannel (i.e. $C^{(1234)} \geq 0$ and satisfies causality constraints) there exist two CPTP maps $\mathcal{E}$ and Ω such that

$$C^{(1234)} = \mathrm{Choi}(\Omega \circ \mathcal{E}) = W^{(34)} * \eta^{(12)}$$

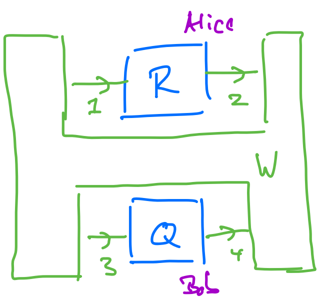
Graphically:



This can fail for objects with "more slots"

Examples: Higher order maps that map pairs of channels onto channels.

- process matrix

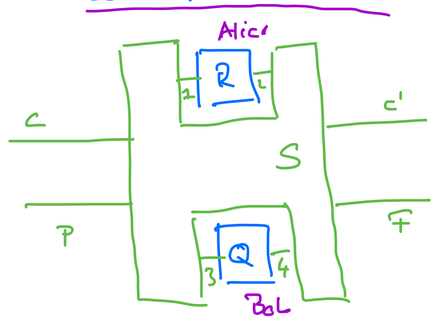


$$(R^{(12)} \otimes Q^{(34)}) * W^{(1234)} = 1$$

$\forall$ CPTP R, Q

$W^{(1234)} \geq 0$

Quantum Switch



$$\mathrm{tr}_{c'F}\left[(R^{(12)} \otimes Q^{(34)}) * S^{(1234cPFc')} \right] = \mathbb{1}^{(cP)}$$

$$S^{(1234cPFc')} \geq 0$$