Cylindrical Packing of Foam Cells

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(Received December 19; 1994; Accepted March 26, 1995)

Keywords: Foam, Structure Transition, Bubbles, Phyllotaxis, Cylindrical Packing

Abstract. We present results of observations of the structures found in a cylindrical foam of equal bubbles. This work greatly extends the preliminary findings of Weaire et al. Photographs of many of these structures are included.

1. Introduction

Weaire et al. (1992) have shown that when bubbles of equal size form a foam within a cylindrical tube the surface structure is generally a perfect hexagonal honeycomb. The variety of spiral structures which they observed recall the phyllotactic patterns of plant growth (Thompson, 1942) as well as analogous effects in superconductivity (Levithov, 1991). However, since the published results were very preliminary, we have repeated the experiment to obtain more extensive data, as presented here. Such data should provide a more reliable point of comparison for a theory of such structures, which we intend to develop in subsequent papers.

As before, bubble structures have been observed and recorded for a sequence of values of \( \lambda \), the ratio \( D/d \) of tube diameter \( D \) to bubble diameter \( d \). Except as noted below, the experimental set-up was precisely as previously described:

According to the standard notation used in phyllotaxis, all structures are annotated by the parameters \( k, l, m \) \((k \geq l \geq m)\). These parameters relate to the description of the structure as a spiral: a strip whose width comprises \( k, l \) or \( m \) cells can be wound on the cylinder to create the \( k, l, m \) structure. The phyllotaxis describes the arrangement of identical florets in composite flowers or plants (daisy, pine cone, cauliflower, pineapple, etc.). Every bubble at the surface of the tube (every floret on the flower) has 6 neighbours and belongs to 3 different visible spirals. These spirals belong to 3 families with \( k, l \) and \( m \) distinct spirals in each family \( k > l > m \) in phyllotaxis; \( k \geq l \geq m \) in our observations. \( k \) is the number of the steepest spirals, \( l \) and \( m \) the numbers of less steep spirals.

If one labels the bubbles by the integers \( z = 1, 2, ..., N \), \( z = 1 \) being the lowest and \( z = N \) the highest bubble in the tube, then the neighbours of bubble \( z \) are indicated by \( z + k \),
$z + l$ and $z + m$. It follows immediately that

$$k = l + m,$$

since one can go from bubble $z$ to $z + k$ directly or with a jog through bubble $z + m$ (RIVIER et al., 1984). In the case of non-phyllotactic structures $(k, k, 0$ or $2m, m, m)$, these rules are still valid provided that the same $z$ is given to all bubbles with the same height.

In the present work we have traced more carefully the structural transitions which occur as $\lambda$ is varied, and made some notes of internal structure, which previously had been ignored.

2. Essential Features of the Structures

Figure 1 shows the different ranges of $\lambda$ for the observed structures. As in the previous discussion, the structure was imagined to be rolled out to form an undistorted hexagonal lattice, and the vector $V$ identified as the smallest which takes a given cell to an equivalent one. Its magnitude is given by

$$V^2 = (k - m / 2)^2 + (3 / 4)m^2 = l^2 + lm + m^2.$$

It is approximately equal to $\pi \lambda$ since, if we wrap the lattice around the cylinder, the displacement described by $V$ follows its circumference.

For $\lambda$ up to 2.5 or $V$ up to 6.0 (6, 0, 0) the number of observations greatly exceeds that of the original experiment. This explains why some structures, which were not found by WEAIRE et al. (1992) (specifically 2, 2, 0; 3, 3, 0; 4, 3, 1; 4, 4, 0; 6, 4, 3; 6, 5, 1), have been observed here. On the other hand 5, 4, 1 (which was observed by Weaire et al.) was not seen. The new structures are comparatively rare, that is, they occur in narrow ranges of $\lambda$.

Photographs of the structures are shown in Fig. 2, and sketches of the topological structure of some surface cells are shown in Fig. 3. In some cases several internal structures are possible for given $k, l, m$.

Table 1 provides further information and comments on the observations. As $\lambda$ increases, the first appearance of an internal bubble is in the 5, 3, 2 structure. In all of the cases reported here there is at most a single chain of connected internal bubbles. This is indicated by the number 1 for the parameter $n$ in the table. When necessary a further letter is added to discriminate between different structures.

