Topic 5: Rational Expectations, Consumption & Asset Prices

We will now explore how the methods used to derive the previous models of stock prices can also be used to derive rational-expectations-based models of household consumption. We will also briefly discuss the interaction between consumption behaviour and asset returns.

The Household Budget Constraint

We start with an identity describing the evolution of the stock of assets owned by households. Letting A_t be household assets, Y_t be labour income, and C_t stand for consumption spending, this identity is

$$A_{t+1} = (1 + r_{t+1})(A_t + Y_t - C_t) \tag{1}$$

where r_{t+1} is the return on household assets at time $t + 1$. Note that Y_t is *labour* income (income earned from working) not total income because total income also includes the capital income earned on assets (i.e. total income is $Y_t + r_{t+1}A_t$.) Note, we are assuming that Y_t is take-home labour income, so it can considered net of taxes.

As with the equation for the return on stocks, this can be written as a first-order difference equation in our standard form

$$A_t = C_t - Y_t + \frac{A_{t+1}}{1 + r_{t+1}} \tag{2}$$

We will assume that agents have rational expectations. Also, for the moment, we will assume that the expected return on assets equals a constant, r, (as in our original treatment of stock prices). This implies

$$A_t = C_t - Y_t + \frac{1}{1 + r} E_t A_{t+1} \tag{3}$$

Using the same repeated substitution methods as before this can be solved to give

$$A_t = \sum_{k=0}^{\infty} \frac{E_t (C_{t+k} - Y_{t+k})}{(1 + r)^k} \tag{4}$$

Note that we have again imposed the so-called "transversality condition" — in this case, it is that the terminal term $\frac{E_t A_{t+k}}{(1+r)^k}$ goes to zero as k gets large.

One way to understand this equation comes from re-writing it as

$$\sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{(1 + r)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1 + r)^k} \tag{5}$$

This is usually called the *intertemporal budget constraint*. It states that the present value sum of current and future household consumption must equal the current stock of financial assets plus the present value sum of current and future labour income.

A consumption function relationship can be derived from this equation by positing some theoretical relationship between the expected future consumption values, $E_t C_{t+k}$, and the current value of consumption. This is done by appealing to the optimising behaviour of the consumer.

Optimising Behaviour by the Consumer

We will assume that consumers wish to maximize a welfare function of the form

$$W = \sum_{k=0}^{\infty} \left(\frac{1}{1+\beta} \right)^k U\left(C_{t+k}\right) \tag{6}$$

where $U(C_t)$ is the instantaneous utility obtained at time t, and β is a positive number that describes the fact that households prefer a unit of consumption today to a unit tomorrow. If the future path of labour income is known, consumers who want to maximize this welfare function subject to the constraints imposed by the intertemporal budget constraint must solve the following Lagrangian problem:

$$L\left(C_t, C_{t+1}, ...\right) = \sum_{k=0}^{\infty} \left(\frac{1}{1+\beta} \right)^k U\left(C_{t+k}\right) + \lambda \left[A_t + \sum_{k=0}^{\infty} \frac{Y_{t+k}}{(1+r)^k} - \sum_{k=0}^{\infty} \frac{C_{t+k}}{(1+r)^k} \right] \tag{7}$$

For every current and future value of consumption, C_{t+k}, this yields a first-order condition of the form

$$\left(\frac{1}{1+\beta} \right)^k U'\left(C_{t+k}\right) - \frac{\lambda}{(1+r)^k} = 0 \tag{8}$$

For $k = 0$, this implies

$$U'\left(C_t\right) = \lambda \tag{9}$$

For $k = 1$, it implies

$$U'\left(C_{t+1}\right) = \left(\frac{1+\beta}{1+r} \right) \lambda \tag{10}$$

Putting these two equations together, we get the following relationship between consumption today and consumption tomorrow:

$$U'\left(C_t\right) = \left(\frac{1+r}{1+\beta} \right) U'\left(C_{t+1}\right) \tag{11}$$

