1. (a) Consider the Jaynes-Cummings model at resonance, 
\[ H = \frac{\hbar \omega}{2} (|e\rangle\langle e| - |g\rangle\langle g|) + \hbar \omega \hat{a}^\dagger \hat{a} + \frac{\hbar \Omega}{2} (|e\rangle\langle g| + \hat{a}^\dagger |g\rangle\langle e|). \]

The eigenstates of this model are of the form \( |\psi\rangle = u|n, g\rangle + v|n - 1, e\rangle \). Insert this state into the time-independent Schrödinger equation 
\[ \hat{H} |\psi\rangle = E |\psi\rangle, \]
and take matrix elements to obtain linear equations for \( u, v \). Hence calculate the eigenstates and eigenenergies.

(b) Suppose that at time 0 the model is in a Fock state of \( n \) photons and the atom in the ground state, \( |\phi(0)\rangle = |n, g\rangle \).
At some time \( t > 0 \) the atomic state is measured, and this experiment is repeated many times. Show that the average number of atoms found in the excited state is
\[ \langle n_e \rangle = \langle \phi(t)|(|e\rangle\langle e|)|\phi(t)\rangle = \sin^2 \left( \frac{\Omega \sqrt{n} t}{2} \right). \]

(c) What is the variance in the number of atoms found in the excited state, \( \langle n_e^2 \rangle - \langle n_e \rangle^2 \)?

(d) Explain, in terms of the eigenenergies of the model, why the frequency in the oscillations of the atomic population depends on \( n \).

2. (a) Suppose that at time 0 the model above is prepared in an arbitrary state of the field, with the atom in the ground state. This state may be written as a superposition of the states discussed above
\[ |\phi(0)\rangle = \sum_n c_n |n, g\rangle, \]
for some coefficients \( c_n \). Show, by generalising your argument above, that the probability that the atom is excited at a later time is
\[ \langle n_e \rangle = \sum_n |c_n|^2 \sin^2 \left( \frac{\Omega \sqrt{n} t}{2} \right). \]

(b) For the coherent state, \( |c_n|^2 \) is the Poisson distribution, characterised by the mean number of photons \( \bar{n} \). The figure below shows a numerical evaluation of the resulting population, when \( \bar{n} = 10 \). Explain why there are oscillations at short times, and why these oscillations decay, in terms of the ranges of frequencies appearing in the sum above. (Optional) Explain the reappearance of the oscillations, and estimate the time for such a “revival”.

![Probability of excited atom vs time and frequency](image-url)