It will undoubtedly be possible to create many more structures with more complex ordered internal arrangements, as $\lambda$ is further increased. The present notation will then have to be replaced with something more akin to the standard crystallographic nomenclature. Recent advances in the analysis of photographs of foam structures (THOMAS et al., 1995) promise to facilitate the progression to this level of precision by providing quantitative geometrical information.
Fig. 1. Observations of different surface structures, most of which are listed in Table 1. Examples are shown as photographs in Fig. 2.
Fig. 2. Photographs of various cylindrical foam structures.
Fig. 2. (continued).

Table 1.

<table>
<thead>
<tr>
<th>k, l, m - n (n = 1 denotes internal bubbles)</th>
<th>V</th>
<th>Range of $\lambda$</th>
<th>Occurrences</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1, 0 - 0</td>
<td>1.000</td>
<td>0.44–1.25</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>2, 1, 1 - 0</td>
<td>1.732</td>
<td>1.05–1.53</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>2, 2, 0 - 0</td>
<td>2.000</td>
<td>1.10–1.24</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3, 2, 1 - 0</td>
<td>2.646</td>
<td>1.25–1.83</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>3, 3, 0 - 0</td>
<td>3.000</td>
<td>1.53–1.68</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4, 2, 2 - 0</td>
<td>3.464</td>
<td>1.25–1.98</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>4, 3, 1 - 0</td>
<td>3.606</td>
<td>1.978–1.984</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4, 4, 0 - 0</td>
<td>4.000</td>
<td>1.58</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5, 3, 2 - 0</td>
<td>4.359</td>
<td>1.55–2.34</td>
<td>11</td>
<td>the transition 5/3/2/1 to 5/3/2/0 occurs at $\lambda = 2.14$</td>
</tr>
<tr>
<td>5, 3, 2 - 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 4, 1 - ?</td>
<td>4.583</td>
<td></td>
<td></td>
<td>has been observed only under particular conditions</td>
</tr>
<tr>
<td>5, 5, 0 - 1</td>
<td>5.000</td>
<td>1.55–2.38</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>6, 3, 3 - 1t</td>
<td>5.169</td>
<td>1.76–2.56</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>6, 3, 3 - 1k</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

never observed pentagonal dodecahedron
6/3/3/1t has been observed with an irregular internal cell and is a transitional structure before 6/3/3/1k which contains a Kelvin cell as internal cell.
Fig. 3. Sketches of the internal topology of surface cells. The structure $5, 3, 2 - 1$ can be described by an arrangement of $P_1$, $P_2$ and $P_3$ cells. $6, 3, 3 - 1t$ is described by $P_1$ and $P_2$. $6, 3, 3 - 1k$ needs only $P_1$.

3. Structural Transitions

We can explore when structural transitions occurs for increasing $\lambda$, by adjusting the gas supply. Each step in Fig. 4(a) corresponds to a structural transition, which is not necessarily the limit of stability of the relevant structure. Arrows are drawn to show the
Fig. 4. (a) Structural transitions observed on increasing/decreasing the bubble-tube size ratio $\lambda$. (b) Structural transitions observed with the modified apparatus shown in Fig. 5.
Fig. 5. Variation of the apparatus to include a conical funnel.

direction of variation of $\lambda$, increasing ($\lambda$ from 1.1 to 2.5) and then decreasing ($\lambda$ from 2.3 to 1.1). The increase or the decrease of $\lambda$ gives transitions to structures which are not necessarily the closest, in terms of the magnitude of $V$. The same sequence is observed in both directions, with a large degree of hysteresis.

A similar experiment has been performed using a special tube with a funnel (Fig. 5) instead of the usual cylindrical glass tube. When the conical part is filled up, bubbles are forced into the cylindrical part with an additional pressure made by the bubbles in the conical part. Under such conditions we have increased $\lambda$ from 0.8 to 1.8. As shown in Fig. 4(b), this method presents us with many more structures, for reasons which remain obscure; only by using it did we find the structure 2, 2, 0.

Structural transitions can be induced by introducing a steady drainage which can increase the liquid fraction according to Weaire et al. (1993). In this way we were able to observe a transition from 2, 1, 1 to 1, 1, 0. A video recording of this is available.

4. Conclusion

Although it seems that all or almost all surface structures $k, l, m$ occur, they are by no means equally common. Explanation of the preference for some structures over others must await a theory. One thing is clear, however, that there is no trace of the celebrated preference for Fibonacci numbers which is such a feature of phyllotaxis in plants.

This research was supported by the EU HCM Programme, FOAMPHYS Network, Contract ERBCHRXCT940542. N. Pittet is supported by the Swiss OFES, Contract OFES94.0066.
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