Topic 2: AK Models

While the Solow model is widely used as a baseline model of economic growth, it is still considered by many to be unsatisfactory as a description of the process leading to economic growth. This is because the model views improvements in total factor productivity (technological progress) to be the ultimate source of growth in output per worker, but does not provide an explanation as to where these improvements come from. In the language of economists, long-run growth is determined by something that is *exogenous* in the model. In the next two handouts, we will examine two different theories in which growth is *endogenous*, meaning determined by the actions of the economic agents described in the model. The two models achieve this quality using two very different routes: The one we look at now does so by getting rid of the assumption of diminishing marginal returns to capital accumulation, the second by endogenising the rate of technological progress.

The AK Model

Recall that our production function in the Solow model was of the form

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \qquad 0 < \alpha < 1 \tag{1}$$

and the parameter α being less than one implied diminishing marginal returns to capital accumulation, which had important implications for the model. This suggests that one route to "endogenising" the growth rate is to dispense with the dimishing marginal returns assumption altogether. The simplest such model just sets $\alpha = 1$ which gives

$$Y_t = A_t K_t \tag{2}$$

For obvious reasons, this class of models—first discussed in Sergio Rebelo's 1990 *JPE* paper—are known as "AK" models.

We can use similar methods to those used before to think about what a steady-state growth path might look like in this economy. Taking logs and derivatives, the growth rate of output is determined by

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} + \frac{\dot{K}_t}{K_t} \tag{3}$$

Capital accumulation is still given by

$$\dot{K}_t = sY_t - \delta K_t \tag{4}$$

So the growth rate of capital input is

$$\frac{\dot{K}_t}{K_t} = \frac{sY_t}{K_t} - \delta \quad (5)$$
$$= sA_t - \delta \quad (6)$$

Thus, the growth rate of output is

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} + sA_t - \delta \quad (7)$$

Note that in this model, the growth rate of output depends not only on the growth rate of A but also on the level. This means tha an increasing A will lead to an explosive path for output, so steady growth requires $\frac{\dot{A}_t}{A_t} = 0$. For this reason, the steady-state growth rate consistent with the AK model is

$$\left(\frac{\dot{Y}_t}{Y_t}\right)^{ss} = sA - \delta \quad (8)$$

This simple model shows that getting rid of diminishing marginal returns to capital accumulation has a dramatic effect on the model's predictions for the sources of growth. The steady-state growth rate depends positively on the savings rate and negatively on the depreciation rate, neither of which had any effect on long-run growth in the Solow model. Also, the fact that the *level* of technological efficiency has an effect on the growth rate has important implications. While it may be impossible (if this model were the correct one) for A to grow without bounds, it may still be possible for government policy to affect the level of A, for instance through regulatory policies. Thus, the AK model presents a dramatically different picture of growth, and one in which the link between government actions and growth is much more obvious than in the Solow model.

A More Realistic AK Model

There is an obvious criticism of the simple AK model: Where are the people? Clearly one needs human input to create the goods and services that constitute GDP. The following model shows, however, that one can extend the AK model to a case in which there is labour input as well as physical capital. The key here, however, that will distinguish this model from the Solow model is that the effect of labour input is determined by the stock of what is termed, *human capital*. Thus, a more skilled individual will be assumed to produce more output than an unskilled individual, and the total stock of such "skills" is called human capital. Crucially, human capital can be accumulated through education. Thus, both types of capital can be accumulated—this turns out to imply that the model has similar properties to the AK model.

Formally, this model views output as determined by

$$Y_t = A_t K_t^{\alpha} H_t^{1-\alpha} \tag{9}$$

where K_t is physical capital and H_t is human capital. So, growth is determined by

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} + \alpha \frac{\dot{K}_t}{K_t} + (1-\alpha)\frac{\dot{H}_t}{H_t} \tag{10}$$

We will assume that each type of capital accumulates according to

$$\dot{K}_t = s_K Y - \delta K \tag{11}$$

$$\dot{H}_t = s_H Y - \delta H \tag{12}$$

Note that I've assumed the same depreciation rate for both types of capital. This simplifies some of the calculations for the model without changing any of the relevant conclusions.

