Important application of the Fast Fourier transform: X-ray tomography. Also referred to as CAT scan (Computerized Axial Tomography).

This has revolutionized medical diagnosis. Invented in the 1960's and 70's. Nobel price in medicine for Cormack and Hounsfield in 1979.

Applications in Science
Also used in scientific research in order to infer the structure of opaque materials non-invasively (e.g. metal foams).

X-Rays
X-rays discovered in 1895 by Wilhelm Roentgen (won the first Nobel price in Physics 1901). X-rays are electromagnetic waves with wavelengths \( \approx 0.01 - 10 \) nm. X-Rays are absorbed/scattered from high Z-nuclei who have a higher electron density (such as bones).
How does tomography work?

Basic idea: Reconstruct 3D information by recording the sample slice by slice. 3D image created by combining the slices.

Basic problem: How do you reconstruct one slice?

Reconstructing the slice

In the following we will see how the Fourier transform is used to do image reconstruction for one particular slice.

When X-rays pass through a uniform slab of material its intensity $I$ decreases exponentially (attenuation):

$$ I = I_0 e^{-\lambda d} $$

(1)

where $I_0$ is the initial intensity and $\lambda$ is the absorption coefficient (many processes such as scattering and photoelectric effect can contribute to $\lambda$).

The reconstruction problem

As X-rays pass through the sample, their intensity decreases, depending on the local attenuation function $f(x, y)$. The X-ray detector is constructed in such a way that X-rays that enter at an angle are rejected. The attenuation will be of the form

$$ I = I_0 e^{-\int f(x, y) d\eta} $$

(2)

where $d\eta$ is an element along the path.

The projection at angle $\phi$

Each parallel beam offset by $\xi$ from the center traverses a different portion of the sample and is attenuated according to eq. (2). The attenuated signal at different $\xi$ is then recorded at the detector, which is the projection at a given angle $\phi$:

$$ p(\xi; \phi) = -\ln \left( \frac{I}{I_0} \right) = \int f(x, y) d\eta $$

(3)

Goal

Reconstruct $f(x, y)$ from the various projections $p(\xi; \phi)$ obtained at different angles $\phi$.
Relation between coordinate systems

The two coordinate systems are related by a simple rotation

\[ \zeta = x \cos(\phi) + y \sin(\phi) \tag{4} \]
\[ \eta = -x \sin(\phi) + y \cos(\phi) \tag{5} \]

Likewise,

\[ x = \zeta \cos(\phi) - \eta \sin(\phi) \tag{6} \]
\[ y = \zeta \sin(\phi) + \eta \cos(\phi) \tag{7} \]

Fourier transforming the attenuation

Let’s look at the Fourier transform of \( f(x, y) \)

\[ \mathcal{F}[f(x, y)] = \left( \frac{1}{\sqrt{2\pi}} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(k_x x + k_y y)} \, dx \, dy \] \tag{8}

Want to express \( k_x, k_y \) in rotated frame \( k_{\zeta}, k_{\eta} \).

\[ \begin{align*}
  k_{\zeta} &= k_x \cos(\phi) + k_y \sin(\phi) \\
  k_{\eta} &= -k_x \sin(\phi) + k_y \cos(\phi)
\end{align*} \tag{9} \tag{10} \]

\[ \begin{align*}
  k_x &= k_{\zeta} \cos(\phi) - k_{\eta} \sin(\phi) \\
  k_y &= k_{\zeta} \sin(\phi) + k_{\eta} \cos(\phi)
\end{align*} \tag{11} \tag{12} \]

Using these coordinate transformations, we can transform \( \mathcal{F}[f(x, y)] \) to the \( (\zeta, \eta) \) coordinate system.

Can easily show that the exponential appearing in the FT of \( f(x, y) \) is simply

\[ e^{-i(k_x x + k_y y)} = e^{-i(k_{\zeta} \zeta + k_{\eta} \eta)} \] \tag{13}

and \( dx \, dy = d\zeta \, d\eta \) (The Jacobian is 1)

Therefore,

\[ F(k_{\zeta}, k_{\eta}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(k_{\zeta} \zeta + k_{\eta} \eta)} \, d\zeta \, d\eta \]
The Projection theorem

We will now restrict ourselves. We will keep the angle \( \phi \) fixed and only evaluate the Fourier transform along the \( k_\xi \) axis, so \( k_\eta = 0 \). This corresponds to integrating along the radial coordinate, except that the integral extends from \(-\infty\) to \(+\infty\):

\[
\mathbb{F} \{ f(k_\xi \cos(\phi), k_\xi \sin(\phi)) \} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x, y) e^{-ik_\xi \xi} d\eta \quad (14)
\]

Now recall, that the projection \( p(\xi; \phi) \) is given by eq.(3):

\[
p(\xi; \phi) = \int f(x, y) d\eta \quad (15)
\]

Combining the last two equations we obtain the projection theorem:

\[
F(k_\xi \cos(\phi), k_\xi \sin(\phi)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p(\xi; \phi) e^{-ik_\xi \xi} d\xi \quad (16)
\]

The Fourier Transform of the projection \( p(\xi; \phi) \) at a given angle \( \phi \) is the Fourier transform of the attenuation function along \( \phi \) in transform space.

Measuring the projections at various angles \( \phi \) will yield the FT of the attenuation function throughout the \( k \)-plane, one radial line at a time.

Reconstruction problem

However, there is a practical problem implementing it via the DFT/FFT as illustrated below.