When there is uncertainty about future labour income, this optimality condition can just be re-written as

$$U'(C_t) = E_t\left[\left(\frac{1+r}{1+\beta}\right)U'(C_{t+1})\right] \tag{12}$$

This implication of the first-order conditions for consumption is sometimes known as an *Euler equation.*

In an important 1978 paper, Robert Hall proposed a specific case of this equation.[1] Hall's special case assumed that

$$U(C_t) = aC_t + bC_t^2 \tag{13}$$

$$r = \beta \tag{14}$$

In other words, Hall assumed that the utility function was quadratic and that the real interest rate equalled the household discount rate. In this case, the Euler equation becomes

$$a + 2bC_t = E_t[a + 2bC_{t+1}] \tag{15}$$

which simplifies to

$$C_t = E_t C_{t+1} \tag{16}$$

This states that the optimal solution involves next period's expected value of consumption equalling the current value. Because, the Euler equation holds for all time periods, we have

$$E_t C_{t+k} = E_t C_{t+k+1} \qquad k = 1, 2, 3, ... \tag{17}$$

So, we can apply repeated iteration to get

$$C_t = E_t(C_{t+k}) \qquad k = 1, 2, 3, ... \tag{18}$$

In other words, all future expected values of consumption equal the current value. Because it implies that changes in consumption are unpredictable, this is sometimes called the *random walk* theory of consumption.

[1] "Stochastic Implications of the Life-Cycle Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy*, December 1978.

The Rational Expectations Permanent Income Hypothesis

Hall's random walk hypothesis has attracted a lot of attention in its own right, but rather than focus on what should be unpredictable (changes in consumption), we are interested in deriving an explicit formula for what consumption should equal.

To do this, insert $E_t C_{t+k} = C_t$ into the intertemporal budget constraint, (5), to get

$$\sum_{k=0}^{\infty} \frac{C_t}{(1+r)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k} \tag{19}$$

Now we can use the geometric sum formula to turn this into a more intuitive formulation:

$$\sum_{k=0}^{\infty} \frac{1}{(1+r)^k} = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r} \tag{20}$$

So, Hall's assumptions imply the following equation, which we will term the *Rational Expectations Permanent Income Hypothesis*:

$$C_t = \frac{r}{1+r} A_t + \frac{r}{1+r} \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k} \tag{21}$$

Let's look at this equation closely. It states that the current value of consumption is driven by three factors:

- The expected present discounted sum of current and future labour income.
- The current value of household assets. This "wealth effect" is likely to be an important channel through which financial markets affect the macroeconomy.
- The expected return on assets: This determines the coefficient, $\frac{r}{1+r}$, that multiplies both assets and the expected present value of labour income. In this model, an increase in this expected return raises this coefficient, and thus boosts consumption.

A Concrete Example: Constant Expected Growth in Labour Income

This RE-PIH model can be made more concrete by making specific assumptions about expectations concerning future growth in labour income. Suppose, for instance, that households expect labour income to grow at a constant rate g in the future:

$$E_t Y_{t+k} = (1+g)^k Y_t \tag{22}$$

This implies

$$C_t = \frac{r}{1+r}A_t + \frac{rY_t}{1+r}\sum_{k=0}^{\infty}\left(\frac{1+g}{1+r}\right)^k \tag{23}$$

As long as $g < r$ (and we will assume it is) then we can use the geometric sum formula to simplify this expression

$$\sum_{k=0}^{\infty}\left(\frac{1+g}{1+r}\right)^k = \frac{1}{1-\frac{1+g}{1+r}} \tag{24}$$

$$= \frac{1+r}{r-g} \tag{25}$$

This implies a consumption function of the form

$$C_t = \frac{r}{1+r}A_t + \frac{r}{r-g}Y_t \tag{26}$$

Note that the higher is expected future growth in labour income g, the larger is the coefficient on today's labour income and thus the higher is consumption.

