$\Rightarrow$ Such supermaps cannot necessarily be represented by a causally ordered circuit.

But they do not lead to logical inconsistencies and allow for the study of indefinite causal order.

Ex.: Quantum Switch

Higher order map that allows one to superpose perfect signalling in two opposite orders.

Recall: $\text{Choi}(I_{\alpha\to\beta}) = |I^{(\alpha\beta)}\rangle\langle I^{(\alpha\beta)}|$ (where $\mathcal{H}_\alpha \cong \mathcal{H}_\beta$)

$\Rightarrow$ Switch is given by $S := |S\rangle\langle S|$, with

$$|S\rangle = \underbrace{|00\rangle_{cc'}}_{\text{control qubit}} \underbrace{|I^{(P1)}\rangle}_{\substack{\text{identity channel}\\ P\to 1}} \underbrace{|I^{(13)}\rangle}_{\substack{\text{identity channel}\\ 2\to 3}} |I^{(4F)}\rangle + |11\rangle_{cc'} |I^{(P3)}\rangle |I^{(41)}\rangle |I^{(2F)}\rangle$$

- - - -

Depending on the input on the control, the Switch has different signalling directions:

$$|0\rangle\langle 0|^{(c)} * S = |0\rangle\langle 0|^{(c')} \otimes I^{(P1)} \otimes I^{(23)} \otimes I^{(4F)}$$ (first Alice then Bob)

$$|1\rangle\langle 1|^{(c)} * S = |1\rangle\langle 1|^{(c')} \otimes I^{(P3)} \otimes I^{(41)} \otimes I^{(2F)}$$ (first Bob then Alice)

$|+\rangle\langle+|^{(c)} * S$ = superposition of causal orders (not representable in a causally ordered circuit).

<u>But:</u> $R^{(12)} * Q^{(34)} * S^{(1234CPFC')}$ is CPTP, since

$S \geq 0$ and $\mathrm{tr}_{FC'}[S] = |0\rangle\langle 0|^{(c)} \otimes \pi^{(P1)} \otimes \pi^{(23)} \otimes \mathbb{1}^{(4)} + |1\rangle\langle 1|^{(c)} \otimes \pi^{(P3)} \otimes \pi^{(41)} \otimes \mathbb{1}^{(2)}$

$\Rightarrow \mathrm{tr}_{C'F}[(R^{(12)} \otimes Q^{(34)}) * S] = \dots = \mathbb{1}^{(PC)}$

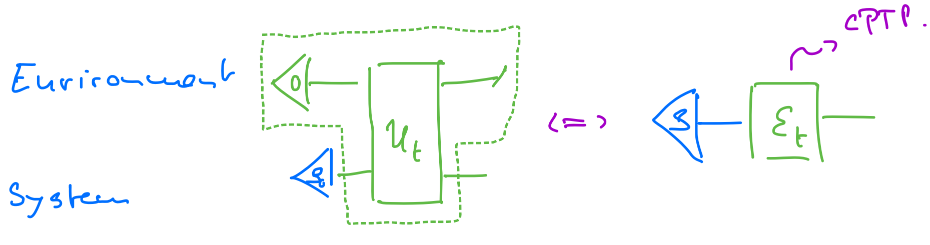
$\Rightarrow$ Quantum Switch is "valid" from an axiomatic point of view, still under debate from a physical perspective.

<u>Higher order quantum maps and open quantum system dynamics</u>

In general, any system of interest interacts in some un-controllable way with its environment.

↳ Dynamics are not unitary anymore but have to be described by (a family of) CPTP maps.

<u>Graphically:</u>



Basic assumption: Initially, system and environment are uncorrelated.

Then: $\varrho_t = \mathrm{tr}_E [\, \mathcal{U}_t [\varrho \otimes |0\rangle\langle 0|_E]\,] = \mathcal{E}_t[\varrho]$

Question: How do we describe the dynamics of a system that is not initially uncorrelated from its environment.

Ex.: System and Environment might have been uncorrelated initially, but they are generally correlated at a later time, i.e.

$$\varrho^{SE}_{t'} := \mathcal{U}_{t'} [\varrho \otimes |0\rangle\langle 0|] = \varrho^{SE}_{t'} \neq \varrho_{t'} \otimes \eta^{E}_{t'}$$

How would we describe the dynamics from t' to some later time t?

↳ $\exists$? $\tilde{\mathcal{E}}_{t,t'}$, such that $\varrho_t = \tilde{\mathcal{E}}_{t,t'}[\varrho_{t'}]$, where $\varrho_{t'} = \mathrm{tr}_E[\varrho^{SE}_{t'}]$.

