Fourier Analysis

Useful information

Books:
- "A first course in computational physics", Paul devVries
- "Computational Physics", Nicholas J. Giordano and Hisao Nakanishi
- "Partial Differential Equations - AN Introduction", Walter A. Strauss


Lecture Notes and Problem Sets:
http://www.tcd.ie/Physics/People/Matthias.Moebius/teaching/

Fourier Analysis and its applications

Fourier analysis originated from the study of heat conduction: Jean Baptiste Joseph Fourier (1768-1830).
Fourier analysis enables a function (signal) to be decomposed into its frequency components.
Wide range of applications:
- Spectrum analysis
- Digital filtering (e.g. electronics, image processing), Deconvolution
- Audio compression (MP3 etc.)
- Solving differential equations

Definition of Fourier series

Consider a periodic function $f(t)$ with period $T = 2\pi$ such that

$$f(t + 2\pi) = f(t). \quad (1)$$

Any periodic function can be expressed as an infinite Fourier series of $\sin$ and $\cos$ functions with frequencies that are integer multiples of $\omega = 2\pi/T = 1$, the frequency of the function:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)). \quad (2)$$

where $a_0, a_1, \ldots, b_1, \ldots, b_n$ are the Fourier coefficients.
The Fourier series can be viewed as an expansion of orthogonal functions that form a complete set over any $2\pi$ interval:

\[
\int_{-\pi}^{\pi} \sin(mt) \sin(nt) dt = \begin{cases} \pi \delta_{m,n} & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases} 
\]

\[
\int_{-\pi}^{\pi} \cos(mt) \cos(nt) dt = \begin{cases} \pi \delta_{m,n} & \text{if } m \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases} 
\]

\[
\int_{-\pi}^{\pi} \sin(mt) \cos(nt) dt = 0 \text{ all integral } m \text{ and } n 
\]

From these orthogonality relations we can derive the Fourier coefficients $a_n$ and $b_n$.

The Fourier coefficients are given by

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt 
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt 
\]

You can prove this by substituting the Fourier series for $f(t)$ (eqn. (2)) into the above relations and use the orthogonality equations (eqns. (3,4,5)).

Consider the unit step function (i.e square wave):

\[
f(t) = \begin{cases} -1 & \text{for } -\pi < t < 0 \\ +1 & \text{for } 0 < t < \pi \end{cases}
\]

Since the function is odd, all $a_n$’s are zero. The $b_n$’s are given by

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \sin(nt) dt + \frac{1}{\pi} \int_{0}^{\pi} (+1) \sin(nt) dt 
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} \sin(nt) dt = \frac{2}{n\pi} \left[ -\cos(nt) \right]_{0}^{\pi} 
\]

\[
= \begin{cases} 0, & n = 2, 4, 6, \ldots \\ 4/n\pi, & n = 1, 3, 5, \ldots \end{cases}
\]
Fourier Analysis

The Fourier series

The Fourier transform

Fourier series of square wave

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PY4C01 - Numerical Methods II

Fourier series of square wave: $N=1$

Notes

Fourier series of square wave: $N=5$

Notes

Fourier series of square wave: $N=50$

Notes
The amplitude of the high frequency sine waves falls of $1/n$. Therefore convergence is slow.

In general: For discontinuous functions, like the square wave, the coefficients decrease as $1/n$. For continuous functions with discontinuous slopes (e.g. full wave rectifier), coefficients typically decrease as $1/n^2$.

Overshoot at discontinuities is called “Gibbs phenomenon”.

In electronics, square wave pulses are common. If apparatus does not pass the high frequencies, square wave will be rounded off.

Fourier series of complex functions

Definition of Fourier series can easily be extended to complex functions. Using the identity $e^{ix} = \cos(x) + i\sin(x)$ we can rewrite Fourier series (eqn. (2)) as

$$f(t) = \sum_{n=\infty}^{\infty} c_n e^{int},$$  \hspace{1cm} (9)

where

$$c_n = \begin{cases} \frac{(a_n - ib_n)}{2}, & n > 0, \\ \frac{a_0}{2}, & n = 0, \\ \frac{(a_n + ib_n)}{2}, & n < 0 \end{cases}$$  \hspace{1cm} (10)

which can be obtained by integration

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$  \hspace{1cm} (11)

Fourier series of functions with arbitrary period

So far, considered functions periodic on a $2\pi$ interval. Can easily extend to functions with any period $T$.

