1. Urn A contains two white and four red balls, urn B contains three white and three red balls. The urns are not labelled so that it is not known which is which. One of the urns is selected at random (each with probability $\frac{1}{2}$) and a ball is drawn from it.

(a) Compute the probability that this ball is red.

(b) If the ball drawn is red, what is the probability that the urn from which it was drawn is A?

(c) The first ball drawn is red, this is left out. A second ball is drawn from the same urn. What is the probability that the second ball is red?

(d) If two draws result in red balls, what is the probability that the urn from which they were drawn is A?

(e) In a game involving these urns your objective is to draw two red balls. The first ball drawn is red, you now have a choice which urn to draw the second ball from, the same urn or the other urn. What should you do?
2. (a) Three fair six sided dice (3D6) are thrown.

(i) What is the number of equally likely outcomes?

(ii) Compute the probability that no 6 appears on any of the three dice.

(iii) Let $Y_3$ be the maximum number showing of the three dice, compute the pdf of $Y_3$. ($P(Y_3 = y)$ is required for all possible values of $y$.)

(b) Let $Y_2$ be the maximum on two dice (2D6). Compute the pdf of $Y_2$ and hence $P(Y_3 > Y_2)$, where $Y_3$ is as in (a) above. Assume $Y_3$ and $Y_2$ are independent.

(c) In the "Really Small Lottery" three numbers are drawn from the numbers 1 to 36. Compute the probabilities of matching one, two and three numbers. One thousand coupons are filled in randomly. Compute the approximate probability that there are no "match three" coupons. What is the expected number of such coupons?

3. A random variable $X$ has the pdf:

$$f_X(x) = \begin{cases} (a+1) x^a & 0 \leq x \leq 1, a > -1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the distribution function of $X$, $F_X(x) = P(X \leq x)$. Compute $E(X)$.

(b) If $a=2$, compute $P(X>0.2)$, $P(X>0.5 \mid X>0.2)$ and $P(X>E(X))$.

(c) $X$ is used as a model of the proportion of metal contained in a sample of ore. e.g. $X=0.3$ means that 30% of the weight of the sample is metal and 70% useless rock.

(i) Samples from location A have $a = -0.5$, samples from location B have $a = 0.5$. Where would you dig the mine?  

Contd.
(ii) A sample of 100 grammes of ore has been obtained from A. Compute the probability that originally this sample contained at least 20 grammes of metal.

(iii) Given that 20 grammes of metal have been extracted from a sample originally weighing 100 grammes compute:

1. the probability that the sample contains at least another 30 grammes of metal;
2. \( P(Y \leq y) \), the pdf of \( Y \) and \( E(Y) \), where \( Y \) is the amount of metal remaining in this sample.

(d) \( n \) independent observations \( X_1, X_2, \ldots, X_n \) of metal contents were obtained from the same location. Compute the likelihood of these data and obtain an expression for the maximum likelihood estimator for \( a \).

4.
(a) The distribution function of a discrete random variable \( X \) is:

\[
F_X(x) = \frac{x}{x + 1} \quad x = 0, 1, 2, \ldots
\]

(i) Compute \( P(X=x) \).

(ii) Demonstrate that \( E(X) = \infty \).

(b) \( Z_1 \) is a Bernoulli random variable with parameter \( p_1 \), so that:

\[
P(Z_1 = 1) = p_1, \\
P(Z_1 = 0) = 1 - p_1.
\]

Compute the probability generating function (PGF) of \( Z_1 \), \( G_{Z_1}(\theta) = E(\theta^X) \).

\( Z_2 \) is an independent Bernoulli random variable with parameter \( p_2 \).

Compute the PGF of \( Z_1 + Z_2 \).

(c) \( W \) is a three valued random variable with probabilities described by the table overleaf:
<table>
<thead>
<tr>
<th>w</th>
<th>P(W=w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.08</td>
</tr>
<tr>
<td>1</td>
<td>0.44</td>
</tr>
<tr>
<td>2</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Compute the PGF of W, hence or otherwise show that W can be thought of as a sum of two independent Bernoulli random variables. Obtain their parameters $p_1$ and $p_2$.

5. (a) Prove that if $X$ and $Y$ are independent random variables then $E(h_1(X)h_2(Y)) = E(h_1(X))E(h_2(Y))$, where $h_1$ and $h_2$ are some real valued functions such that $-\infty < E(h_1(X)) < \infty$ and $-\infty < E(h_2(X)) < \infty$. You may assume that $X$ and $Y$ are proper continuous random variables.

(b) Prove that if $X_1$ and $X_2$ are i.i.d. random variables with moment generating function $\Psi_X(t)$ and $Z = X_1 + X_2$ then:

$$\Psi_Z(t) = (\Psi_X(t))^2$$

Obtain the cumulant generating function (CGF) of the average of $X_1$ and $X_2$, $W = \frac{1}{2}Z$.

The coefficient of skewness is defined to be:

$$\gamma(W) = \frac{E((W-E(W))^3)}{V(W)^{3/2}}$$

Show that $\gamma(W) < \gamma(X)$.

6. (a) Compute the pdf of the maximum of two i.i.d. exponential random variables with parameter $\lambda$. Compute the mean.

(b) A system of type A consists of two parallel components with independent exponential lifetimes with parameter $\lambda$. The system only fails when both components fail.
A type B system contains a single longer lasting component with exponential lifetime with parameter $\frac{\lambda}{2}$.

A type C system again consists of two components with parameter $\lambda$, however in this case the second component only switches on when the first one has failed.

With $\lambda=1$ (in appropriate time units), for each of A, B and C compute:

(i) the probability the system lasts longer than 1 time unit.

(ii) given the system has lasted at least 1 time unit, the probability that it lasts at least another 1 time unit.

(iii) the mean life time of the system.

For systems B and C compute the probabilities that exactly 2 components are required for continuous operation over 2 time units.

7. The joint pdf of $X$ and $Y$ is:

$$f_{XY}(x,y) = \begin{cases} 2(x + y - 2xy) & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $X$ and $Y$ are uniform $U(0,1)$ random variables.

Compute $f_{X|Y=0.5}(x)$.

Compute the covariance $\text{Cov}(X,Y)$. (Note: you may do this here or use the general result derived in (b) below.)

(b) Show that if the joint pdf of $X$ and $Y$ is:

$$f_{XY}(x,y) = (c+1)(x^c + y^c - (c+1)x^c y^c) \quad 0 \leq x \leq 1, 0 \leq y \leq 1;$$

$$0 \quad \text{otherwise};$$

then $X$ and $Y$ have $U(0,1)$ distributions. Why can we not have $c=2$?

Compute the covariance of $X$ and $Y$. 

8. The vector \((X,Y,Z)\) is trivariate normal with:

\[
\mu = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}.
\]

Compute the means and variances of:

(a) \(X - Y - Z\).
(b) \((X,W)\) where \(W = X + Y\).
(c) \(X\) given \(Y = 1\) and \(Z = 1\).
(d) \(X + Y\) given \(X + Z = 3\).