![Figure 6.21 The Fourier transform is known on the polar grid, indicated by the black data, while the straightforward FFT requires the data to be known on a Cartesian grid.](image)

Using the projection theorem, we obtain the transform of \( p(\xi; \phi) \) in Fourier space, but not on a cartesian grid as required by the discrete Fourier transform.

In principle one could use interpolation to approximate the Fourier transform on the Cartesian grid. In practice this is usually not done for two reasons:

1. Interpolation introduces rounding errors that propagate through the whole image when the inverse DFT is applied to the transform of the projection.
2. Interpolation requires additional computational resources.

To circumvent these problems, so called back projection methods are employed. We will eventually need an interpolation, but it will be done in the coordinate space.

Notes
In the following we will make a change to polar coordinates to the inverse FT that gives us $f(x, y)$:

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y$$  \hfill (17)

We express $x, y$ in terms of the polar coordinates $r, \theta$ and the conjugate variables $k_x, k_y$ in terms of $\rho, \phi$:

Therefore, $f(x, y)$ becomes

$$= \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} F(\rho \cos(\phi), \rho \sin(\phi)) e^{i\rho r \cos(\theta - \phi)} \rho d\rho d\phi$$

$$- \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} F(\rho \cos(\phi), \rho \sin(\phi)) e^{i\rho r \cos(\theta - \phi)} \rho d\rho d\phi$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} F(\rho \cos(\phi), \rho \sin(\phi)) e^{i\rho r \cos(\theta - \phi)} \rho d\rho d\phi$$

$$- \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} F(-\rho \cos(\phi), -\rho \sin(\phi)) e^{-i\rho r \cos(\theta - \phi)} \rho d\rho d\phi$$

We express $x, y$ in terms of the polar coordinates $r, \theta$ and the conjugate variables $k_x, k_y$ in terms of $\rho, \phi$:

Therefore, $f(x, y)$ becomes

$$f(r \cos(\theta), r \sin(\theta)) = \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} F(\rho \cos(\phi), \rho \sin(\phi)) e^{i\rho r \cos(\theta - \phi)} \rho d\rho d\phi$$

$$- \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} F(-\rho \cos(\phi), -\rho \sin(\phi)) e^{-i\rho r \cos(\theta - \phi)} \rho d\rho d\phi$$

If we do the $\phi$ integration last, then $\rho$ integration is done at constant $\phi$. At fixed $\phi$, however, $k_x$ lies along $\rho$.

Therefore, make the following substitutions: $\rho \rightarrow k_x$ in the first integral and $\rho \rightarrow -k_x$ in the second (and reverse limits of integration).

Also, through the choice of our coordinate system we have

$$r \cos(\theta - \phi) = \zeta$$  \hfill (18)
Finally we obtain $f(r \cos(\theta), r \sin(\theta))$

$$= \frac{1}{2\pi} \int_0^\infty \left( \int_0^\infty F(k_x \cos(\phi), k_x \sin(\phi)) e^{ik_x \xi} k_x \, dk_x \right) \, d\phi$$

$$+ \int_0^\infty F(k_x \cos(\phi), k_x \sin(\phi)) e^{ik_x (-\xi)} k_x \, dk_x \, d\phi$$

$$= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty F(k_x \cos(\phi), k_x \sin(\phi)) e^{ik_x \xi} k_x \, dk_x \, d\phi$$

So

$$f(r \cos(\theta), r \sin(\theta)) = \int_0^\pi \tilde{p}(\xi; \phi) \, d\phi \quad (19)$$

where

$$\tilde{p}(\xi; \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x \cos(\phi), k_x \sin(\phi)) e^{ik_x \xi} \, dk_x \quad (20)$$

is the modified projection at $\phi$, which is just the inverse Fourier transform of $F(k_x \cos(\phi), k_x \sin(\phi))$. This can be computed numerically with the inverse DFT.

Let’s summarize the steps one has to take to reconstruct the slice:

1. Initialize everything. The outermost loop will integrate over the various angles $\phi$ (eq. 19).
2. Measure and obtain the projection $p(\xi; \phi)$ at the current angle $\phi$. It will be measured at discrete points $\xi_i$.
3. Using the FFT, obtain $F(k_x \cos(\phi), k_x \sin(\phi)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\xi; \phi) e^{-ik_x \xi} \, d\xi$ (Projection theorem).
4. Using the inverse FFT compute the modified projection $\tilde{p}(\xi; \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x \cos(\phi), k_x \sin(\phi)) e^{ik_x \xi} \, dk_x$.
5. Evaluate the contribution to the $\phi$ integral: $f(r \cos(\theta), r \sin(\theta)) = \int_0^\pi \tilde{p}(\xi; \phi) \, d\phi$.
6. For this we need to determine $\xi$ for a particular $x, y$ coordinate through the coordinate transformation and then find the value of the modified projection at $\xi$ by the interpolation of $\tilde{p}(\xi; \phi)$.

Tomography - summary

- Powerful tool to probe matter non-invasively in three dimensions.
- Mathematically well defined by use of the projection theorem.
- Numerical implementation is more complicated: Data is taken at discrete points in a polar coordinate system, but the discrete Fourier transform assumes cartesian coordinates.
- Interpolation in Fourier space would lead to image artifacts in the reconstruction.
- Back projection in Fourier space would lead to image artifacts in the reconstruction.
- Even though interpolation is still required, it is done in coordinate space which is less prone in introducing artifacts.