Consider a periodic function $f(t)$ with period $T$, and corresponding angular frequency $\omega = 2\pi/T$.

Let $t \to \omega t = 2\pi t/T$. Then the complex Fourier series (eqn. (9)) becomes

$$f(t) = \sum_{n=\infty}^{\infty} c_n e^{int},$$  \hspace{1cm} (12)

where,

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-int} dt$$  \hspace{1cm} (13)

Fourier series of functions with arbitrary period

Similarly for the Fourier series of real functions (eqn. (2)) becomes:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos(n \omega t) + b_m \sin(n \omega t) \right),$$  \hspace{1cm} (14)

where,

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n \omega t) dt$$  \hspace{1cm} (15)

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n \omega t) dt$$  \hspace{1cm} (16)

Function is expressed as a series of $\sin$ and $\cos$ functions with frequencies that are integer multiples of $\omega$, the frequency of the function.
Fourier Analysis

The Fourier series

The Fourier transform

Fourier series - summary

- Fourier series decomposes periodic signal into its frequency components.
- Even discontinuous functions or functions with discontinuous slopes (where Taylor expansion fails) can be expressed in a converging Fourier series.
- Solving differential equations. e.g. harmonic oscillator with some periodic driving force.

Fourier series to Fourier transform

Concept of Fourier series can be expanded to non-periodic functions. Consider the Fourier series for functions with period $T$ in complex representation:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\Delta \omega t}, \quad (17)$$

with

$$c_n = \frac{\Delta \omega}{2\pi} \int_{-T/2}^{T/2} f(t) e^{-in\Delta \omega t} dt, \quad (18)$$

where we have written $\omega = \Delta \omega$, since the discrete frequencies $n\omega$ are separated by $\Delta \omega = 2\pi / T$.

Now define

$$c_n = \frac{\Delta \omega}{\sqrt{2\pi}} g(n\Delta \omega), \quad (19)$$

so that

$$g(n\Delta \omega) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} f(t) e^{-in\Delta \omega t} dt, \quad (20)$$

$$f(t) = \sum_{n=-\infty}^{\infty} \Delta \omega g(n\Delta \omega) e^{in\Delta \omega t}. \quad (21)$$

Take the limit as $T \to \infty$. Then, $n\Delta \omega$ becomes the continuous variable $\omega$ and the summation becomes an integral as $\Delta \omega = 2\pi / T \to d\omega$.

The Fourier transform

"time domain" $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega. \quad (22)$

"frequency domain" $g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (23)$

We define $g(\omega)$ to be the Fourier transform of $f(t)$:

Fourier transform $\mathcal{F}[f(t)] = g(\omega) \quad (24)$

inverse transform $\mathcal{F}^{-1}[g(\omega)] = f(t) \quad (25)$
Remarks:

- As with the Fourier series, the Fourier transform is used to obtain information on the frequency spectrum. However, \( \mathcal{F}[f(t)] = g(\omega) \) is complex in general, even if \( f(t) \) is real. Fourier transform is often represented in terms of its magnitude and phase: \( g(\omega) = |g(\omega)| e^{i\theta(\omega)} \), where \( \theta = \tan^{-1}(\text{Im}(g(\omega))/\text{Re}(g(\omega))) \).
- Prefactor \( (1/\sqrt{2\pi}) \) of the transforms depends on convention. Here we chose them to be distributed symmetrically.
- Often, the Fourier transform is used to go from the time domain to the frequency domain. However, mathematically, the original transform could be considered the inverse and vice versa. Therefore, the sign in the exponential also depends on the convention.

