Attempt 6 Questions

1. Given a set of \( n \) pairs of numbers \((x_i, y_i)\), for \( i = 0, \ldots, n - 1 \), the least-squares straight line which fits this data is defined by \( y = ax + b \) where

\[
a = \frac{\bar{x} \bar{y} - \bar{x} \bar{y}}{\bar{x}^2 - (\bar{x})^2} \quad \text{and} \quad b = \bar{y} - a \bar{x}
\]

Write a C program which reads in \( n \) pairs of numbers \((x_i, y_i)\), and calculates and prints the values of \( a \) and \( b \) which define this least squares straight line fit.

2. Convert the following numbers between base 10 (decimal), and base 16 (hex): 

\((231)_{10}, -(ab)_{16}, (23.46)_{10}, \text{ and } (c.a3)_{16}\).

Assuming integers are stored in 2's complement format using 4 bytes per number, and reals are stored in IEEE double precision format, determine the HEX patterns by which each of these numbers is represented in the computer.

What is the largest positive real number (excluding infinity) which can be stored in IEEE single precision format? How is \(+\infty\) stored in IEEE single precision format?
3. Write C functions which execute the following tasks:

(a) Given \( x \), evaluates and returns \( f(x) = (3 \sin(x) + x^{109})/((2x - 1)(\cos(x))) \)

(b) Given a real value \( x \) and an integer value \( n > 0 \), evaluates the sum of the first \( n \) terms in the Taylor series expansion for \( \sin(x) \).

(c) Prompts for and reads in two double values \( x \) and \( y \), and an integer value \( i \). After your function reads in these values, arrange that it checks that \( x > 1.0 \), \( y < 8.0 \), and \( 0 <= i <= 100 \). If any value is outside its range, your function should print an error message and exit. If all values are in range your function should return the values to the calling program.

Write a small C program which calls each of the three functions above in sequence.

4. Indicate the order of evaluation of the following C expressions by adding parenthesis, (for example \( x = a + b \rightarrow (x = (a+b)) \)),

\[
\begin{align*}
1 + j + x \\
x + y * i / j * x \\
x > 6 && y < 5 || i/j > 0 \\
x = y = i <= 7
\end{align*}
\]

If \( x \) and \( y \) are type double and \( i \) and \( j \) are type int. what values do these expressions return if \( x = 3.0, y = 5.0, i = 3, \) and \( j = 4 \).

Describe the memory map which results when the following program is compiled. Trace what happens in memory when the program executes.

```c
#define N 4
double x[N], y[N];
main()
{
    int i;
    double *t_p, y[N];
    for ( i = 0; i < N; i++ )
    {
        x[i] = i+1;
        y[i] = 2 * i * i;
    }
    t_p = &x[0];
    *t_p = y[0];
    t_p = t_p + 1;
    *t_p = y[1];
    t_p = y;
    *t_p = 3.5;
}
```
5. Describe the Bracketing and Bisection Method and the Newton method to find the root of a function \( f(x) \).

If \( x_i \) is the estimate of the root that either method produces at iteration \( i \), prove that

\[
(x_{i+1} - x_i) \propto (x_i - x_t)/2 \quad \text{for Bracketing and Bisection}
\]
\[
(x_{i+1} - x_i) \propto (x_i - x_t)^2 \quad \text{for Newton}
\]

where \( x_t \) is the true root.

Apply either the Bracketing and Bisection method or the Newton method to find two different roots of the function

\[
f(x) = x^2 - 3x - 5
\]
correct to three decimal places.

6. A sufficient condition for a continuous function \( f(x) \) to have a minimum in an interval \([a, c]\) is that there exists a \( b \) with \( a < b < c \) such that \( f(a) > f(b) \) and \( f(c) > f(b) \). Describe how this observation can be used to define an algorithm to find a minimum of \( f(x) \).

Show that the optimum implementation of this algorithm makes use of the factor \( R \) which is defined by the equation

\[
\frac{R}{1 - R} = \frac{1 - R}{1}
\]

Find the numerical value of \( R \).

Apply this algorithm to find the minimum of the function

\[
f(x) = x^2 - 3x - 5
\]
correct to 3 decimal places.

7. Convert the second order differential equation

\[
\frac{d^2x}{dt^2}(t) = 1 + x(t)
\]

with initial conditions

\[
x(0) = a \quad \frac{dx}{dt} = 0
\]

to two coupled first order equations.

Define the Euler algorithm to integrate these coupled equations, and apply this algorithm for \( n \) steps, each of size \( h \) to find \( x(T = nh) \).

Compare the answer you obtain with the answer obtained by solving the differential equation exactly, and comment on any differences.
8. The trapezoidal rule approximates

\[ I = \int_{x}^{x+h} f(x) \, dx \]

with the formula

\[ I = \frac{h}{2} (f(x) + f(x + h)) \]

Show that the error of this approximation is \( O(h^3) \).

The \( n \) step extended trapezoidal rule divides the interval from \( a \) to \( b \) into \( n \) equal subintervals, and uses the trapezoidal rule to evaluate the integral on each subinterval. Estimate the error made in approximating

\[ \int_{a}^{b} f(x) \, dx \]

with this extended rule. This error estimate should be given in terms of \( b - a \) and \( n \).

Write a C function to calculate \( \int_{a}^{b} f(x) \, dx \) using an \( n \) step extended trapezoidal rule.