The Lucas Critique

The fact that the coefficients of so-called *reduced-form* relationships, such as the consumption function equation (26), depend on expectations about the future is an important theme in modern macroeconomics. In particular, in a famous paper, rational expectations pioneer Robert Lucas pointed out that the assumption of rational expectations implied that these coefficients would change if expectations about the future changed.[2] In our example, the MPC from current income will change if expectations about future growth in labour income change.

Lucas's paper focused on potential problems in using econometrically-estimated reduced-form regressions to assess the impact of policy changes. He pointed out that changes in policy may change expectations about future values of important variables, and that these changes in expectations may change the coefficients of reduced-form relationships. This type of problem could make reduced-form econometric models based on historical data useless for policy analysis. This problem is now known as the *Lucas critique* of econometric models.

To give a specific example, suppose the government is thinking of introducing a temporary tax cut on labour income. As noted above, we can consider Y_t to be after-tax labour

[2] Robert Lucas, "Econometric Policy Evaluation: A Critique," *Carnegie-Rochester Series on Public Policy*, Vol. 1, pages 19-46.

income, so it would be temporarily boosted by the tax cut. Now suppose the policy-maker wants an estimate of the likely effect on consumption of the tax cut. They may get their economic advisers to run a regression of consumption on assets and after-tax labour income. If, in the past, consumers had generally expected income growth of g, then the econometric regressions will report a coefficient of approximately $\frac{r}{r-g}$ on labour income. So, the economic adviser might conclude that for each extra dollar of labour income produced by the tax cut, there will be an increase in consumption of $\frac{r}{r-g}$ dollars.

However, if households have rational expectations and operate according to equation (21) then the true effect of the tax cut could be a lot smaller. For instance, if the tax cut is only expected to boost this period's income, and to disappear tomorrow, then each dollar of tax cut will produce only $\frac{r}{1+r}$ dollars of extra consumption. The difference between the true effect and the economic advisor's supposedly "scientific" regression-based forecast could be substantial. For instance, plugging in some numbers, suppose $r = 0.06$ and $g = 0.02$. In this case, the economic advisor concludes that the effect of a dollar of tax cuts is an extra 1.5 $(=\frac{.06}{.06-.02})$ dollars of consumption. In reality, the tax cut will produce only an extra 0.057 $(=\frac{.06}{1.06})$ dollars of extra consumption. This is a big difference.

The Lucas critique has played an important role in the increased popularity of rational expectations economics. Examples like this one show the benefit in using a formulation such as equation (21) that explicitly takes expectations into account, instead of relying only on reduced-form econometric regressions.

Incorporating Time-Varying Asset Returns

One simplification that we have made up to now is that consumers expect a constant return on assets. Here, we allow expected asset returns to vary. The first thing to note here is that one can still obtain an intertemporal budget constraint via the repeated substitution method. This now takes the form

$$\sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{\left(\prod_{m=1}^{k+1} (1 + r_{t+m})\right)} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{\left(\prod_{m=1}^{k+1} (1 + r_{t+m})\right)} \tag{27}$$

where $\prod_{n=1}^{h} x_i$ means the product of x_1, x_2 .... x_h. The steps to derive this are identical to the steps used to derive equation (52) in handout 4.

The optimisation problem of the consumer does not change much. This problem now has the Lagrangian

$$L\left(C_t, C_{t+1}, ...\right) = \sum_{k=0}^{\infty} \left(\frac{1}{1+\beta}\right)^k U\left(C_{t+k}\right) + \lambda \left[A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{\left(\prod_{m=1}^{k+1} \left(1+r_{t+m}\right)\right)} - \sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{\left(\prod_{m=1}^{k+1} \left(1+r_{t+m}\right)\right)} \right]$$

And instead of the simple Euler equation (12), we get

$$U'\left(C_t\right) = E_t \left[\left(\frac{1+r_{t+1}}{1+\beta} \right) U'\left(C_{t+1}\right) \right] \tag{28}$$

or, letting

$$R_t = 1 + r_t \tag{29}$$

we can re-write this as

$$U'\left(C_t\right) = E_t \left[\left(\frac{R_{t+1}}{1+\beta} \right) U'\left(C_{t+1}\right) \right] \tag{30}$$

