Reimbedding and the Schoenflies Conjecture

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Setting: \( S^3 \cong_{PL/DIFF} P \subset S^4 \) divides \( S^4 \) into \( X \) and \( Y \).

**Conjecture (Schoenflies Conjecture)**

\[ X \text{ and } Y \cong_{PL/DIFF} D^4. \]

**Observations:**

- \( X \cong_{PL/DIFF} D^4 \iff Y \cong_{PL/DIFF} D^4 \).
- Handles of \( P \) are level and appear with non-decreasing index.
- A handle into \( X \) only affects topology of \( Y \) (and vice versa).
- Sample lemma: If all all 0,1-handles on same side, then \( X \) and \( Y \cong_{PL/DIFF} D^4 \).
$P$ in Kearton-Lickorish position $\implies S_0^3 \cap P$ Heegaard splits $P$.

The genus of the embedding is the genus of this Heegaard surface.
Example: Schoenflies Conjecture obviously true for genus 1:
If 1-handle knotted, where can 2-handle go?

This would give non-separating 2-sphere $\Rightarrow$ not $S^3$.
If 1-handle **unknotted**, where can 2-handle go?

This leaves $X$ standard. (All handles lie in $X$.)
Some philosophical observations/questions:

- Argument hardest when $S_0^3 \cap P$ Heegaard splits $S_0^3$ also.


- Agol-Freedman: if allowed to stabilize (increase genus) then any embedding with one max can be isotoped to Heegaard embedding: every generic $S_t^3 \cap P$ Heegaard splits $S_t^3$.

- So focus on special case of Heegaard embeddings of $P$?
Suppose $P \subset S^4$ is genus 2 embedding.

**Possibility 1:** One 0-handle, two 1-handles, all lying in $X$, say. Then all handles on same side $\implies$ done.

**Possibility 2:** One 0-handle, two 1-handles, one each in $X$ and $Y$:

Generic case:

![Generic case diagram]

Solution: where’s the next 2-handle?
Special case for second 1-handle:

Many possibilities for next 2-handle...

Regardless, know $X$ will have only one 1-handle and one 2-handle (as will $Y$).
Lucky break - can invoke Gabai’s celebrated Property R:

**Theorem (Property R Gabai 1987)**

If surgery on knot $K \subset S^3$ gives $S^1 \times S^2$, then $K$ is the unknot.

Relevance: Consider trace of this surgery, i.e. associated 4-manifold cobordism between $S^3$ and $S^1 \times S^2$.

**Corollary**

Suppose $U^4$ is homology ball, with $\partial U \cong S^3$. If $U$ has a single 1-handle and a single 2-handle, then $U \cong B^4$. 
Proof: \((0\text{-handle } \cup 1\text{-handle})\) is \(S^1 \times D^3\), so boundary is \(S^1 \times S^2\). Adding 2-handle surgeries boundary to \(S^3\).

In reverse direction: have surgered \(S^3\) to get \(S^1 \times S^2\). Gabai says \(K \subset S^3\) is just \(S^1 \subset S^3\).

More general point: big 3-manifold theorem is useful for this 4-dimensional question.
Side observation: may as well assume first 1-handle is unknotted, by reimbedding:

- All 2-handles will be trapped in knotted torus, so reimbedding extends thru 2-handles.
- Reimbedding changes $X$ (becomes $X'$ say) but $Y$ unchanged.
- If can show $X' \cong D^4$ then: $\implies Y \cong D^4 \implies X \cong D^4$.
- After the reimbedding, $S_0^2 \cap P$ Heegaard splits $S_0^2$. 
Possibility 3: Two 0-handles, three 1-handles, one connecting:

Use reimbedding trick.
Not a Heegaard surface, but at least one side $Y$ is a handlebody.

For genus two, reimbedding always can make one side a handlebody; here's why:

- $S_0^t \cap P$ compresses in either $X$ or $Y$, say $Y$
- If compresses completely $\implies Y$ a handlebody.
- If doesn't compress then get knotted companion tori, which describe reimbedding.
Let

- $X_0 = X \cap S^3_0$
- $X_- = X \cap S^3 \times [-1, 0]$
- $X_+ = X \cap S^3 \times [0, 1]$

Similarly denote $Y_0, Y_-, Y_+$, with $Y_0 = Y_- \cap Y_+$
Is it possible to reimbed $Y$ so $X_0$ is a handlebody (or vice versa)? Would such a reimbedding be useful?

Reimbedding possibility: Observation 1

**Theorem (Fox 1948)**

Any compact connected 3-dimensional submanifold of $S^3$ can be re-imbedded as the complement of handlebodies.

Hence always $Y_0$ can be reimbedded so $X_0$ becomes a handlebody.

But how to do this so that $Y_t, t \neq 0$ can also be reimbedded? Need reimbedding to extend over 2-handles. (Companion tori guaranteed this in genus 2 case above.)
Observation 2:
Suffices to construct a sequence of reimbeddings:

1. Reimbed $Y$ so $X_0$ is more compressible than it was
2. Reimbed new $X$ so $Y_0$ is more compressible than it was
3. Reimbed new $Y$...

If eventually get $X_0$ (or $Y_0$) a handlebody, and if this property can be used to show $X$ or $Y$ is now $D^4$, then done.

Observation 3:
Deep theorems in 3-manifold theory (using Gabai’s sutured manifold theory) shows that reimbedding sequence to handlebody works in genus 3 case.
Observation 4:

**Theorem (Agol-Freedman 2013)**

*If P has single maximum then can isotope \( P \subset S^4 \) so that each \( X_t \) and \( Y_t \) are *both* handlebodies. (Does not require \( P = S^3 \).)*

Each \( P_t \) is Heegaard surface in \( S^3_t \), so it’s a **Heegaard embedding**. (This may have been known to Eaton.)

**Sketch:**

- For knotted 1-handle in \( Y_t \) (say), introduce extra 1-handles (’intervention arcs’ = IA) so \( Y_t \) remains handlebody.
- With even more IA’s, the IA’s can be added or deleted one-by-one.
- Delete IA’s by handle-cancellation. Then result isotopic (not level-preserving) with original
Toy example: $X = S^1 \times D^3$, $Y = D^2 \times S^2$, each $X_t$ a handlebody

Clearly isotopic if, near end, just add canceling 1 & 2-handle
Introduce 1-handle earlier

Now each $Y_t$ also a handlebody.
Problem? Requires massive stabilization of the embedding - no control on genus.
Another source of intervention arcs: Handle-slides in Heegaard theory... are replaced with arc additions and deletions: