Topic 6: Investment With Adjustment Costs

Over the past few weeks we have seen a number examples of forward-looking first-order stochastic difference equations of the form

$$y_t = ax_t + bE_t y_{t+1} \tag{1}$$

The solution that we have derived has been of the form

$$y_t = a\sum_{k=0}^{\infty} b^k E_t x_{t+k} \tag{2}$$

so that y_t is a completely forward-looking variable. Note that this means that y_t does not depend at all on its own past values. We will now turn to an example which does not correspond to this case.

Specifically, we will look at a theory of the determination of the capital stock (and thus investment). Empirical studies show that the capital stock does not change very much from period to period. Economists usually rationalise this by assuming that there are some form of "adjustment costs" that prevent firms from changing their capital stock too quickly. In this handout, we will consider a model of investment with adjustment costs, show that it implies a *second-order* stochastic difference equation, and examine the methods used to solve these types of equations.

The Firm's Problem

Consider now the following model of firm investment. We will assume that, each period, there is a level of the log of the capital stock, k_t^*, that the firm would choose if there were no adjustment costs. We will call this the *frictionless optimal capital stock.* With adjustment costs the firm has to choose a planned sequence of capital stocks $E_t\{k_t, k_{t+1}, k_{t+2}, ...\}$ minimise the following "loss function"

$$L(k_t, k_{t+1}, t_{t+2}, ...) = E_t\left[\sum_{m=0}^{\infty} \theta^m \left\{\left(k_{t+m} - k_{t+m}^*\right)^2 + \alpha\left(k_{t+m} - k_{t+m-1}\right)^2\right\}\right] \tag{3}$$

This might look a bit intimidating but it's not too complicated:

- Firstly, for each period, $t + m$, there is a term $\left(k_{t+m} - k_{t+m}^*\right)^2$ that describes the loss in profits suffered by the firm from not having its capital stock equated with the frictionless optimal level.

- Secondly, there is a term $\alpha \left(k_{t+m} - k_{t+m-1}\right)$ which describes the concept of adjustment costs formally: *Ceteris paribus* changes in the capital stock have a negative effect on firm profits.
- The reason we are assuming that k_t is actually the log of the stock, as opposed to the stock itself, is that this way these losses can be viewed in percentage terms: It is the percentage gap between capital and its frictionless optimal that matters and also the percentage change in the stock. This makes more sense than levels of these gaps mattering because economic growth will make levels of these variables grow over time.
- Finally, the parameter θ is a discount rate less than one, which tells us that firms care more about profits today than profits tomorrow.

This loss function can be re-written as

$$\begin{aligned} L\left(k_t, k_{t+1}, t_{t+2}, ...\right) &= \left(k_t - k_t^*\right)^2 + \alpha\left(k_t - k_{t-1}\right)^2 + \theta E_t\left[\left(k_{t+1} - k_{t+1}^*\right)^2 + \alpha\left(k_{t+1} - k_t\right)^2\right] \\ &\quad + \theta^2 E_t\left[\left(k_{t+2} - k_{t+2}^*\right)^2 + \alpha\left(k_{t+2} - k_{t+1}\right)^2\right] + ... \end{aligned} \tag{4}$$

An optimal plan is arrived at by differentiating this with respect to each of the capital stock terms k_{t+m} and setting these derivatives equal to zero. Consider first differentiating with respect to k_t. This gives

$$2\left(k_t - k_t^*\right) + 2\alpha\left(k_t - k_{t-1}\right) - 2\alpha\theta E_t\left(k_{t+1} - k_t\right) = 0 \tag{5}$$

Again, try differentiating with respect to k_{t+1}. This gives

$$E_t\left[2\theta\left(k_{t+1} - k_{t+1}^*\right) + 2\alpha\theta\left(k_{t+1} - k_t\right) - 2\alpha\theta^2\left(k_{t+2} - k_{t+1}\right)\right] = 0 \tag{6}$$

This is the exact same as the previous first-order condition, only shifted forward one period. In fact one can show that all of the FOCs describing the optimal dynamics of the capital are consistent with the same second-order stochastic difference equation

$$E_t\left[\left(k_t - k_t^*\right) + \alpha\left(k_t - k_{t-1}\right) - \alpha\theta\left(k_{t+1} - k_t\right)\right] = 0 \tag{7}$$

Drawing terms together, this gives

$$-\alpha\theta E_t k_{t+1} + \left(1 + \alpha + \alpha\theta\right) k_t - \alpha k_{t-1} = k_t^* \tag{8}$$

which can be re-written as

$$E_t k_{t+1} - \left(1 + \frac{1}{\theta} + \frac{1}{\alpha\theta}\right) k_t + \frac{1}{\theta} k_{t-1} = -\frac{1}{\alpha\theta} k_t^* \tag{9}$$

Because the maximum difference between time subscripts is two, this is a second-order stochastic difference equation. There are two different methods that are commonly used to solve equations of this form. I will discuss the so-called *factorization* method. For completeness, I have also attached the derivation of the solution using the other method known as the *method of undetermined coefficients*, but you can ignore this if you wish.

Lag Operators

The factorization method makes use what are known as *lag* and *forward* operators. These are commonly used in calculations relating to time series, and they work as follows. The lag operator turns a variable dated time t into a variable dated time $t-1$:

$$Ly_t = y_{t-1} \tag{10}$$

Lag operators can be multiplied and added just like normal variables. So, for instance, one can write

$$L^k y_t = y_{t-k} \tag{11}$$

The forward operator has the reverse effect of the lag operator

$$F^k y_t = y_{t+k} \tag{12}$$

Lag and forward operators also obey a form of the geometric sum formula. Recall that for $-1 < \beta < 1$, we have

$$\sum_{m=0}^{\infty} \beta^m = \frac{1}{1-\beta} \tag{13}$$

Recall also that if $-1 < \beta < 1$ and

$$y_t = \beta E_t y_{t+1} + x_t \tag{14}$$

then the solution is

$$y_t = \sum_{m=0}^{\infty} \beta^m E_t x_{t+k} \tag{15}$$

Equation (14) can be re-written as

$$y_t = E_t \left[\frac{1}{1-\beta F} x_t\right] \tag{16}$$

So equation (15) means that

$$\frac{1}{1-\beta F}=\sum_{m=0}^{\infty}\beta^m F^m \tag{17}$$

The same applies for lag operators

$$\frac{1}{1-\beta L}=\sum_{m=0}^{\infty}\beta^m L^m \tag{18}$$

To verify that this is the case, note that if

$$y_t=\beta y_{t-1}+x_t \tag{19}$$

then one can apply repeated substitution to re-write this as

$$y_t=x_t+\beta x_{t-1}+\beta^2 x_{t-2}+\beta^3 x_{t-3}+..... \tag{20}$$

Armed with this knowledge of lag and forward operators we can solve the second-order stochastic difference equation using the factorization method.

Solution via Factorization

This method first re-writes equation (9) in terms of lag and forward operators. Written this way it is

$$E_t\left[\left(F-\left(1+\frac{1}{\theta}+\frac{1}{\alpha\theta}\right)+\frac{1}{\theta}L\right)k_t\right]=-\frac{1}{\alpha\theta}k_t^* \tag{21}$$

Next, the method re-expresses the left-hand-side in terms of a quadratic equation in F multiplied by L:

$$E_t\left[\left(F^2-\left(1+\frac{1}{\theta}+\frac{1}{\alpha\theta}\right)F+\frac{1}{\theta}\right)Lk_t\right]=-\frac{1}{\alpha\theta}k_t^* \tag{22}$$

Now, you may recall that polynominals of the form

$$g(x)=x^2+bx+c \tag{23}$$

can be re-written in terms of their roots as

$$g(x)=(x-\lambda_1)(x-\lambda_2) \tag{24}$$

where

$$\lambda_1+\lambda_2 = -b \tag{25}$$

$$\lambda_1\lambda_2 = c \tag{26}$$

In this case, one can show that the polynomial

$$x^2 - \left(1 + \frac{1}{\theta} + \frac{1}{\alpha\theta}\right) x + \frac{1}{\theta} \tag{27}$$

has two roots such that one root (λ) is between zero and one while the other equals $\frac{1}{\theta\lambda}$. This means that the optimality condition for the capital stock can be re-expressed as

$$E_t \left[(F - \lambda) \left(F - \frac{1}{\theta\lambda} \right) Lk_t \right] = -\frac{1}{\alpha\theta} k_t^* \tag{28}$$

Dividing across by $\left(F - \frac{1}{\theta\lambda}\right)$, this becomes

$$E_t \left[(F - \lambda) Lk_t \right] = -\frac{1}{\alpha\theta} E_t \left[\frac{1}{F - \frac{1}{\theta\lambda}} k_t^* \right] \tag{29}$$

Now we can use the properties of lag operators just derived to show that

$$\frac{1}{F - \frac{1}{\theta\lambda}} = \frac{-\theta\lambda}{1 - \theta\lambda F} = -\theta\lambda \sum_{k=0}^{\infty} (\theta\lambda)^k F^k \tag{30}$$

So, the capital stock process has a solution of the form

$$k_t = \lambda k_{t-1} + \frac{\lambda}{\alpha} E_t \left[\sum_{n=0}^{\infty} (\theta\lambda)^n k_{t+n}^* \right] \tag{31}$$

Note now how adding adjustment costs changes the solution for a rational expectations model. This produces a second-order difference equation, and the solution is no longer completely forward-looking. Instead, the capital stock has a forward-looking component, which is a geometric discounted sum, but it also has a backward-looking component, whereby it depends on its own lagged value.

An Example: Investment, Output, and the Cost of Capital

The model can be fleshed out by stating what are the determinants of the frictionless optimal capital stock. For instance, if the production function was of the Cobb-Douglas form, then the optimal capital stock would take the form

$$K_t^* = \frac{Y_t}{C_t} \tag{32}$$

where Y_t is output and C_t is the cost of capital. Using lower-case letters to denote logs, this can be written as

$$k_t^* = y_t - c_t \tag{33}$$

So, the capital stock is determined by

$$k_t = \lambda k_{t-1} + \frac{\lambda}{\alpha} E_t \left[\sum_{n=0}^{\infty} (\theta\lambda)^n (y_{t+n} - c_{t+n}) \right] \tag{34}$$

Now assume that output and the cost of capital both follow AR(1) processes

$$y_t = \rho_y y_{t-1} + \epsilon_t^y \tag{35}$$

$$c_t = \rho_c c_{t-1} + \epsilon_t^c \tag{36}$$

The infinite sum component of the solution can now be written as

$$\begin{aligned} E_t \sum_{n=0}^{\infty} (\theta\lambda)^n y_{t+n} &= \left[\sum_{n=0}^{\infty} (\theta\lambda\rho_y)^n \right] y_t \\ &= \frac{1}{1-\theta\lambda\rho_y} y_t \end{aligned} \tag{37}$$

while

$$E_t \sum_{n=0}^{\infty} (\theta\lambda)^n c_{t+n} = \frac{1}{1-\theta\lambda\rho_c} c_t \tag{38}$$

So, the capital stock process is

$$k_t = \lambda k_{t-1} + \frac{\lambda}{\alpha} \frac{1}{1-\theta\lambda\rho_y} y_t - \frac{\lambda}{\alpha} \frac{1}{1-\theta\lambda\rho_c} c_t \tag{39}$$

This gives us a "reduced-form" relationship between the capital stock, the lagged capital stock, output and the cost of capital.