- Is $\tilde{\mathcal{E}}_{t,t'}$ still CP? TP? Linear?

- NB: Correlations can be seen as a memory of past interactions
  ↳ open system dynamics with memory / non-Markovian dynamics.

Open System dynamics via assignment maps?

Uncorrelated case: $S_t = \mathcal{E}_t[S]$

$$\begin{array}{cccc} \text{Environment} & S\otimes|0\rangle\langle 0| & \longrightarrow & \mathcal{U}_t[S\otimes|0\rangle\langle 0|] \\ & \mathcal{A}\uparrow & & \downarrow \mathrm{tr}_E \\ \text{System} & S & \longrightarrow & \mathcal{E}_t[S] \end{array}$$

Mathematically, the dynamics of the system can be understood as a concatenation of three maps:

$$\mathcal{E}_t[S] = \mathrm{tr}_E \circ \mathcal{U}_t \circ \mathcal{A}[S]$$

discarding the environment

closed system dynamics

"Assignment" map

$\mathcal{A}[\cdot] = \cdot \otimes |0\rangle\langle 0|$

Does this approach also work for initially correlated system-environment states?

Minimal requirements one would demand from an assignment map:

$\mathcal{A}$ should be
(i) linear, i.e. $\mathcal{A}[\alpha S_1 + \beta S_2] = \alpha \mathcal{A}[S_1] + \beta \mathcal{A}[S_2]$
(ii) faithful, i.e. $\mathrm{tr}_E[\mathcal{A}[S]] = S$
(iii) positive and TP.

Thm. (Pechukas 1994):

Any assignment map $\mathcal{A} : L(\mathcal{H}_S) \to L(\mathcal{H}_S \otimes \mathcal{H}_E)$ that satisfies (i) - (iii) is of product form, i.e.

$$\mathcal{A}[\varsigma] = \varsigma \otimes \tau \quad \text{for } \underline{\text{fixed}} \text{ quantum state } \tau \in L(\mathcal{H}_E).$$

Proof: From faithfulness and positivity: for any pure state $|\psi_i\rangle\langle\psi_i| \in L(\mathcal{H}_S)$ we must have

$$\mathcal{A}[|\psi_i\rangle\langle\psi_i|] = |\psi_i\rangle\langle\psi_i| \otimes \tau_i \quad , \text{ where } \tau_i \text{ might}$$

depend on ψ_i.

From positivity and TP we see that $\tau_i \geq 0$ and $\mathrm{tr}(\tau_i) = 1$ holds for all τ_i.

Now, let $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^{d_S^2}$ be a (non-orthogonal) basis of $L(\mathcal{H}_S)$.

$\Rightarrow$ Any pure state $|\phi\rangle\langle\phi| \in L(\mathcal{H}_S)$ can be represented

as $|\phi\rangle\langle\phi| = \sum_{i=1}^{d_S^2} \alpha_i \, |\psi_i\rangle\langle\psi_i|$ and from linearity,

we have $\mathcal{A}[|\phi\rangle\langle\phi|] =: |\phi\rangle\langle\phi| \otimes \tau = \sum_i \alpha_i \, |\psi_i\rangle\langle\psi_i| \otimes \tau_i$

$\hookrightarrow$ Let $\{\Delta_j\}_{j=1}^{d_S^2}$ be the dual set of $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^{d_S^2}$, i.e.

$$\mathrm{tr}(\Delta_j \, |\psi_i\rangle\langle\psi_i|) = \delta_{ij} \quad \forall i, j.$$

$$\Rightarrow \mathrm{tr}_S[\Delta_j \mathcal{A}[|\phi\rangle\langle\phi|]] = \mathrm{tr}[\Delta_j |\phi\rangle\langle\phi|]\,\tau = \alpha_j \tau_j$$

$$\Rightarrow \tau = \tau_j \quad \text{for all } j$$

$$\Rightarrow \mathcal{A}[\varrho] = \varrho \otimes \tau \qquad \forall\, \varrho \in \mathcal{L}(\mathcal{H}_S) \qquad \boxtimes$$

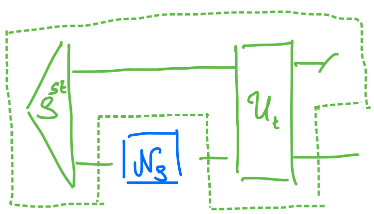
An assignment map that satisfies (i) - (iii) cannot account for initial system-environment correlations.

Two options: - Drop one of the assumptions (i) - (iii)

- Since the assignment map $\mathcal{A}$ is unphysical, we can actually resolve these issues by turning to higher order maps.