Asset Returns and the Equity Premium Puzzle

Previously, we had used an equation like this to derive the behaviour of consumption, given an assumption about the determination of asset returns. However, Euler equations have taken on a double role in modern economics because they are also used to consider the determination of asset returns, taking the path of consumption as given. The Euler equation also takes on greater importance than it might seem based on our relatively simple calculations because, once one extends the model to allow the consumer to allocate their wealth across multiple asset types, it turns out that equation (30) must hold for *all* of these assets. This means that for a set of different asset returns $R_{i,t}$, we must have

$$U'\left(C_t\right) = E_t \left[\left(\frac{R_{i,t+1}}{1+\beta} \right) U'\left(C_{t+1}\right) \right] \tag{31}$$

for each of the assets.

So, for example, consider a pure risk-free asset that pays a guaranteed rate of return next period. The nearest example in the real-world is a short-term US treasury bill. Because there is no uncertainty about this rate of return, call it $R_{f,t}$, or the discount rate, these terms can be taken outside the expectation term, and the

$$U'\left(C_t\right) = \frac{R_{f,t+1}}{1+\beta} E_t \left[U'\left(C_{t+1}\right) \right] \tag{32}$$

So, the risk-free interest rate should be determined as

$$R_{f,t+1} = \frac{(1+\beta)\, U'(C_t)}{E_t\left[U'(C_{t+1})\right]} \tag{33}$$

To think about the relationship between risk-free rates and returns on other assets, it is useful to use a well-known result from statistical theory, namely

$$E(XY) = E(X)E(Y) + Cov(X,Y) \tag{34}$$

The expectation of a product of two variables equals the product of the expectations plus the covariance between the two variables. This allows one to re-write (31) as

$$U'(C_t) = \frac{1}{1+\beta}\left[E_t\left(R_{i,t+1}\right) E_t\left(U'(C_{t+1})\right) + Cov\left(R_{i,t+1}, U'(C_{t+1})\right)\right] \tag{35}$$

This can be re-arranged to give

$$\frac{(1+\beta)\, U'(C_t)}{E_t\left[U'(C_{t+1})\right]} = E_t\left(R_{i,t+1}\right) + \frac{Cov\left(R_{i,t+1}, U'(C_{t+1})\right)}{E_t\left[U'(C_{t+1})\right]} \tag{36}$$

Note now that, by equation (33), the left-hand-side of this equation equals the risk-free rate. So, we have

$$E_t\left(R_{i,t+1}\right) = R_{f,t+1} - \frac{Cov\left(R_{i,t+1}, U'(C_{t+1})\right)}{E_t\left[U'(C_{t+1})\right]} \tag{37}$$

This equation tells us that expected rate of return on risky assets equals the risk-free rate *minus* a term that depends on the covariance of the risky return with the marginal utility of consumption. This equation is known as the *Consumption Capital Asset Pricing Model* or Consumption CAPM, and it plays an important role in modern finance.

In theory, the consumption CAPM should be able to explain to us why some assets, such as stocks, tend to have such high returns. However, it turns out that it has some difficulty in doing so. Empirical calculations tend to implement the model using some specific utility function such as $U(C_t) = C_t^{\theta}$. But these calculations usually tell us that the equity premium—the difference between the average return on stocks and the average risk-free rate—could only be consistent with the consumption CAPM if the parameter θ implied extraordinarily high (and generally incredible) levels of risk aversion. There is now a very large literature dedicated to solving the so-called equity premium puzzle, but as of yet no agreed best solution.[3]

[3] The paper that started this whole literature is Rajnish Mehra and Edward Prescott, "The Equity Premium: A Puzzle" *Journal of Monetary Economics*, 15, 145-161. For a review, see Narayana Kocherlakota, "The Equity Premium: It's Still a Puzzle" *Journal of Economic Literature*, 34, 42-71.