The Dirac delta function

If Fourier transform are consistent, then inserting \( f(t) \) into \( g(\omega) \) should lead to an identity:

\[
g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega') e^{i\omega't} d\omega' \right) e^{-i\omega t} dt \tag{26}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( 1 \right) e^{i(\omega' - \omega) t} dt \tag{27}
\]

The term in the square brackets \([\ldots]\) has to be the Dirac delta function:

\[
\delta(\omega' - \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega' - \omega) t} dt \tag{28}
\]

Consider the same integral with finite limits:

\[
\delta(\omega' - \omega) = \frac{1}{2\pi} \int_{-\tau}^{\tau} e^{i(\omega' - \omega) t} dt = \frac{\sin((\omega' - \omega)\tau)}{\pi(\omega' - \omega)}
\]

For \((\omega' - \omega) \approx 0, \) this can be approximated as

\[
\frac{\sin((\omega' - \omega)\tau)}{\pi(\omega' - \omega)} \approx \frac{1}{\pi(\omega' - \omega)} \left[ \tau(\omega' - \omega) - \frac{\tau^3(\omega' - \omega)^3}{3!} + \ldots \right]
\]

\[
\approx \frac{\tau}{\pi} - \frac{\tau^3}{3!\pi}(\omega' - \omega)^2 + \ldots
\]

The peak height is \( \propto \tau \) and the width is \( \propto 1/\tau \), therefore the area under the curve remains (approximately) constant.
Eqns. (27) and (28) have to hold for any function $g(\omega)$. Set $g(\omega) = 1$:

$$1 = \int_{-\infty}^{\infty} \delta(\omega' - \omega) d\omega'$$  \hspace{0.5cm} (29)

However, according to eqn. (28), $\delta(0) = \infty$.

Delta function is a distribution, not a function, and has only a meaning inside an integrand!

### Properties of the Fourier transform

**Linearity**: $\mathfrak{F}[f_1(t) + f_2(t)] = \mathfrak{F}[f_1(t)] + \mathfrak{F}[f_2(t)]$.

**Shifting**: $\mathfrak{F}[f(t - t_0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t} dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega(t_0 + \tau)} d\tau$$

$$= e^{-i\omega t_0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega \tau} d\tau$$

where $\tau = t - t_0$. Similarly, $\mathfrak{F}^{-1}[g(\omega - \omega_0)] = e^{i\omega_0 t} f(t)$

**Scaling**: Let $\mathfrak{F}[f(t)] = g(\omega)$ and $\alpha > 0$. Then,

$$\mathfrak{F}[f(\alpha t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \int_{-\infty}^{\infty} f(t') e^{-i\omega \alpha t'} dt'$$

$$= \frac{1}{\alpha} g(\omega/\alpha),$$

where substitution $t' = \alpha t$ was made. For $\alpha < 0$, we obtain

$$\mathfrak{F}[f(\alpha t)] = -\frac{1}{|\alpha|} g(\omega/\alpha)$$

**This is a crucial relation**. As function $f(t)$ becomes more localized, its Fourier transform becomes broader in frequency domain.

![Diagram](image_url)
Consider the integral \( I = \int_{-\infty}^{\infty} f_1(t) \overline{f_2(t)} \, dt \). Substitute Fourier transforms of \( f_1(t) \) and \( f_2(t) \):

\[
I = \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_1(\omega) e^{i\omega t} \, d\omega \right] \overline{\left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_2(\omega') e^{i\omega' t} \, d\omega' \right]} \, dt
\]

For \( f_1(t) = f_2(t) \), we obtain Parseval’s theorem:

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\overline{g(\omega)}|^2 \, d\omega \quad (30)
\]

**Higher dimensions**

So far considered 1D Fourier transform. Generalization to higher dimensions as follows:

**2D:**

\[
\mathcal{F}[f(x, y)] = g(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(k_x x + k_y y)} \, dx \, dy
\]

**3D:**

\[
\mathcal{F}[f(x, y, z)] = g(k_x, k_y, k_z) = \left( \frac{1}{2\pi} \right)^\frac{3}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-i(k_x x + k_y y + k_z z)} \, dx \, dy \, dz
\]

\[
= g(\vec{k}) = \left( \frac{1}{2\pi} \right)^\frac{3}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} \, d\vec{r}
\]