As before, we can define capital output ratios

$$x_K = \frac{K}{Y} \tag{13}$$

$$x_H = \frac{H}{Y} \tag{14}$$

and, as before, we can write the growth rates of the two capital inputs as functions of the relevant capital-output ratios

$$\frac{\dot{K}}{K} = \frac{s_K}{x_K} - \delta \tag{15}$$

$$\frac{\dot{H}}{H} = \frac{s_H}{x_H} - \delta \tag{16}$$

With the simple AK model, we showed that it was not possible to have steady-state growth unless A was constant. The same turns out to be true for this model, but demonstrating it is a bit more subtle. First, note that one can re-write the production function as

$$Y = A\left(x_K Y\right)^{\alpha}\left(x_H Y\right)^{1-\alpha} \tag{17}$$

Dividing across by Y this becomes

$$1 = A x_K^{\alpha} x_H^{1-\alpha} \tag{18}$$

Taking logs and derivatives of this, we get

$$\frac{\dot{A}}{A} = -\alpha \frac{\dot{x_K}}{x_K} - (1-\alpha)\frac{\dot{x_H}}{x_H} \tag{19}$$

So, positive steady-state growth in A is only possible, if a weighted average of the growth rates of the capital-output ratios was negative. But, falling values of x_K and x_H would imply exploding values for the growth rates of K and H (see equations 15 and 16) so this is not consistent with steady-state growth.

Equations 15 and 16 imply that the steady-state consistent with constant growth in K and H features both x_K and x_H being constant, and thus both type of capital growing at the same rate as output. This implies

$$\frac{\dot{K}_t}{K_t} = \frac{s_K}{x_K} - \delta = \frac{s_H}{x_H} - \delta = \frac{\dot{H}_t}{H_t} \tag{20}$$

This tells us that

$$\frac{s_K}{x_K} = \frac{s_H}{x_H} \Longrightarrow \frac{x_K}{x_H} = \frac{s_K}{s_H} \Longrightarrow \frac{K}{H} = \frac{s_K}{s_H} \tag{21}$$

So, along a steady-state growth path, the stock of human capital can be written as

$$H = \frac{s_H}{s_K} K \tag{22}$$

We can use this fact to re-write the level of output on the steady-state path as

$$\begin{aligned} Y &= AK^{\alpha}\left(\frac{s_H}{s_K}K\right)^{1-\alpha} && (23)\\ &= A\left(\frac{s_H}{s_K}\right)^{1-\alpha} K && (24) \end{aligned}$$

and the growth rate of output is

$$\begin{aligned} \frac{\dot{Y}}{Y} &= \frac{\dot{K}}{K} && (25)\\ &= \frac{s_K Y}{K} - \delta && (26)\\ &= A s_K \left(\frac{s_H}{s_K}\right)^{1-\alpha} - \delta && (27)\\ &= A s_K^{\alpha} s_H^{1-\alpha} - \delta && (28) \end{aligned}$$

So, we see that allowing for both types of inputs—physical and human capital—to be continuously accumulated produces a model that has the same long-run implications as the basic AK model. The steady-state growth rate is now $As_K^{\alpha}s_H^{1-\alpha} - \delta$ instead of $As - \delta$, so we have replaced the single savings rate with a geometric average of the two savings rate in the two-factor model. But substantively, the model's implications are unchanged.

Discussion of AK Model

The preceding model showed that AK models cannot be dismissed as easily as one might first have thought. That said, there are still some reasons to doubt the predictions about long-run growth generated by this class of models:

- *Non-Accumulable Factors*: In the real world, there are factors of production that are in fixed supply, such as land, or that cannot simply be accumulated indefinitely such as energy. Remember that the AK model results are of a knife-edge variety: Any move away from all factors being accumulable, and we move back to the Solow model results.
- *Treatment of Human Capital*: The strict parallel between human capital and physical capital in the model just described is probably not completely accurate. For instance, not all expenditures on education will produce the same effect on output. The marginal boost to aggregate output of teaching someone how to read and write is presumably greater than that of a masters in economics! Thus, there may be limits to which one can increase growth just by boosting educational enrollment.

It may also be reasonable to assume some role for non-skilled labour in the production function. For instance, in a well-known paper, Greg Mankiw, David Romer, and David Weil (*Quarterly Journal of Economics*, May 1992) put forward a model in which the production function is

$$Y_t = A_t K_t^{\alpha} H_t^{\beta} L_t^{1-\alpha-\beta} \qquad 0 < \alpha < 1 \qquad 0 < \beta < 1 \tag{29}$$

In this case, there is a role for human capital, H, which can be accumulated but also for unskilled labour, L. Not all factors in this model can be accumulated, so its long-run growth rate is determined by the growth rate in A_t, as in the Solow model. However, the model behaves more like a Solow model with a higher "capital share" parameter, (i.e. a higher value of the parameter α in the last handout). This implies larger "level effects" of changes in saving on output per worker, and also slower convergence speeds.

The moral of this type of model is that the AK approach may be strictly wrong about savings rates affecting long-run growth rates, but they may have larger level effects than in the basic Solow model. Also, while the AK models may be wrong about there being no "convergence dynamics" towards a steady-state level of output, these dynamics may be slower than the Solow model predicts.