Note that the magnitudes of the coefficients on output and the cost of capital depend positively on the *persistence* of these variables. If ρ_y is close to one, then the coefficient on output will be high, with the same applying for ρ_c and the cost of capital. One example of an application of this type of reasoning is in the Tevlin-Whelan *JMCB* paper on the reading list. That paper reports much larger coefficients on the cost of capital for computers than for non-computing equipment, and uses a model of this sort to provide an explanation. The cost of capital for computing equipment is largely determined by the very persistent shocks than lead to ever-decreasing computer prices. In contrast, for non-computing equipment, the cost of capital depends on a set of less persistent variables such as interest rates and tax incentives. This suggests that the cost of capital should have a smaller coefficient in a regression for the non-computing capital stock.

Completely Optional Appendix: The Undetermined Coefficients Method

The other method used to solve these models starts by assuming that one knows the form of the solution. So, one "guesses" that the solution is of the form

$$k_t = \lambda_1 k_{t-1} + \gamma E_t \left[\sum_{n=0}^{\infty} \lambda_2^n k_{t+n}^* \right]$$

From there, one goes on to figure out a unique set of values for λ_1, λ_2 andγ that are consistent with this equation, and with the optimality conditions for the capital stock. In this case

$$E_t k_{t+1} = \lambda_1 k_t + \gamma E_t \left[\sum_{n=0}^{\infty} \lambda_2^n k_{t+n+1}^* \right]$$

So, we have

$$\begin{aligned} -\alpha\theta \left[\lambda_1 k_t + \gamma E_t \left[\sum_{n=0}^{\infty} \lambda_2^n k_{t+n+1}^* \right] \right] + (1+\alpha+\alpha\theta) k_t - \alpha k_{t-1} &= k_t^* \\ (1+\alpha+\alpha\theta-\alpha\theta\lambda_1) k_t &= \alpha k_{t-1} + k_t^* + \alpha\theta\gamma E_t \left[\sum_{n=0}^{\infty} \lambda_2^n k_{t+n+1}^* \right] \end{aligned}$$

This can be re-written as

$$k_t = \frac{\alpha}{(1+\alpha+\alpha\theta-\alpha\theta\lambda_1)} k_{t-1} + \frac{k_t^*}{(1+\alpha+\alpha\theta-\alpha\theta\lambda_1)} + \frac{\alpha\theta\gamma}{(1+\alpha+\alpha\theta-\alpha\theta\lambda_1)} E_t \left[\sum_{n=0}^{\infty} \lambda_2^n k_{t+n+1}^* \right]$$

So, one can begin to make inferences about the coefficients:

$$\begin{aligned} \lambda_1 &= \frac{\alpha}{(1+\alpha+\alpha\theta-\alpha\theta\lambda_1)} \\ \gamma &= \frac{1}{(1+\alpha+\alpha\theta-\alpha\theta\lambda_1)} = \frac{\lambda_1}{\alpha} \\ \lambda_2 &= \alpha\theta\gamma = \theta\lambda_1 \end{aligned}$$

The solution is

$$k_t = \lambda k_{t-1} + \frac{\lambda}{\alpha} E_t \left[\sum_{n=0}^{\infty} (\theta\lambda)^n k_{t+n}^* \right] \tag{40}$$

where λ solves

$$\lambda (1+\alpha+\alpha\theta-\alpha\theta\lambda) = \alpha \tag{41}$$

This can be re-written as

$$\lambda^2 - \left(1 + \frac{1}{\theta} + \frac{1}{\alpha\theta}\right) \lambda + \frac{1}{\theta} = 0 \tag{42}$$

so the solution is the same as that derived from the factorization method above.

Personally, I am less fond of this method because it involves guessing the form of the solution, which is a bit of a cheat, because it is still quite algebra-intensive, and because it becomes impractical to apply once one moves to higher-order difference equations. In contrast, the factorization method can be used to characterize the solutions of difference equations of any order.