Superchannels for open system dynamics

How does one obtain input states when there are initial correlations?



$\mathcal{N}_S$: "preparation" map that acts as $[\mathcal{N}_S \otimes \mathcal{I}_E][\varrho^{SE}] = \varrho \otimes \tau_{\mathcal{N}}$

generally depends on $\mathcal{N}$

Ex.: Preparation of $|0\rangle\langle 0|_S$ through measurement in the computational basis, i.e. $(\mathcal{N}_{|0\rangle\langle 0|} \otimes I_E)[\rho^{SE}] =$

$$= |0\rangle\langle 0|_S \otimes \mathrm{tr}_S(\rho^{SE} |0\rangle\langle 0|_S)$$

$\Rightarrow$ final state at time t:

$$\rho_t = \underbrace{\mathrm{tr}_E[\mathcal{U}_t[(\mathcal{N}_S \otimes I_E)[\rho_{SE}]]]}_{\text{Not necessarily linear in } \rho}$$

However, the above is linear in $\mathcal{N}_S$. In Choi form, we have

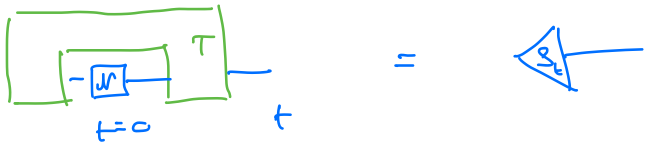
$$\rho_t = (\underbrace{\mathbb{1}_E}_{\substack{\text{Choi state} \\ \text{of the} \\ \text{Environment} \\ \text{trace}}} * \underbrace{U_t}_{\substack{\text{Choi state} \\ \text{of } \mathcal{U}_t}} * \rho^{SE}) * \underbrace{N_S}_{\substack{\text{Choi state} \\ \text{of } \mathcal{N}_S}} =: \underbrace{T}_{\substack{\text{Choi state} \\ \text{of a} \\ \text{"process tensor"}}} * N_S$$

$\Rightarrow$ Easy to see: $T \geq 0$ and $\mathrm{tr}(T * N) = 1$ for all N CPTP

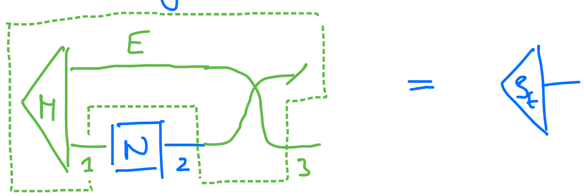
2) T maps all preparations $\mathcal{N}$ to the correct final state.

(NB: The preparations $\mathcal{N}$ can be arbitrary CP maps)

Graphically:



Ex.: Initially maximally entangled state and a SWAP gate.



$\Rightarrow$ ρ_t should be the partial trace of the Choi state of $\mathcal{N}$.

Indeed: $\rho_t = T^{(123)} * N^{(12)} =$

$$= M^{(1E)} * N^{(12)} * \underbrace{M^{(E3)}}_{\text{Choi matrix of relabelling } E\to 3} * \mathbb{1}^{(2)}$$

$$= \underbrace{\left(M^{(1E)} * M^{(E3)} \right)}_{M^{(13)}} * \operatorname{tr}_2 (N^{(12)}) = \operatorname{tr}_2 (N^{(32)}) \qquad \blacksquare$$

The uncorrelated case is included:

Let $\rho^{SE} = \rho^S \otimes |0\rangle\langle 0|^E$

$\Rightarrow T = \mathbb{1}^E * U * (\rho^S \otimes |0\rangle\langle 0|^E)$

$$= \rho^S \otimes \underbrace{(\mathbb{1}^E * U * |0\rangle\langle 0|^E)}_{\text{Choi state of a CPTP map}} = \rho^S \otimes \underbrace{\Upsilon_E}_{\text{Choi of CPTP.}}$$

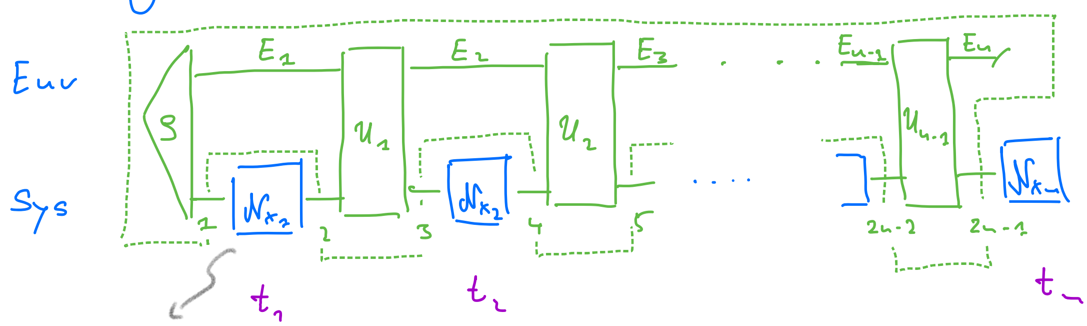
$\Rightarrow$ A superchannel stemming from an initially uncorrelated system environment state is of product form, i.e.,

Can this be extended to more times?

What if we don't just want to perform an initial preparation, but sequential measurements, say to deduce a joint probability distribution of the form

$$\mathbb{P}(x_n, t_n; x_{n-1}, t_{n-1}; \dots; x_1, t_1).$$

Graphically:



CP map corresponding to outcome x_i

$$P(x_n, t_n; \dots; x_1, t_1) = \left(S^{(1E_1)} * U_1^{(2E_1 3E_1)} * \dots * U_{n-1}^{(2n-2\, E_{n-1}\, 2n-1\, E_n)} \right) * \left(N_{x_1} \otimes N_{x_2} \otimes \dots \otimes N_{x_n} \right)$$

$$=: \underbrace{T^{(12\dots 2n-1)}}_{\text{"process tensor"}} * \left(N_{x_1} \otimes \dots \otimes N_{x_n} \right)$$

→ contains all statistical information that can be probed on the system at times $t_1, \dots, t_n$.

⇒ Proper description of multi-time experiments in QM.

↳ $T \geq 0$ (completely positive in the "correct" sense)

$T * (N_1 \otimes \dots \otimes N_n) = 1$ (for all $N_1, \dots, N_n$ TP).

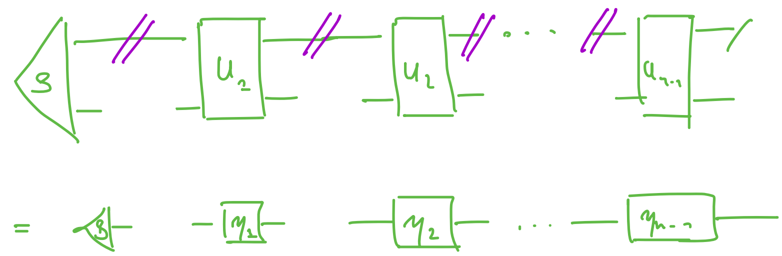
+ hierarchy of causality constraints.

What about memory, i.e. information that is transported through the environment?

$\boxed{\begin{array}{l}\text{Thm: A process with corresponding process tensor } T^{(1\dots 2n-1)} \\ \text{is memoryless (Markovian) iff it is of product form, i.e.} \\ \\ T^{(1\dots 2n-1)} = S^{(1)} \otimes \eta_1^{(23)} \otimes \eta_2^{(45)} \otimes \dots \otimes \eta_{n-1}^{(2n-2\, 2n-1)},\end{array}}$

where every η_i is the Choi state of a CPTP map.

Graphically: $\rightsquigarrow$ no information flow through the environment



$\Rightarrow$ Reduces to the known memoryless case, but can describe quantum processes with memory effects and allow for the quantification of said memory.

Are process tensors physical?

$\hookrightarrow$ Analogously to superchannels, there is a dilation thm. for process tensors, i.e., each process tensor can be represented by an underlying circuit.

Without proof:

Properties of a process tensor coming from a circuit:

$$T \geq 0, \quad \mathrm{tr}_{2n-1} T^{(1\dots 2n-1)} = \mathbb{1}_{2n-2} \otimes T^{(1\dots 2n-3)}, \quad \mathrm{tr}_{2n-3} T^{(1\dots 2n-3)} =$$

$$= \mathbb{1}_{2n-4} \otimes T^{(1\dots 2n-5)}, \dots, \mathrm{tr}_3 T^{(123)} = \mathbb{1}_2 \otimes T^{(1)}, \quad \mathrm{tr}\, T^{(1)} = 1$$

(**)

Analogously to superchannels, every process tensor admits a physical realisation:

Thm: (Stinespring dilation for process tensors)
Every object T that satisfies the conditions $(**)$ can be understood as coming from a causally ordered underlying circuit, i.e.

$$T = \mathcal{S}_{SE} * \mathcal{U}_1 * \mathcal{U}_2 * \cdots * \mathcal{U}_{n-1} * \mathbb{1}_E,$$

where $\mathcal{S}_{SE}$ is a system-environment state and $\mathcal{U}_i$ are system-environment unitaries

Graphically: